Use of GeoGebra in explorative, illustrative and demonstrative moments

Uso de GeoGebra en los momentos de exploración, ilustración y demostración

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Abstract

The aim of this work is to present a set of GeoGebra constructions designed to study general properties over triangles, selected from Secondary Education curriculum. In one hand, these constructions allow a class management centered on properties, rather than on individual examples. On the other hand, there will be a discussion whether these constructions help to cross the "inductive-deductive" gap from explanation to formal proof in geometry.
***Keywords**: geometry; illusion of transparency; teacher training; formal proof.*

Resumen

Se presenta una serie de construcciones con GeoGebra en las cuales se trabajan propiedades generales de geometría plana de contenidos propios de Educación Secundaria. Por un lado, estas construcciones permiten una gestión de aula centrada en las propiedades y no en los ejemplos particulares. Por otro lado, se discute si estas construcciones permiten pasar del razonamiento inductivo (explicación) al deductivo (demostración formal) en geometría.
***Palabras clave**: geometría; ilusión de la transparencia; formación del profesorado; demostración formal.*

GeoGebra for instruction of Primary and Secondary School Teachers

In recent years, GeoGebra (GGB) has displaced Cabri II Plus at Spanish Universities, in Primary School Teacher Grades and Secondary School Teacher Masters (LASA, SÁENZ DE CABEZÓN and WILHELMI, 2009, 2010). Some motivations are purely pragmatic (language, technical and economic) and some others epistemological and educational; namely: GGB presents tools for integrated development of notions, processes and meanings on Geometry, Algebra and Functions Theory, that highlights the essentially relational aspect of mathematics. In addition, since version 4.0, a statistical package contributes to its versatility, and the 3D version is on the roadmap for software developers. All this justifies why its use is gradually spreading at Primary and Secondary schools.

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Despite these advantages, the widespread use of the program at Secondary Schools is far from been a done deal. Therefore, centres for teacher assistance carry out concrete activities in order to increase their digital competence in this area. These activities are reinforced by those organized by the various GGB Institutes in Spain: discussion forums (wiki-GeoGebra), attendance seminars, training courses for teachers and classroom activities for students (SÁENZ DE CABEZÓN, LASA and WILHELMI, 2009, 2010).

Therefore, in this context, it's pertinent to include GGB at University Education. The use of GGB has been adapted to new Primary School Teacher Grades and Secondary School Teacher Masters (SSTM). One of the aims of these latest studies is to instruct prospective Secondary School teachers on the basic handling of the programme, so they can evaluate, use and, ultimately, independently design applets and situations with GGB for learning and teaching processes in mathematics.

1. GGB and illusion of transparency in geometry classroom

SSTM students within Mathematics speciality must develop geometry topics, among others, both for exposure and for solving exercises and theorem proving. GGB has been used together with digital whiteboard to complement these developments and visualize geometry results related to Secondary School curriculum.

The use of traditional blackboard in the illustration of geometry contents has the obvious limitation of not being able to display more than one example, or a few, of geometrical representations during each session. These examples can generate a phenomenon of *illusion of transparency*, i.e., "the phenomenon whereby while teachers interpret an example as a model or as a representative of a class, students only see such an example". This phenomenon is an example of the distance in the dynamics of construction and communication of mathematics, as scientific knowledge and as a crystallized and labeled teaching object at schools. It is therefore essential to identify means that allow students and teachers 'talk the same language'". How can GGB contribute to overcome this phenomenon? i.e., how could GGB shorten the distance between the interpretation of an example as an "isolated object" and as a "representative of a class"? (LASA and WILHELMI, 2012). Answering these questions is essential in any process of generalization, in which *intensive* (general) and *extensive* (particular) objects are involved.

> “As a result of a generalization process we obtain a type of mathematical object we call intensive object, which becomes the rule that generates the class (collection or a set) of generalized objects and that enables the identification of particular elements as representative of a class (GODINO, FONT, WILHELMI and LURDUY, 2011). Through particularization processes new objects are obtained that we call extensive (particular) objects. A finite set or collection of particular objects simply listed should not be considered as an intensive until the subject shows the rule applied to delimit the constituent elements of the set. Then the set becomes something new, different from the constituent elements, as a unitary entity emerging from the set. Therefore, besides the generalization process giving rise to the set, there is a process of unitization.” (AKÉ, GODINO, GONZATO and WILHELMI, 2013)

Therefore, generalization processes are complex by nature and specific educational decisions may be taken to allow students develop their knowledge from particular elements to determination of classes.

2. Three basic moments to apply GGB

There are three moments on mathematical activity at Secondary School where the use of GGB is pertinent: exploration, illustration and demonstration of a property. On these three moments the *example-class* duality is essential (WILHELMI, GODINO and FONT, 2007), since geometric objects are then accepted as models of particular kinds of situations. In general, GGB is currently used to illustrate a particular property using an example; therefore, explorative and demonstrative moments exceed its principal use.

2.1. Exploration

Dynamic geometry software allows the construction of explorative models for solving exercises and problems. These models serve to the purpose of inferring properties from a geometric figure or construction so far unknown.

The goal is to design a construction that satisfies the restrictions of a proposition or the initial conditions of a problem. After manipulating the construction, students deduce its properties.

Normally, these constructions are not made by students. Teachers previously design the constructions or take them from a catalogue, such as the Gauss Project (http://recursostic.educacion.es/gauss/web/).

2.2. Illustration of a property

As mentioned before, the widespread use of dynamic geometry software –in particular, GGB–, consists in giving examples of properties by means of concrete cases selected *ad hoc*. A construction is presented which shows the veracity of a given property. This construction serves as a manipulative model and its use may be complemented by a digital whiteboard. Thus, for example, to study properties of a triangle, the dynamic software can generate multitude of triangles, instead of only a few of them as in the ordinary slate.

This widespread use should motivate new examples that would improve students' confidence in the formulated conjectures. Hölz (2001), cited by Burke and Kennedy (2011), states that in an environment of dynamic geometry software, student who observe the truth of a conjecture has the urge to know the reason for the claim. After all, the illustration of a property is just a "picture" of it. At Secondary Education, material and temporal constraints, or student cognitive restrictions may lead to the didactical decision of ignoring the formal demonstration of the property, limiting the activity to an illustrative presentation of it. However, there may be differences between the illustrative construction and formal proofs of these same properties.

2.3. Demonstration of a property

Traditionally, the step-by-step formal proof of a geometrical property it's carried out on blackboard. However, since ordinary blackboards are substituted by digital whiteboards and dynamic geometry software, these formal proofs are left out; illustrative constructions are not designed considering elements of the formal proof, and sometimes, computing steps differ from pure logical reasoning. It's teacher's job to select situations which permit to join both reasoning; inductive reasoning due to dynamic geometry software, and deductive reasoning, traditionally linked to formal proofs –pencil and paper proofs.

Most authors agree that inductive arguments should come first, since this type of reasoning serves to motivate students. In fact, students prefer pragmatic proofs rather than intellectual ones (BALLACHEFF, 1987). When assisting a mathematical production with a dynamic model, an empirical proof may be enough for students, since they can construct an inductive argument with a sufficient number of trials. Therefore, students do not have a practice in proving, and don't see a necessity to justify any mathematical process they use (DREYFUS, 1999). In addition, dynamic software such as GGB or Sketchpad present an environment where is easy to find counterexamples, therefore, the notion of axiom is extended and few propositions require a formal proof; if you fail to find a counterexample with the dynamic model, they're believed to be true (DE VILLIERS, 2004).

In this context, different authors present a number of classifications for roles demonstration may play, such us explanation (a statement made to clarify something and make it understandable), verbal argument (a process of reasoning, intended to convince) or formal proof (mathematic demonstration) (DREYFUS, 1999); there is an ultimate role for proof which is usually not considered at school, that is, the role of *a posteriori* systematization, i.e., to organize unrelated results into a unified whole (DE VILLIERS, 2004). Actually, one of the challenges in mathematics education is that of evaluation, i.e., to evaluate student's proof-like productions, deciding whether an argument is accurate enough to be considered a proof.

We do not wish to defend the traditional transmission-model. Nevertheless, in this work, we will consider demonstration as "a formal proof". Since it's clear from literature that GGB serves to the purposes of illustration, we would like to make a step forward and discuss which characteristics should a GGB construction have in order to aid a formal argument. In fact, the aim of section 3 is to present examples of GGB constructions and discuss whether they help the articulation of a formal proof.

3. Examples of GGB constructions in geometry

In this section we show examples of constructions of the three moments described in the previous section. All examples refer to constructions over triangles taken from geometry curriculum at Secondary Education. All examples will be given according to the same scheme:

- Statement.
- Description of the construction.
- Utility.
- Justification or formal proof.

3.1. Example of model for experimentation

- *Statement*. Given a triangle, $\widehat{ABC}$, show that $\hat{A}$ is a right angle if and only if the median length from *A* equals half the length $\overline{BC}$ of the triangle.
- *Description of the construction*. The construction represents the elements of the statement (triangle $\widehat{ABC}$, median $\overline{AA'}$, segments that define the midpoint A' on the side *a*, and the angle α we intend to study) and allows to explore the situation. For any disposition of the triangle with obtuse angle α, the median length is smaller than half the segment $\overline{BC}$, while to any disposition of the triangle with acute angle α, the median length is greater than half the segment $\overline{BC}$.
- *Utility*. The construction helps to make a conjecture. In addition, gives the possibility to think a way to prove the property, showing the involved elements and certain results that may serve the purpose of completing the proof. Figure 1 show two moments of this exploration. After the explorative moment, the activity turns to find a logical argument to complete the proof.
- *Justification or formal proof*. Suppose $\widehat{A}$ right; given that $\widehat{B} + \widehat{C} = 90$, we divide angle $\widehat{A}$ in two angles, $\widehat{B}$ and $\widehat{C}$; triangles $\widehat{AA'C}$ and $\widehat{AA'B}$ are isosceles, having each two equal angles; therefore, segments $\overline{AA'}$, $\overline{A'C}$ and $\overline{A'B}$ are of equal length, and it follows that point A' is in fact the midpoint of the segment $\overline{BC}$. We could reverse the argument to obtain the second implication.

Figure 1a:

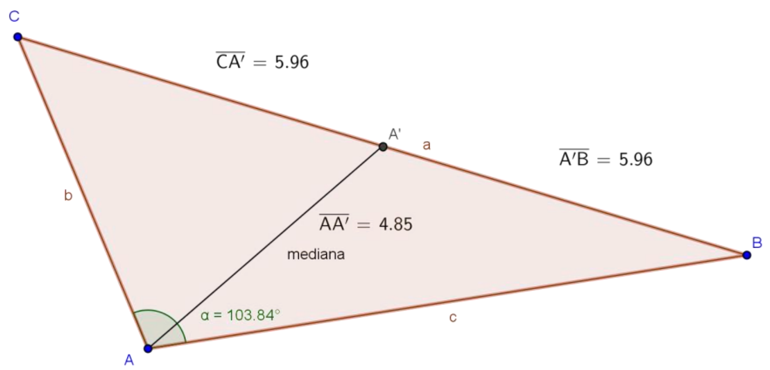

Figure 1b:

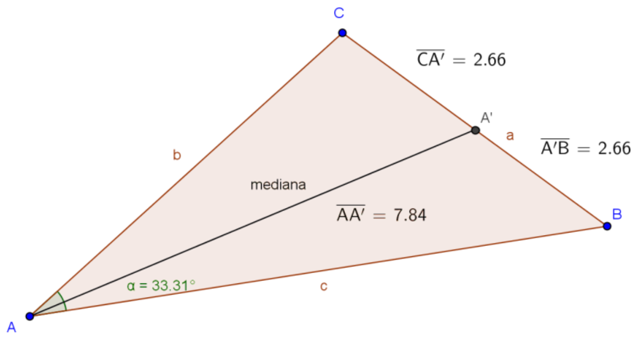

FIGURE 1: Explorative moment

3.2. Example of illustration of a property

- *Statement*. Given a triangle, if one side is less than another, then the opposite angles to each of these sides satisfy the same inequality.
- *Description of the construction*. The geometric construction effectively explains the property described, but gives no clue to prove the veracity of the claim. Text boxes display the change in inequalities for angles, when the length of sides satisfies the same inequality (figure 2).
- *Utility*. GGB is a handy tool to explain the property, since it allows you to create a "numerical" model. However, it gives no clue to justify what is observed.
- *Justification or* formal *proof*. When the triangle is equilateral or isosceles, clearly, $A = B$ if and only if $a = b$. Suppose then an scalene triangle with $a < b$. In this case, draw an arc of radius a centered at vertex C, which intersects b in A'. The triangle $\widehat{A'BC}$ is isosceles, with equal angles $\widehat{B}'$ and $\widehat{A}'$ (vertex B on the original triangle corresponds to vertex B' on the inner triangle). Since $\hat{B} > \widehat{B}'$, $\widehat{B}' = \widehat{A}'$ and $\widehat{A}' > \hat{A}$, then we get $\hat{A} < \hat{B}$.

Figure 2a:

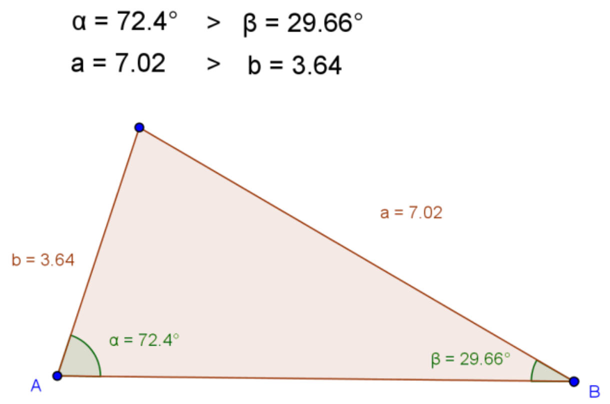

Figure 2b:

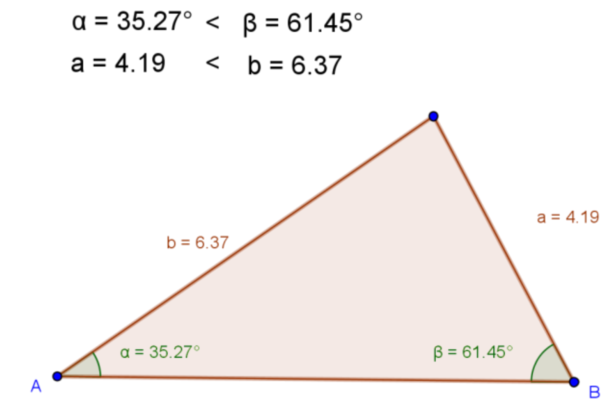

FIGURE 2: Explorative moment

3.3. Example of demonstration of a property

We present three examples of classic constructions with triangles. For each of them, we'll compare the illustrative and demonstrative constructions.

3.3.1. Intersection of bisectors of a triangle: incenter

- *Statement*. The three bisectors of a triangle intersect at a single point, called incenter.
- *Description of the construction*. The illustrative construction (figure 3a) shows elements that participate in the proof of the property. Also, some other elements are quoted, which are conclusion of the property. With the help of checkboxes to show or hide, as required, the bisectors and their intersections, the inscribed circumference, it's radius to each side of the triangle, and three

inner triangles defined by these radii/heights.

The demonstrative construction (figure 3b) only shows those elements that are necessary for the logic proof: two bisectors, through which we conclude that the third bisector necessarily have to intersect at the same point where the first two do.

Figure 3a:

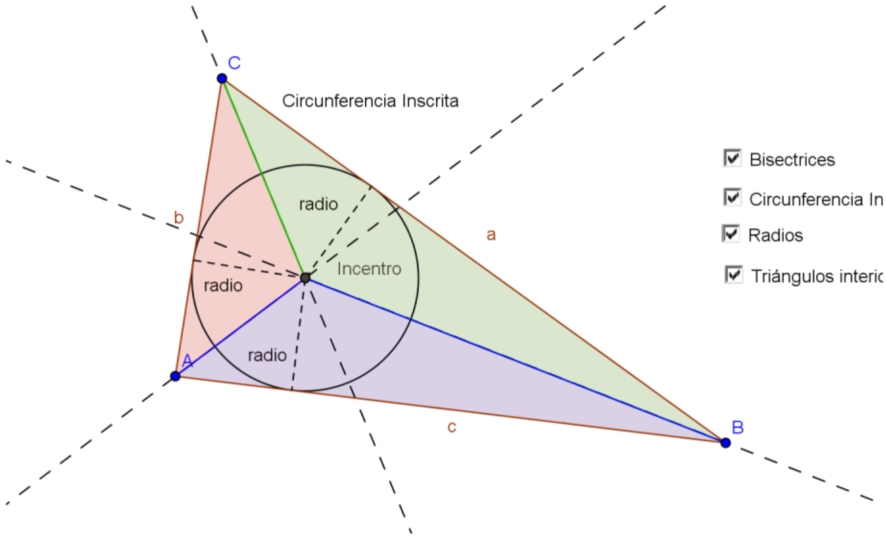

Figure 3b:

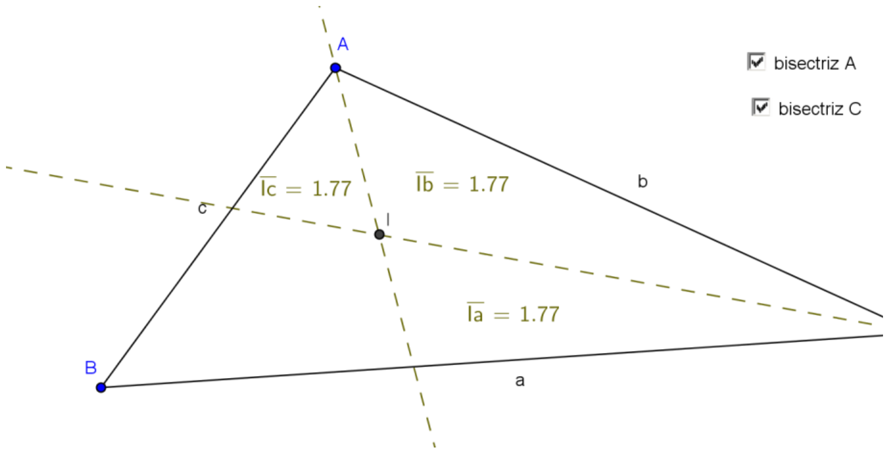

FIGURE 3: Incenter

- *Utility*. The illustrative construction is interesting since synthesizes many contents concerning the incenter, and shows the relationship between these elements. However, in the study of plane geometry, the illustrative construction does not always go hand in hand with the formal proof. The proof needs formal reasoning, and the austere demonstrative construction leads to this path.
- *Justification or formal proof*. The bisector of the angle $\hat{A}$ is equidistant from sides b and c; in turn, the bisector of angle $\hat{C}$ is equidistant from sides a and

b; therefore, the intersecting point of the two segments, I, is necessarily equidistant from sides a and c, and it's necessarily on the bisector of angle $\hat{B}$.

We will not go into details, but a similar situation arises when comparing the illustrative and demonstrative constructions associated to the study of the circumcenter and the circumcircle (figure 4).

Figure 4a:

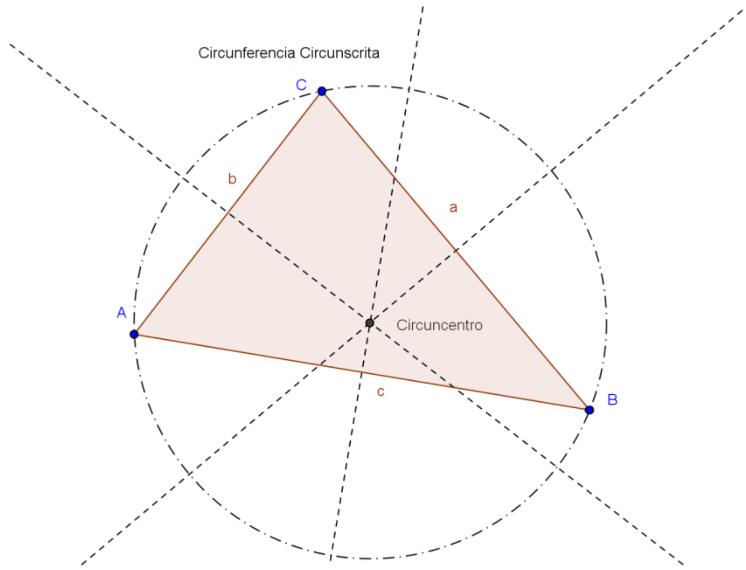

Figure 4b:

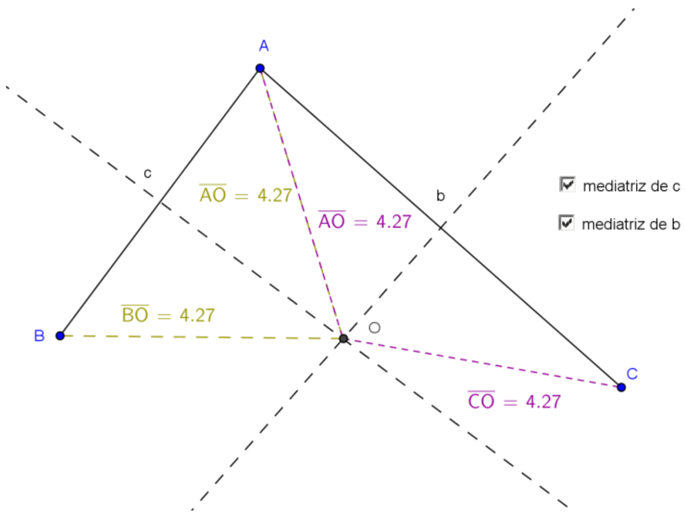

FIGURE 4: Circumcenter

3.3.2. Intersection of the medians of a triangle: barycentre

- *Statement*. The three medians of a triangle intersect at a single point, called barycentre.
- *Description of the construction*. The *illustrative construction* (figure 5a)

merely indicates the intersection of the medians and the ratio of the segments that are formed, i.e., the barycentre cuts each median into two pieces of ratio 1:2. In fact, the step of marking the medians according to this ratio belongs to the proof and it's one of the arguments we need to conclude that medians do intersect at a single point; it's not a conclusion that arises from the property, as one may think when observing the illustrative construction. The *proof construction* shows otherwise (figure 5b). This second construction uses a slider to show, step-by-step, the argument which justifies the fact that three medians have to intersect at a single point.

Figure 5a:

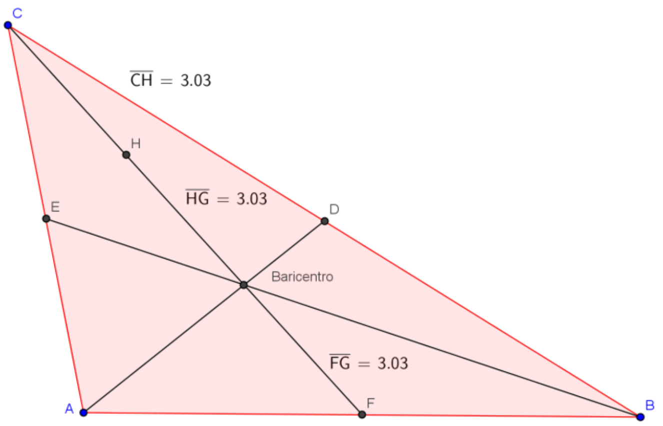

Figure 5b:

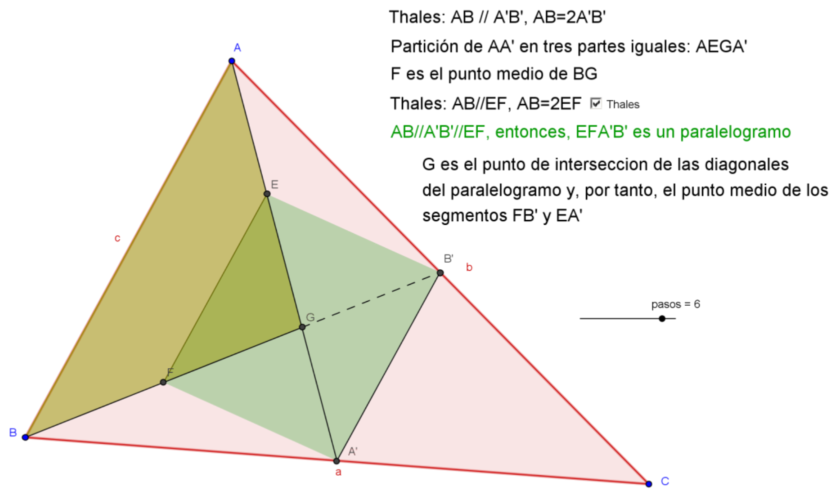

FIGURE 5: Barycentre

- *Utility*. The study of properties of the barycentre of a triangle is an extreme example of an illustration that does not correspond to the proof of the property. In fact, at Secondary School, they do explain that the barycentre is the point of intersection of the three medians of the triangle, without proving the truth of that claim.

- *Justification or formal proof*. Let be A the midpoint in segment a and draw segment $\overline{AA'}$; divide segment $\overline{AA'}$ into three equal parts, $\overline{AE} = \overline{EG} = \overline{GA'}$; let F be the midpoint in segment $\overline{BG}$; by Thales, $\overline{AB} \parallel \overline{EF}$ and $\overline{AB} = 2\overline{EF}$; we conclude that $EFA'B'$ is a parallelogram; G is the point of intersection of the diagonals of the parallelogram, and thus, the midpoint of segment $\overline{FB'}$ and $\overline{EA'}$; we conclude that G is the point of intersection of the medians.

Educational implications

When working geometry on traditional blackboard, teachers use a geometric drawing or diagram to explain the steps of logical reasoning, in order to prove a property.

With the gradual introduction of dynamic geometry software, we proceed to illustrate the geometrical property through a dynamic construct on screen or digital whiteboard. The computer construction can skip steps of logical reasoning that justifies the property, since the design of the construction is determined by the sequence of tools from the computer program. The property “is seen”, and there exists a risk: the dynamic model may become a goal instead of a mean; i.e., it has a phenomenon of *metacognitive slipping* (BROUSSEAU, 1997): GGB becomes the finale goal of the teaching process.

University professors committed in the organization of Mathematical Olympiad tests in its various phases, alert of a fact with roots in the use of dynamic geometry software: students don’t use anymore Euclidean reasoning, and therefore, they are not able to perform geometric formal proofs or arguments requiring several steps because, based on their mathematical experience, they don’t have such necessity or instruction.

Mathematical tasks designed to be solved by dynamic models should contemplate two phases. In the first one, a dynamic illustrative model should be used to study the details of the problem. The inductive process carried out by dynamic models is essential to analyze particular elements, but this activity is partly worthless if no effort is made to highlight the *element-class* duality, i.e., the dynamic model has to open a path of reasoning to complete the formal proof of the property. Therefore, in a second phase, logical arguments should take a leading role. Otherwise, the property “is seen” but remains unproven.

This involves designing dynamic models which do not only cover the first phase of illustration, providing just an inductive proof based on the lake of counterexamples. The model should also show the way to prove the property, for which a second GGB construction can be used, probably a simpler one, which indicates the steps of the logical demonstration.

References

AKÉ, L.P., GODINO, J.D., GONZATO, M., WILHELMI, M.R. (2013). Proto-algebraic levels of mathematical thinking. In Lindmeier, A.M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for Psychology of Mathematics Education*, Vol. 2, pp. 1–8. Kiel, Germany: PME.

BALACHEFF, N. (1987). Processus de prevue et situations de validation. *Educational Studies in Mathematics*, 18(2), 147–176.

BROUSSEAU, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

BURKE, M., KENNEDY, P. (2011). GeoGebra: From simulation to formalization in teacher preparation and inservice programs. In L. Bu and R. Schoen (Eds.), *Model-Centered Learning*. (pp. 57–72) Rotterdam, NED: Sense Publishers.

DE VILLIERS, M. (2004). Using dynamic geometry to expand mathematics teacher's understanding of proof. *International Journal of Mathematics Education in Science and Technology*, 35(5), 703–724.

DREYFUS, T. (1999). Why Johnny Can't Prove. *Educational Studies in Mathematics*, 38(1–3), 85–109.

GODINO, J.D., FONT, V., WILHELMI, M.R., LURDUY, O. (2011). Why is the learning of elementary arithmetic concepts difficult? Semiotic tools for understanding the nature of mathematical objects. *Educational Studies in Mathematics*, 77(2), 247–265.

LASA, A., SAÉNZ DE CABEZÓN, A., WILHELMI, M. R. (2009). Restrictions in the lecture of geometry with GeoGebra in Primary School Teacher Education. *International GeoGebra Conference*, 14–15 July. Linz: University of Upper Austria.

LASA, A., SAÉNZ DE CABEZÓN, A., WILHELMI, M. R. (2010). Uso de GeoGebra en la Titulación de Maestro. *Epsilon*, 27(1), 7–14.

LASA, A., WILHELMI, M.R. (2012). GeoGebra en la formación de profesorado de E.S.O y Bachillerato. *Cónica.* N.3, en prensa. [Disponible en http://acgeogebra.cat/joomla/index.php]

SAÉNZ DE CABEZÓN, A., LASA, A., WILHELMI, M. R. (2009). La gestión del tiempo en la enseñanza de la geometría. *International GeoGebra Conference*, 14–15 July. Linz: University of Upper Austria.

SAÉNZ DE CABEZÓN, A; LASA, A., WILHELMI, M. R. (2010). Utilización de GeoGebra en proporcionalidad geométrica en 2º ESO. *Epsilon*, 27(1), 21–32.

WILHELMI, M. R., GODINO, J.D., FONT, V. (2007). Bases empíricas de modelos teóricos en didáctica de las matemáticas: reflexiones sobre la teoría de las situaciones didácticas y el enfoque ontológico y semiótico. En M. J. Alderete y M. L. Porcar (Eds.), *Temas de Didáctica de las Matemáticas*. (pp. 1–20) Mendoza, Argentina: Universidad de Cuyo. [Disponible en: http://www.webpersonal.net/vfont/Burdeos.pdf]