Long live triangles!
Dynamic models for trigonometry

¡Larga vida a los triángulos!
Modelos dinámicos en trigonometría

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Abstract

We present a GGB-Book to assist learning and teaching situations for trigonometry. First, we justify the choice to use dynamic geometry software as an instrument to organize mathematical activity, based on a classification of dynamic models based on the "moment of the mathematical activity". Second, we present a detailed theoretical proposal with guidelines to use it. Finally, we show some results of a practical experience with secondary school students (age 15-16).
Keywords*: trigonometry; explorative model; ostensive representation of mathematical objects; inductive arguments.*

Resumen

Se presenta en este trabajo un Libro-GGB para asistir situaciones de enseñanza y aprendizaje de la trigonometría. En primer lugar, se justifica la decisión de utilizar el software de geometría dinámica como instrumento para organizar la actividad matemática, basada en la clasificación de modelos dinámicos por "momentos de la actividad matemática". En segundo lugar, se presenta una propuesta teórica detallada junto con indicaciones de uso. Finalmente, se muestran los resultados de una experiencia práctica con estudiantes de educación secundaria (15-16 años).
Palabras clave: *trigonometría; modelo de exploración; representación ostensiva de objetos matemáticos; argumentos inductivos.*

Introduction

The second middle of the XX century was influenced by structuralism and the paradigm that the comprehension of general structures and reasoning would lead students to a better understanding of mathematics. The curriculum was reformed and mathematics where taught in Europe in terms of *modern mathematics*, which included a strong presence of logic, set theory and the use of symbols in early educational stages. For example, in Early Childhood and Primary Education, operative use of arithmetic and

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descriptive plane geometry where postponed and the inner structure of mathematics was strengthen instead (LACASTA & WILHELMI, 2012).

The approach to general mathematics even brought decisions to exclude particular representations of mathematical objects. For example, in their aim to rebuild the foundations of mathematics, the Bourbaki group presents geometry as a general algebra, without a single drawing[4].

> Mathematics were abstracted to the point where it became a complicated but ultimately meaningless game of moving symbols on paper according to algorithmic rules – meaningless in that this intellectual game was not meant to have signification in terms of the physical world or the situations of everyday life (GEROFSKY, 2016).

Bourbakis' general approach is stated clearly in their manifesto: "The organizing principle will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular" (MATHIAS, 1992, 6). This example is indicative of the importance and the status intensive objects and their ostensive (codified) representations have in mathematics practice and education.

Driven by Bourbakis paradigm, a rigorous argument must be necessarily abstract, and so, mathematics must be discussed using ostensive codified representations of intensive objects. For example, mathematicians may consider n-dimensional geometry to create general constructions. But, as Stewart (1994) remarks, there are geometric results related to plane geometry that do not arise from n-dimension geometry, such as the presence of number π in many different areas of mathematics and physics. Thus, the "general to particular" point of view fails to discuss special cases or non-algebraic topics.

Nevertheless, rigor is not abstraction, and mathematicians use many types of ostensive representations of mathematical objects (drawings, diagrams, natural language, etc.) to solve problems or model situations. Mathematic Education theories based on semiotic grounds classify ostensive representations as linguistic elements used in mathematical practice in their various registers (FONT, GODINO & GALLARDO, 2013).

In particular, gestures are didactically analyzed within *embodiment* and *multimodality* approaches, as being not only the first step in concept formation (Piagetian sensory-

[4] Leading Bourbaki mathematician Jean Dieudonné loads the famous sentence onto his shoulders: "Down with Euclid! Death to all triangles!" Bourbaki refused to use diagrams and other ostensive representations of particular mathematical objects; hence, the title for this article.

motor experiences), but an extension of the semiotic ground that unifies body and mental processes (ARZARELLO & SABENA, 2014). In this direction, neuroscience indicates that brain-functions work as a neuronal cloud or network, with no central brain location responsible for sense-making nor brain-modules controlling different brain areas devoted to different sensorial modalities.

> We don't store images as facsimiles of things, events, words or sentences […] since any facsimile-like storing system would cause insuperable capacity problems […] explicit memory-images take place in the mind when some neuronal lighting-patterns activate, simultaneously for an instant. (DAMASIO, 2012, 154-156)

Brain activity includes biological regulation, and control over emotions and feelings. Those aspects are relevant to operative and discursive mathematical practice too, since intuition or analogies are important factors to guide quests on problem solving heuristics (GUZMAN, 1991).

Bourbaki claimed a revolution against pictorial representations: "Death to all triangles!" We claim the opposite, the eternal resurrection of the ostensive realm as a previous and necessary stage to think about non-ostensive (ideal) mathematical objects: "Triangles are dead, long live triangles!"

1. Use of ostensive representations

While *modern mathematics* generated new results in mathematical research, in Mathematics Education led to the illusion that once the student is exposed to the mathematical structure, then s/he can naturally apply the structure to solve particular problems. However, in order to introduce the general mathematical object to a student, new language and symbols must be introduced too, which need further explanations and the introduction of auxiliary mathematical objects, in a circular process. Brousseau (1997) identifies that illusion as an example of a *metacognitive shift* phenomenon.

Structuralism has been gradually displaced in education, but there are still traces of that period that endure, mostly because of a lack of revision of textbooks (WILHELMI, BELLETICH, LACASTA & LASA, 2013; LASA, 2015, 2016). Books are accessory to learning and teaching processes, but as any other support (manipulative material, paper and pencil, software), a criterion must be applied to select the optimal and convenient combination of supports for each teaching situation, which includes a selection of appropriate ostensive representations.

The Onto-Semiotic Approach to Research in Mathematics Education (OSA) identifies *linguistic elements*, such as terms, expressions, notations or graphs, in their various registers (written, oral or gestural), which conform the ostensive part of operative and discursive mathematical practices. *Linguistic elements* are considered a type of *primary mathematical objects*, with *situations*, *definitions*, *propositions*, *procedures* and *arguments* (FONT, GODINO & GALLARDO, 2013).

> These languages are the ostensive part of a series of concepts / definitions, propositions and procedures that are involved in the elaboration of arguments whose purpose is to decide whether the simple actions of which the practice is composed, and the practice itself as a compound action, are satisfactory. (FONT, GODINO & GALLARDO, 2013, 109)

Each *primary mathematical object* has multiple representations, and mathematical activity includes processes of *treatment* and *conversion* (DUVAL, 1993), where the representation is modified, respectively, within a certain semiotic register, or from one semiotic register to another. *Sense* and *understanding* (RADFORD, 2004) are notions also related to representations: on the one hand, *sense* is the capacity to present different representations of the same mathematical object; on the other hand, the synthesis of these representations leads to the *understanding* of the object (D'AMORE, 2007).

Under these premises, the social perception which sees mathematics as the science of mere manipulation of symbols, slides mathematical activity to the fluent use of algebraic techniques, minimizing or excluding the use of alternative representations. At the same time, the general to particular approach generates new didactical phenomena, such as the *atomization* (LASA, 2016).

> Along the school year, teachers present split up procedures. Many times, this decision is justified by time-managing terms, or under the illusion that if the student is capable of efficient but isolated symbolic manipulations, then she could master them in complex situations. Yet, this decision leads to the disarticulation of different algebraic knowledge, and students lack methods to control their own proposals. (LASA, 2016, 276)

Inner connections of mathematical structures are complex by nature, but the school-system must manage time restrictions and finite resources. Teachers must decide how to spend time without wasting it, and, if time expires, they can decide to give some recipes to the student, to solve particular problems when applied to them. This approach is valid in the short term, but in the long term, students fail to *sense* and *understand* mathematical objects.

2. Explorative, illustrative and demonstrative moments

Manipulative materials, paper and pencil and dynamic software are examples of supports for use in mathematics practice. The late allows a classroom management centered on properties, rather than on individual examples (LASA & WILHELMI, 2013), and therefore, dynamic constructions help to cross the "inductive-deductive" gap from explanation to formal proofs.

> The use of traditional blackboard in the illustration of geometry contents has the obvious limitation of not being able to display more than one example, or a few, of geometrical representations during each session. These examples can generate a phenomenon of illusion of transparency, i.e., "the phenomenon whereby while teachers interpret an example as a model or as a representative of a class, students only see such an example". This phenomenon is an example of the distance in the dynamics of construction and communication of mathematics, as scientific knowledge and as a crystallized and labeled teaching object at schools. It is therefore essential to identify means that allow students and teachers 'talk the same language'". (LASA & WILHELMI, 2013, 53)

GeoGebra contributes to overcome this phenomenon, i.e., shortens the distance between the interpretation of an example as an "isolated object" and as a "representative of a class" (LASA & WILHELMI, 2013). This question is essential in any process of generalization, in which *intensive* (general) and *extensive* (particular) objects are involved. The *example-class* duality is essential in three moments of mathematical activity where the use of GeoGebra is pertinent: *exploration*, *illustration* and *demonstration* of a property (LASA & WILHELMI, 2013). Lasa (2016) joins those three moments up with the classical *action*, *formulation* and *validation* phases (BROUSSEAU, 1997) and articulates an outline to design situations where dynamic geometry software plays the role of *antagonistic milieu*[5]. Frame 1 summarizes this outline.

It should be note that the term *validation* includes a wide rank of types of arguments and functions of proofs, depending on context and needs, which go from "explanation" to "deductive-formal-proof". There is an active research field in Mathematics Education that studies the uses and meanings of proof. Many authors classify and clarify the different roles of proof, and the function of proof which suits better the context of dynamic software is probably that of De Villiers (1998), who notes that explorative

[5] *Antagonistic milieu*. According to TDSM, students who confront a *didactical situation* are able to validate their solution-proposal by means of an immediate feedback from the situation, i.e., a *situation* is a mathematical problem which, "by design", includes elements that give students the essential feedback to be aware of the correctness of their mathematical production. Thus, a dynamic model takes the place of

models aid students to develop intuitions before they formally verify a property.

GeoGebra moment	Action	Formulation	Validation
Explorative	Basic strategy	Concepts-in-act theorems-in-act (VERGNAUD, 1990)	Empirical proofs (trial and error)
Illustrative	Devolution. Eventually, micro-institutionalization in didactical situations with an essential *a didactical* component (BLOCH, 1996)	Explicit formulation of particular properties in concrete cases.	Partial proofs to test out and to convince (oneself and the fellow)
Demonstrative	Interpretation (feedback) to the answers given by the *antagonistic milieu*	Explicit formulation of theorems	Proofs to test out and to convince (oneself and the fellow)

FRAME 1. GeoGebra moments in didactical situations (LASA, 2016, 52)

Moreover, the classical inductive/deductive classification of mathematical arguments is gone out of fashion in official curriculums and textbooks (BON, 2015), and the irruption of automatic-theorem-solving software in mathematic activity modifies the concept of formal-proof itself, since a theorem can be proved true (in a deductive-formal sense) when checked for a finite number of cases (BOTANA ET AL, 2015).

The existence of empirical proofs obliges to revise the classification of arguments and their didactical transposition for every educational level, since the inductive/deductive classification is not accurate anymore from the epistemological point of view. GeoGebra tools include probabilistic numerical approximation modules, such as Are-Equal, Are-Parallel, etc. (figure 1), based on Boolean operations; and automatic-theorem-solving modules (Prove and Prove-Details), which traduce geometric information into algebraic language to compute its veracity by means of a deterministic procedure (BOTANA, 2015).

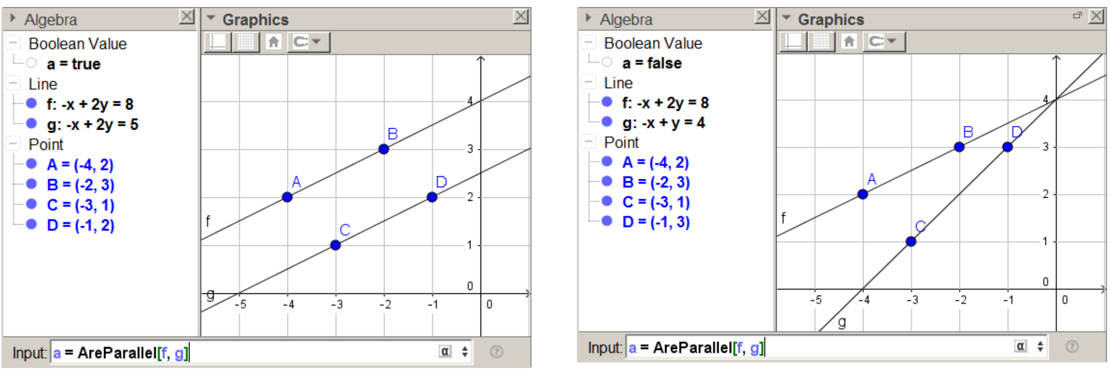

FIGURE 1. Output of GeoGebra PNA tool Are-Parallel

the antagonistic milieu, every time the model gives students that feedback.

Therefore, the use of automatic proofs in school mathematics is an open and interesting research area. Meanwhile, in our context of TDSM and OSA, the term *validation* stands for any type of proof used in the scholar context, regardless of its nature, any time they are useful to please students and teachers communicative processes in school mathematical activities.

In addition, there is empirical evidence (LASA, 2015, 2016) that the integration of dynamic models have a direct effect in student's performance when solving algebraic tasks. Furthermore, the sequence in which different supports are been used and implemented in school practice determine the type of improvement shown by the student:

- On the one hand, students who manipulate a dynamic model before they are introduced to the use of an algebraic technique, obtain the correct and complete algebraic solution more often that those students who are directly instructed in the algebraic technique.
- On the other hand, students who have been instructed in the algebraic technique, are able to articulate better arguments on the instrumental use of the dynamic model of the mathematic situation.

These results suggests a helical outline where the progressive achievement of mathematical knowledge swings from explorative moments (dynamic model) to the consolidation of algebraic techniques (paper and pencil), and back, and optimizes the acquisition of mathematical knowledge in teaching and learning situations.

Student's progressive achievement of algebraic knowledge has been model by OSA. Godino et al (2016) present a theoretical description of *algebraization levels,* which are useful to graduate and describe the uses of algebra in the resolution of mathematical tasks. Students may use basic arithmetical strategies in the total absence of algebra (level 0), algebraic structural properties over numbers (level 1), symbols to represent unknown values (level 2), consolidated algebraic techniques (level 3), etc.

In Chapter 3, we present a series of constructions for trigonometry, whose design follow the theoretical framework summarized on frame 1. Furthermore, the successive continuation of *explorative, illustrative* and *demonstrative* moments leads to a descriptive or *a posteriori* definition of trigonometric ratios: students use the explorative dynamic models in order to build a *personal meaning* (OSA) of trigonometric ratios; later, the teacher takes the responsibility to *institutionalize* (EOS, TDSM) the knowledge that arises from the mathematical activity. In Chapter 4, we briefly present the results of a pilot study (BELLOSO, 2016), where secondary students effectively use those construction in their school activity.

3. Three GeoGebra constructions for trigonometry

In the following sections, we describe a series of dynamic models, compiled in a GGB-Book[6] format and available in four languages (Basque, Italian, English and Spanish). These models are the result of an international collaboration between EHGI[7] and IGT[8]. The aim of the Book is to provide secondary school students (ages 14-17) a material to help them build a gradual meaning for trigonometric notions.

3.1. Trigonometric ratios

Students can use a numerical model as a first approach to *explore* invariant properties of trigonometric ratios. The presented models should be used by students, who begin to introduce themselves to trigonometry, and do not have previous knowledge, other than the classification of triangles (according to sides and angles), and the Pythagorean Theorem.

Using the basic strategy of arithmetic calculations, students seek empirical evidences to state *theorems-in-act* (VERGNAUD, 1990). The first few calculations are executed in paper, to reach students *arithmetical threshold* (LASA & WILHELMI, 2014), but in the long term, as calculations become tedious, a spreadsheet is required. The use of a spreadsheet gives way to the *illustrative* moment. The absence of counterexamples in the study of many cases makes it possible to construct an inductive argument, and students are convinced that the property is true for any particular triangle.

In addition, the dynamic model shows a triangle where the area of the triangle is variable. Therefore, the section of the plane between two rays is variable, even though the angle continues constant. Hence, the dynamic model contributes to overcome a classical epistemological obstacle: the amplitude of an angle does not depend on the area of the inner surface.

The process should end with a deductive argument to prove that trigonometric ratios are invariant, and thereby, they are worth a definition. In this last step, geometrical formal

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[6] The complete GGB-Book is available in the following link: https://www.geogebra.org/m/oOGHNoq8#
[7] Euskal Herriko GeoGebra Institutua, www.ueu.eus
[8] Istituto GeoGebra di Torino, https://www.geogebra.org/istituto+di+geogebra+di+torino

language is required to superimpose the structure of Thales Theorem on the explorative model of the triangle.

- *Sine, explorative moment*.
    - *Description of the model*. The construction[9] shows a dynamic right triangle. The user can select the amplitude of an acute angle (by inserting a numerical value in an "input box") and the length of the base of the triangle (by dragging a vertex, "point on a line").
    - *Steps*. Students work in pairs. First student manipulates the dynamic model in the graphic view to create a particular right triangle for a given acute angle. Second student writes down the values of the triangle from the model and calculates its trigonometric ratios. Then they switch positions and they start again with a new triangle. After a few tries, the *theorem-in-act* should arise.
- *Sine, illustrative moment*.
    - *Description of the model*. The illustrative construction[10] shows the previous explorative model and includes a spreadsheet to aid calculations (figure 2). Students must organize the spreadsheet information in three columns: length of hypotenuse, length of opposite side, free column for calculations.
    - *Steps*. Students still work in pairs, but the use of a spreadsheet facilitates the study of a greater number of cases. Then, students must formulate the explicit property, with the aim of convincing each other, and other teams.
- *Sine, demonstrative moment*.
    - *Description of the model*. Students manipulate a dynamic model[11] for the formal proof of the property, where numerical values are no more visualized (figure 3). Therefore, the construction is essentially different from the previous. The model shows a "control box" to superimpose the dynamic triangle of the explorative and illustrative constructions on a construction of Thales Theorem.
    - *Steps*. The demonstrative model can be used by students, ether independently, ether while teacher explains the deductive argument (in terms of similar right triangles). Students must reformulate their partial arguments in a general theorem, with a construction where the previous triangle overlays Thales Theorem.

[9] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2487311
[10] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2487403
[11] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2487545

FIGURE 2: Illustrative model for sine ratio

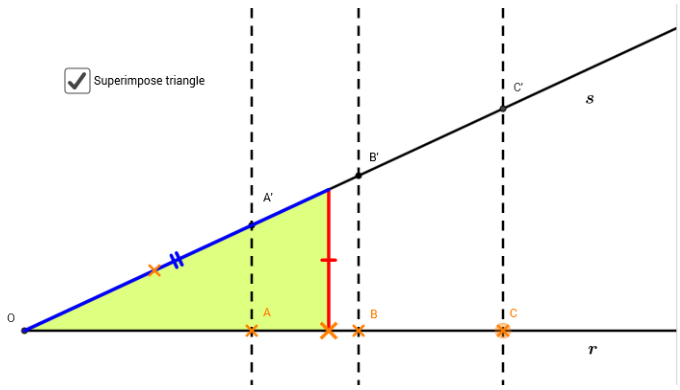


FIGURE 3: Demonstrative model for sine ratio

3.2. Transition to trigonometric functions

Once the notion of trigonometric ratio is presented, a further step is the development of trigonometric functions, as angle α becomes variable x. That is a different interpretation of the notion that has much to do with movement and velocity, i.e., the change of the value y=sin(x) as x goes from 0-to-360.

We present the modification of this perspective by means of an explorative construction, an animation with a "play" button (figure 4). The dynamic model is been designed to facilitate the idea of periodic function, by means of a game, where the student creates a partial function, and the dynamic model reproduces that portion of function every given period.

- *Function, explorative moment*. Individual game[12]. As the student pushes the "play" button, an orange point start to move in real time along the positive direction on the abscise axis. Dragging on the point, student manipulates the ordinate of the point to draw a function.
- *Periodic function, explorative moment*. Individual game[13]. Student pushes the "play" button to put into movement the orange point. When dragging the point, the model shows a second point (dark grey) which moves in translation of the orange one at a certain distance on the abscise axe. The text of the model shows additional information, i.e., the formal symbolic definition of a periodic function.
- *Period of a periodic function, explorative moment*. Individual game[14]. Before putting the model into movement, student must decide a period for the function. The model shows as many dark-grey points as needed, in order to draw a periodic function.

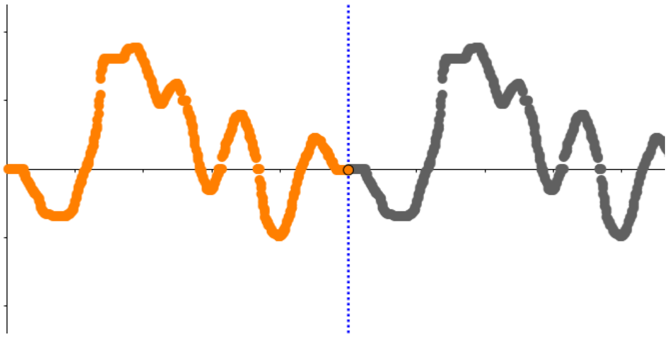


FIGURE 4: Explorative model, periodic function

The way the ordinate of a point in the goniometric circle becomes the ordinate of a point in the Cartesian plane is not natural for a student in the first term. After the explorative moment, once the student is familiar with the periodicity of a function and its graphics, an illustrative model shows how the goniometric circle spreads on the Cartesian plane. This approach follows the premise that natural phenomena, force and velocity can model situations and problems (DEMANA & WAITS, 1990).

Although convenient, the propedeutical introduction of radian angle is not strictly necessary at this stage (ACCOMAZZO, BELTRAMINO & SARGENTI, 2014). The concept of periodicity can be formalize first in the graphical register, before the numerical and symbolical approach, which can then help understand the precise meaning of the radian measure of an angle.

[12] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2487837
[13] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2488139
[14] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2488259

In a first illustrative construction, we show a triangle and its sine ratio within the goniometric circle. In a second illustrative construction (figure 5), the entire goniometric function is putted into movement along the Cartesian plane, in a way where the point showing the ordinate of the triangle and the point traveling along the graphic are actually “the same point”.

- *Goniometric circle, illustrative moment*. The model[15] represents the square triangle within the goniometric circle. The model presents the sine ratio of the triangle in direct relation to the *y* coordinate of the point in the Cartesian plane. Student uses “play” and “restart” buttons to activate the model.
- *y=sin(x) function, illustrative moment*. The model[16] puts into movement the entire goniometric circle, along the abscissa. The orange point represents the ordinate of the triangle vertex in the goniometric circle. The angle on the goniometric circle corresponds to the *x*-value of the function. The orange point paints the *y*=sin(*x*) graphic.

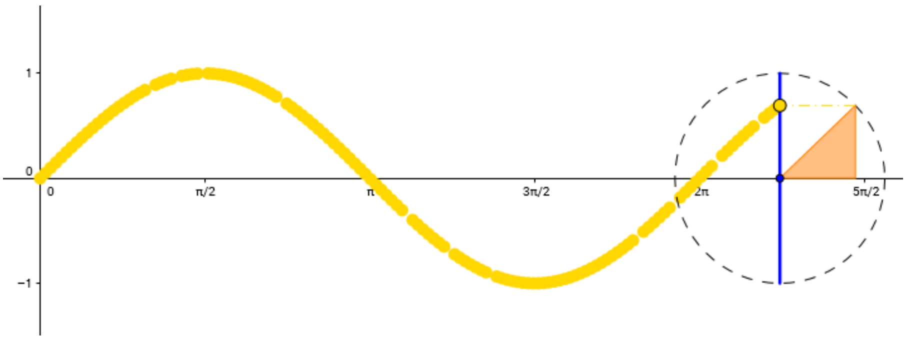


FIGURE 5: Illustrative model, sine function

3.3. Properties of trigonometric functions, the wave function

Trigonometric functions satisfy a great number of properties. Nevertheless, rigorous and formal demonstrations of such equalities are hard to present to students, since they require the introduction of many auxiliary results and complex symbolic calculations. In secondary education, dynamic models can be used to explore, illustrate and sometimes prove trigonometric relations (at least, as we would see, partial proves can be presented for particular cases).

- *Trigonometric projection, explorative model*. The design of the constructions in section 3.1 shows the invariant properties of trigonometric ratios, and visualizes how the independence from side-lengths permits the concretization of a well-formulated definition. In this example, students use

[15] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2494883
[16] The dynamic model for the situation is available in the following link: http://www.geogebra.org/material/simple/id/oOGHNoq8#material/2495119

the explorative model[17] in a similar way, working in pairs and translating information from the graphical view to the spreadsheet, in order to seek patterns on trigonometric projections. The objective is to aid students arrive to relation between lengths in a square triangle, which would be used to demonstrate further properties.

- *Trigonometric properties, demonstrative model*. The dynamic model[18] presents a geometrical situation where each segment length represents a particular trigonometric expression. Students work in pairs. By turns, each student seeks for a particular relation and translates it to the spreadsheet using general (not numerical) expressions: in the model, students introduce a particular numerical value for angle α and there is a geometrical link between angles α and β, but this implicit constraint is unknown and not necessary during the resolution process. Student must organize the collected symbolic information to formulate the property: the model contains two control boxes to show and hide hints using a color code.

All examples shown until know are either numerical or use algebraic codes to represent function variables. These dynamic models are suitable for student who ether are familiar to the use of intensives to codify information and use these intensive expressions for numerical calculations (*algebraization level* 2), or are capable of symbolic calculation with variables (*algebraization level* 3). In more complex algebraic situations, students must deal with parameters (*algebraization level* 4), and the different nature and the distinction between variable and parameter can be problematic. In the design of these dynamic models an explicit decision has been made to use "sliders" to represent parameters, and thus, to give a slider a unique meaning or interpretation.

- *Wave functions, explorative model*. The graphic view on the dynamic model[19] represents two compound functions, distinguished by colors red and blue. A "shuffle" button presents the red function by random, and the spreadsheet shows the numerical values for the "red" parameters. Working in pairs, students must find the values for the "blue" parameters, superimpose both graphics, translate those values to the spreadsheet and find a numerical relation between "red" and "blue" parameters.
- *Wave functions, illustrative model*. After students formulate their conjectures using the explorative model, we use an illustrative model to verify the validity of the aim. The model[20] does not give an argument to demonstrate the property, but shows an alternative geometric structure that "comes together" when correct slider-values are selected, and gives a hint on find the

[17] The dynamic model for the situation is available in the following link: https://www.geogebra.org/m/oOGHNoq8#material/j8kjR7Ow
[18] The dynamic model for the situation is available in the following link: https://www.geogebra.org/m/oOGHNoq8#material/OmnDxOfa
[19] The dynamic model for the situation is available in the following link: https://www.geogebra.org/m/oOGHNoq8#material/YCIuRH4v
[20] The dynamic model for the situation is available in the following link: https://www.geogebra.org/m/oOGHNoq8#material/fHvCHDlp

equation between parameters.

We implemented some of the theoretical designs in concrete school experiences. In the last section of this text, we show some results and empirical data. In particular, we will show a pilot experience where we implemented the first chapter of the GGB-Book, for the *a posteriori* definition of trigonometric ratios.

4. Staging: an experience with secondary school students

Belloso (2016) implements a slightly modified version of the GeoGebra book in chapter 3.1 in a teaching and learning context, with secondary school students (grade DBH4, ages 15-16) in a group of 10 weak students who will not continue higher secondary education. The modification of the book includes a similar dynamic model for *cosine*, and a previous activity where students draw triangles in GeoGebra, given the length of their sides or the amplitude of a given angle. Students are required to manipulate the explorative and illustrative models in the book, to write down their personal conclusions from the outcome data they obtain, by filling a questionnaire (frame 2) in paper, in order to obtain definitions for both trigonometric ratios.

Nº	Questionnaire
1	Construction of triangles: a) draw a triangle in the computer, with the following sizes: a=6cm, b=8cm, c=3cm. Next, draw the triangle in your paper (by hand) and name its sides, angles and vertex; b) change the length of two sides of the triangle (move point A). Write down the values of the lengths of each side and the angles of the triangle (use a table if you find it necessary). Classify each triangle according to their angles and sides (write down all steps); c) try to find a relation that all sides satisfy (write down all steps).
2	Trigonometric ratios: a) follow the steps on the applet and try to find a trigonometric relation. Write down all steps and use a table if necessary.

Frame 2. Questionnaire (BELLOSO, 2016, 109-118)

4.1 Results

There are differences in the number of tries students need to reach conclusions and to convince themselves on the veracity of inductive arguments. Some students find a first conjecture, and try to verify it in a number of tries: the inductive argument arises from the lack of counterexamples; some other students only give an example. For example, one of the first motivational questions is oriented to conclude that inner angles in a triangle add up to 180º. Some students draw many triangles and they fail to find a particular one where angles add up to an alternative number, i.e., they “prove” the property $\alpha+\beta+\gamma=180$; other students don’t hesitate to present just a couple of triangles (figure 6).

c. Saiatu zaitezte angeluen arteko erlazioa zein den ondorioztatzen. (Idatzi jarraitzen dizun pauso guztiak). ~~Bat~~ Hiru angeluak batuz 180° ateratzen da. 93,82° + 56,25° + 29,93° = 180° 38.21° + 60° + 81,79 = 180°	Cuestion: "Tray to conclude a relation for the angles (write down all steps)" Answer: "The trhee angles add up 180º"
A angelua + B angelua + C angelua: 180° ematen du Adibideak 1→ 56,26°+93,82°+29,93°= 180° 2→ 64,42°+82,82°+39,95°= 180° 3→ 72,54°+72,54°+34,82°= 180° 4→ 80,54°+62,72°+36,34°= 180° 5→ 90°+53,13°, 36,87°= 180° 6→ 100,29°+43,53°+36,18=180°	Answer: "Angles A, B and C add up 180º"; then, s/he gives 6 particular examples

FIGURE 6: Examples vs inductive argument

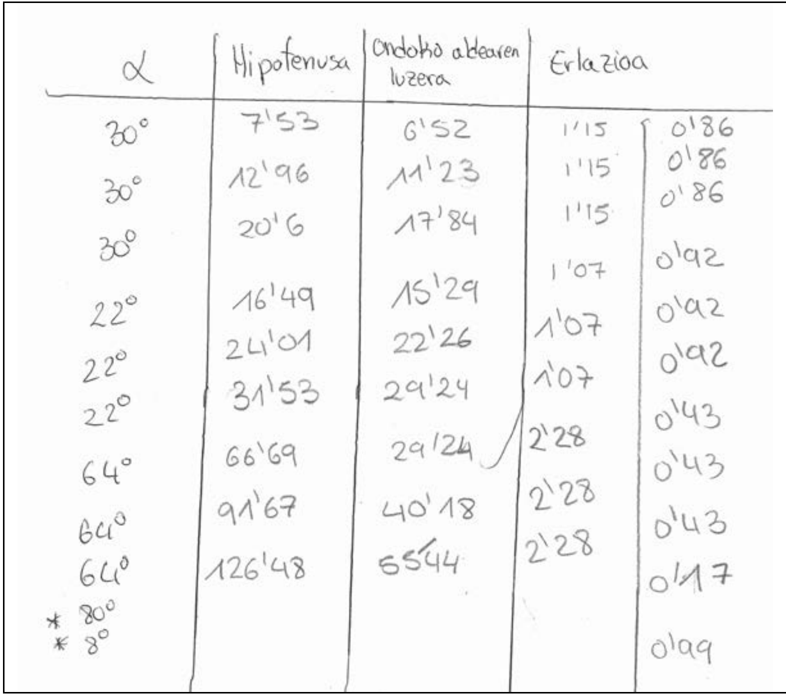
The dynamic model displays decimal numbers in the screen, and students are required to make calculations to two decimal places. Yet, standard mathematical activity at schools and usual textbooks present problems with only integer coefficients and values. This fact produces a phenomenon where students leave decimals out and operate only with the integer part (figure 7).

1→ 120+29+39=188° Zergatik? 2→ 107+32+37=176° 3→ 39+109+35=183° 4→ 27+131+22=180° 5→ 34+119+27=180° 6→ 108+38+35=181° Bi hirukietan 180° ateratzen da eta besteetan ez da ateratzen	Answer: Students give 6 examples ommiting decimals, and conclude the following, "We get 180º in two triangles, but its not general"

FIGURE 7: Pass over decimals

Meanwhile, as a rule, students make a correct use of the dynamic model in the context of the explorative moment, and they achieve the main goal of the activity, i.e., they collect row data from the model, they analyze it and they express their conclusions.

Once the explorative moment is over, students solve a similar situation with the illustrative model, which allows accurate calculations in the spreadsheet; this second moment increases the amount of data, and students validate their first conjecture: trigonometric ratios only depend on the angle (figure 8).

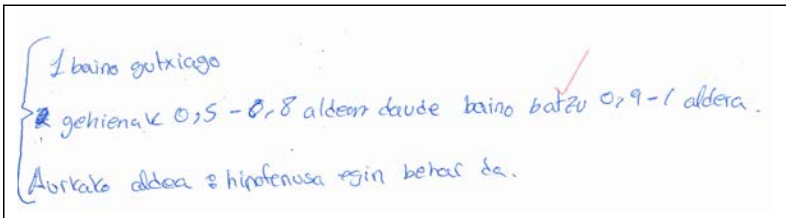


α	Hipotenusa	Ondoko aldearen luzera	Erlazioa	
30°	7'53	6'52	1'15	0'86
30°	12'96	11'23	1'15	0'86
30°	20'6	17'84	1'15	0'86
22°	16'49	15'29	1'07	0'92
22°	24'01	22'26	1'07	0'92
22°	31'53	29'24	1'07	0'92
64°	66'69	29'24	2'28	0'43
64°	91'67	40'18	2'28	0'43
64°	126'48	55'44	2'28	0'43
* 80°				0'17
* 8°				0'99

Answer: Columns correspond to (1) angle, (2) hipotenuse length, (3) adjacent legs length, and (4) the relation students find. All relations solely depend on the value for the angle.

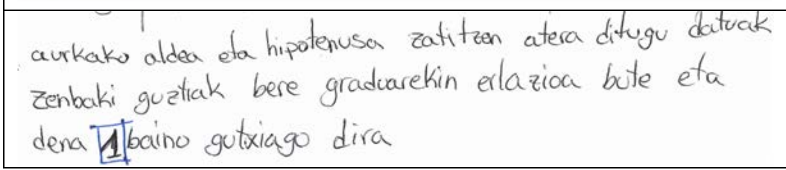
FIGURE 8: Collecting data from the illustrative model: cosine

When attending traditional mathematical lectures, students get from teachers a strong impression that algebra is the most important component in mathematics. This impression comes from the fact that teachers begin their intervention from a particular "nice" formula, and they transform it by means of algebraic transformations to another bright and precise expression. In this activity, students are not familiar with basic algebraic manipulations, and they fail to obtain a symbolic formula for the numerical expression they get, even though they can precisely describe it in natural language (figure 9).



1 baino gutxiago
gehienak 0,5 - 0,8 aldean daude baino batzu 0,9 - 1 aldera.
Aurkako aldea : hipotenusa egin behar da.

Answer: "All [quotients] are less than 1; most of them between 0,5 and 0,8, but there are some between 0,9 and 1; the [invariant] relation arises when dividing the opposite leg by the hipotenuse"



aurkako aldea eta hipotenusa zatitzen atera ditugu datuak zenbaki guztiak bere graduarekin erlazioa bute eta dena 1 baino gutxiago dira

Answer: "The [invariant] values arise from the quotient of the opposite leg and the hipotenuse and all of them are less than 1"

FIGURE 9: Use of natural language to express formulas

4.2 Brief analysis

The activity follows an explorative structure in order to build inductive arguments, and students are required to present their own hypothesis and conjectures. Thereby, each student presents a different conjecture, depending on the empirical data they work with: one student claims that *sinus* is a value between 0.5 and 1; some other proposes a better approach, and claims that the value is a positive value, less than 1 (figure 9). The successive arrangement of *explorative* and *illustrative* models progressively increases accuracy in those calculations. Once the *explorative didactical situation* is over, the teacher is responsible to show the student the *institutional* meaning of the ratio and the correct lower and upper bounds and mathematical values (frame 1).

The use of numerical sets other than integer numbers (rational or real numbers) is problematic, and school mathematical activity often slips coefficients to integer numbers, in order to reduce cognitive impact and avoid didactical obstacles. In consequence, students do not use decimal numbers, and tend to round numbers to their integer part. Dynamic models naturally show and compute decimal numbers, and therefore, promotes the use of decimal numbers, since trigonometric theorems do not arise when skipping the decimal part.

When we request a student to confront an explorative situation, s/he is not supposed to know all linguistic elements involving the mathematical context. When using technological instruments (software, calculators), a student who lacks a particular linguistic element to formulate a conjecture or explain a calculation, would turn to that instrumental interaction. For example, figure 10 shows a production where the student turns to "calculator language" to explain how s/he used an inverse calculation (Shift + Sin) to obtain the angle from its trigonometric ratio, imitating the order in which data is introduced in the calculator.

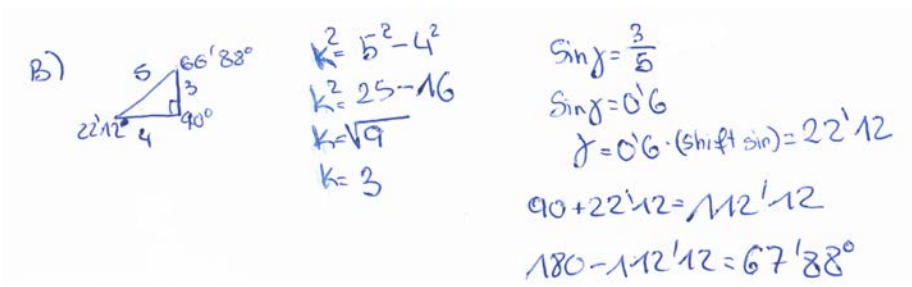


FIGURE 10: Use of "calculator language"

Conclusions

In primary and secondary education, ostensive representations of mathematical objects are primordial for mathematical activity. Dynamic geometry software is a validated instrument to organize mathematical learning and teaching situations, with the capacity to present mathematical objects in many different but interrelated semiotic registers, showing a wide gallery of ostensive representations. There are three basic moments (explorative, illustrative and demonstrative) of the mathematical activity where the use of dynamic models is pertinent. We have described in this article a theoretical proposal of such models in the context of trigonometry and we have shown the results of a particular experience where students successfully use the models. For those teachers who are willing to use the proposal by themselves, we include some guidelines in Annex A.

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ANNEX A

Guidelines and Didactical Information for Teachers

A.1 General Information

Students can use Dynamic Geometry Software in two different basic ways. In the one hand, students can use the software freely to build their own constructions and explore mathematical properties. In the other hand, teachers can use the software to design a *didactical situation*, a closed construction or dynamic model, and students would use it as an *antagonistic medium* to learn mathematics.

The second approach metaphorically takes the name “black box”, since the design of the model remains hidden to the student. The objective of “black box” models is to center classroom activity in the different moments of mathematical activity (to *explore*, *illustrate* and *demonstrate* mathematical properties). Otherwise, when using it in a freeway, students focus their activity in programing, and the mathematics they can develop are restricted to the previous informatics knowledge they have. Programing is also an interesting activity, of course, but if students are required to be *fluent* in the programing instrument as a previous stage in order to do mathematics, the primary objective of the activity (to do mathematics) can suffer a *metacognitive shift* to other type of activities (the instruction on the use of the instrument).

Manipulative materials, paper and pencil and dynamic software are examples of supports that could be use in mathematics practice. The late allows a classroom management centered on properties, rather than on individual examples, and therefore, dynamic constructions help to cross the “inductive-deductive” gap from explanation to formal proofs. Thus, GeoGebra contributes to shorten the distance between the interpretation of an example as an “isolated object” and as a “representative of a class”.

This question is essential in any process of generalization, in which intensive (general) and extensive (particular) objects are involved. The constructions presented in this GG-Book have the following labels to identify the algebraic level required to use them:

- Algebraic level 0: arithmetic; no algebraic knowledge required whatsoever.
- Algebraic level 1: incipient; students recognize intensive objects in natural numeric, iconic or gestural language.
- Algebraic level 2: intermediate; students manipulate isolated variables.
- Algebraic level 3: consolidate; students manipulate algebraic expressions and solve equations.
- Algebraic level 4: advance; students manipulate algebraic expressions with variables and parameters.

In many explorative constructions, the use of “paper and pencil” is also required, since the translation of information between different views on the model (graphical view and spreadsheet) and the translation of information from one support to another (computer and paper) is critical in school mathematical activity.

Link to the GGB-Book: http://www.geogebra.org/book/title/id/oOGHNoq8#

A.2 Chapter 1

GGB-Book chapter 1: Construction of trigonometric ratios, the sine.
Objective: Introduction to trigonometry and trigonometric ratios.

Student previous knowledge: General knowledge on triangles (classification according to sides and angles, Pythagorean Theorem, Thales Theorem), notion of angle, basic arithmetic operations.
Algebraic level: “2”.
Description of models:
The first model is explorative and shows a dynamic square triangle, with three relevant pieces of information: the amplitude of an acute angle, the length of the opposite side to this given angle and the length of the hypotenuse. The model is “two dimensional”, i.e., the user of the model can select the amplitude of the acute angle (by inserting a numerical value in an “input box”) and the length of the base of the triangle (by dragging a vertex, “point on a line”). When the user inserts the acute angle, the “input box” disappears and would not be visible again until the students decides to change to a triangle with a different angle.
The second model is illustrative, shows the same triangle of the explorative model, and includes a spreadsheet to aid calculations. Three columns organize the information in the spreadsheet: length of hypotenuse, length of opposite side, free column for calculations.
The third model is demonstrative. It does not show any numerical information, and therefore, is essentially different from the previous two. A “control box” superimposes the dynamic triangle of the explorative and illustrative constructions on a construction of Thales Theorem.
Use and steps:
In the explorative and illustrative models, student work in pairs and would be names A and B.
1) Student A must prepare the starting ground. She introduces a numerical value for the angle and drags the orange point to modify the base-length. This way, the starting position is not been totally given, and at some point, the student is responsible of the values that would arise in the calculations.
2) Student B writes down the values of the triangle from the model. When using the explorative model, the transcription would be into paper, and calculations made by hand. When using the illustrative model, student would perform both transcription and calculations on the spreadsheet. The task for student B is to perform calculations with the given information.
3) Students switch positions. Student B decides a new position for the orange point (without changing the angle, since the “input box” is gone for the moment), and student A performs new calculations. Student repeat steps 2 and 3 a number of times.
4) Students must formulate a conjecture and the conjecture must be coherent with the obtained data.
5) Students push the “new angle” button, and they start again from step 1 with a new triangle.
6) After repeating the process with a number of triangles, students reformulate their conjecture, which must be coherent with the obtained new data.
Finally, the demonstrative model can be used by students A and B ether independently, ether while teacher explains the deductive argument.
1) Student A decides a position of Thales Theorem.
2) Student B superimposes the triangle on the geometrical construction.
3) Student A and B switch positions, and they repeat steps 1 and 2.
Observations:
Trigonometric ratios are definitions and arise from invariant properties over triangles. Students use a numerical model as a first approach to explore those properties, using the

basic strategy “arithmetic calculations”. The integrate use of “paper and pencil” and “dynamic software” is critical, since the mathematical activity goes from one support to another, and back. Students make calculations until they arrive to their arithmetical threshold, i.e., their arithmetic/algebraic limit. When calculations become tedious, the spreadsheet is required. The absence of counterexamples in the study of many cases makes it possible to construct and inductive argument, and students are convinced that the property is true for any particular triangle. In addition, the dynamic model shows a triangle where the area of the triangle is variable. Therefore, the section of the plane between two rays is variable, even though the angle continues constant. Hence, the dynamic model contributes to overcome a classical epistemological obstacle: the amplitude of an angle does not depend on the area of the inner surface. The process should end with a deductive argument to prove that trigonometric ratios are invariant, and thereby, they are worth a definition. In this last step, geometrical formal language and Thales Theorem are required.

A.3 Chapter 2

GGB-Book chapter 2: Transition to trigonometric functions.
Objective: Presentation of trigonometric functions as a dynamic development of trigonometric ratios.
Student previous knowledge: General knowledge on triangles (classification according to sides and angles, Pythagorean Theorem, trigonometric ratios), notion of angle.
Algebraic level: “2-3”.
Description of models:
The chapter contains five models. The first three models are explorative and the last two illustrative. In the explorative models students play individual games as they visit, little by little, the graphic notions of function, periodic function and period.
In the first explorative model, when the student pushes the “play” button, an orange point start to move in real time along the positive direction on the abscise axis. Dragging on the point, student manipulates the ordinate of the point to draw a function.
In the second explorative model, again, the student pushes the “play” button to put into movement the orange point. When dragging the point, the model shows a second point (dark grey) which moves in translation of the orange one at a certain distance on the abscise axe. The text of the model shows additional information, i.e., the formal symbolic definition of a periodic function.
The third explorative model is similar to the second but includes a small improvement. Before putting the model into movement, the student must decide a period for the function. The model shows as many dark-grey points as needed, in order to draw a periodic function.
After those previous three explorative models, students manipulate the two illustrative models, to see how the goniometric circle becomes a periodic function.
The first illustrative model is a representation of the goniometric circle. The model includes a square triangle inside the goniometric circle. The sine ratio of the triangle represents the ordinate of the point in the Cartesian plane as well. The student can use the “play” and “restart” buttons to activate the model.
Finally, the second illustrative model represents the same goniometric circle of the previous construction, in motion along the abscissa. The orange point represent the ordinate of the triangle vertex in the goniometric circle. The angle on the goniometric circle corresponds to the independent x variable of the function, while the orange point paints the y=sin(x) graphics.

Use and steps:
Students use the explorative models individually. Following the information on the books, students push the “play” button and drag the orange point to draw a function. In the last explorative model, the periodic function construction requires additional information, i.e., the introduction of a number for the period.
The illustrative constructions can be ether used individually by students, or can be used by a teacher while she explains the development of the goniometric circle into a periodic function.
Observations:
Once the notion of trigonometric ratio is presented, a further step is the development of trigonometric functions, as the angle α becomes a variable x. This interpretation of the notion is different from the previous one and has much to do with movement and velocity: the value y=sin(x) continuously changes in relation to $x \in (0,360)$
An animation with a “play” button presents the modification of the perspective in an explorative construction. The notion of periodic function appears by means of a game, where the student creates a partial function, and the dynamic model reproduces that portion of function every given period.
The way the ordinate of a point in the goniometric circle becomes the ordinate of a point in the Cartesian plane is not natural for a student in the first term. After the explorative moment, once the student is familiar with the periodicity of a function and its graphics, we use an illustrative model to show how the goniometric circle spreads on the Cartesian plane. This approach follows the premise that natural phenomena, force and velocity are useful for modeling situations and problems.
The first illustrative construction shows the triangle and its sine ratio within the goniometric circle. The second illustrative construction puts the entire goniometric function into movement along the Cartesian plane, in a way where the point showing the ordinate of the triangle and the point traveling along the graphic are actually “the same point”.

A.4 Chapter 3

GGB-Book chapter 3: Properties of trigonometric functions, the wave function.
Objective: Exploration, illustration and demonstration of trigonometric properties, exemplified in a particular wave function.
Student previous knowledge: Trigonometric ratios and functions.
Algebraic level: “3-4”.
Description of models:
The chapter contains four models. The first two models are explorative and demonstrative models regarding trigonometric projections. The last two models are explorative and illustrative models regarding a particular wave function.
The first model shows an explorative situation where a dynamic square triangle shows all three side-lengths, and the numerical value of an inner acute angle. The position of two orange points represented in a circular arc, change the two dimensions of the triangle (radio and angle). The construction is similar to the illustrative model in chapter 1, and shows both the graphical view and the spreadsheet.
The second model shows a geometrical situation where each segment length represents a particular trigonometric expression. The user is required to introduce the amplitude of an angle in an “input box”, and to decide the location of a mobile orange point in a circular arc. The situation is demonstrative and the target-knowledge of the previous explorative model is necessary to solve it. Students must translate information from the

graphic view to the spreadsheet. There are two additional “control boxes” to show and hide hints to aid the seek properties:

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)$$
$$\cos(\alpha+\beta)=\cos(\beta)\cos(\alpha)-\sin(\alpha)\sin(\beta)$$

The following two models regard a particular aspect of wave functions. Each linear combination of sine and cosine functions admits a unique representation as a modified sine function:

$$a\sin(x)\pm b\cos(x)=A\sin(x\pm\alpha)$$

The explorative model shows a random game. A red function represents the left side of the equation, and a blue function represent the right side, as two different functions. A “shuffle” button presents the red function by random, and the spreadsheet shows the numerical values for the “red” parameters a and b. The user must manipulate the “blue” sliders (parameters A and α) to superimpose the blue graphic on top of the red function. The spreadsheet also shows the “blue” values. When both graphics are the same graphic, students must find the numerical relation between red and blue values.
The final model modifies the previous model to illustrate the solution of the problem. Students can thereby verify their conjectures. The model does not give an argument to demonstrate the property, but shows an alternative geometric structure that “comes together” when the blue sliders take correct values and gives a hint to find the algebraic relation between red and blue parameters.
Use and steps:
The first model explores trigonometric projections. Students use the explorative model in a similar way to explorative model in chapter 1. Students work in pairs and translate information from the graphical view to the spreadsheet, in order to seek patterns on trigonometric projections. The objective is to aid students arrive to the relation between lengths in a square triangle, which would be used to demonstrate further properties. Student A selects positions for both orange points in the circle, thus, changing the value of the radio of the circle and the length of the hypotenuse in the triangle. Then, student B translates the numerical values of the triangle from the graphic view to the spreadsheet and makes calculations. Students A and B interchange their positions and repeat the previous two steps a number of times. Finally, students write down conclusions about the observed numerical relations and translate these conclusions to algebraic language.
The use of the second demonstrative model requires the emerging knowledge of the previous model. By turns, each student seeks for a particular relation and translates it to the spreadsheet using general (not numerical) expressions: in the model, students introduce a particular numerical value for angle α and there is a geometrical link between angles α and β, but this implicit constraint is unknown and not necessary during the resolution process. Students organize the collected symbolic information to formulate the property: the model contains two control boxes to show and hide hints using a color code. Students work in pairs (students A and B). Student A selects a value for the acute angle in the green triangle, (this value must be acute, otherwise, disfigures the geometric structure). Student B selects a position for the orange point. Observe that in the geometric structure: the acute angle in the yellow triangle is always equal to the green value; all triangles are square; hypotenuse in the red triangle is unitary; control boxes show hints about equalities on lengths. By turns, students one and two complete all missing symbolic information in the spreadsheet (remember to write all information

using “quotation marks”). Use expressions such as
sin(α),sin(β),sin(α+β),cos(α),cos(β),cos(α+β). Students write down conclusions about the observed algebraic equalities.
The final two constructions regard an approximation to wave functions. In the explorative construction, students must find the values for the “blue” parameters to superimpose both graphical, translate these values to the spreadsheet and find a numerical relation between “red” and “blue” parameters. In both explorative and illustrative models, students work in pairs. Student A shuffles values a and b, pushing the red button. A new wave function would appear in the graphic view. Student B manipulates sliders to select values A and α, so the blue graphic superimposes the red one. Student A tries to find a numerical relation in the spreadsheet. Students A and B interchange roles, and repeat steps a number of times. Students write down conclusions about the observed numerical and algebraic relations. The last illustrative construction enhances the seek relation in an additional geometric construction that gives a hint of the property.
Observations:
Trigonometric functions satisfy a great number of properties. Nevertheless, rigorous and formal demonstrations of such equalities are hard to present to students, since they require the introduction of many auxiliary results and complex symbolic calculations. In high secondary education, dynamic models can be used to explore, illustrate and sometimes prove trigonometric relations (at least, as we would see, partial proves can be presented for particular cases).
Furthermore, constructions in this last chapter are examples of how you can use dynamic models to enhance the trigonometric representation of points and their radio-angle coordinates.

A.5 Final remarks

All examples shown until know are either numerical or use algebraic codes to represent function variables. These dynamic models are suitable for student who ether are familiar to the use of intensives to codify information and use these intensive expressions for numerical calculations (*algebrization level* 2), or are capable of symbolic calculation with variables (*algebrization level* 3). In more complex algebraic situations, students must deal with parameters (*algebrization level* 4), and the different nature and the distinction between variable and parameter can be problematic. In the design of these dynamic models an explicit decision has been made to use “sliders” to represent parameters, and thus, to give a slider a unique meaning or interpretation.