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The GeoGebra software in the introductory teaching of Dynamic Systems: research with students of Bachelor's Degree in Mathematics[1]

O *software* GeoGebra no ensino introdutório de Sistemas Dinâmicos: uma pesquisa com alunos de Bacharelado em Matemática

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ABSTRACT
The current article presents research that has investigated the contributions from a didactic sequence related to the dynamics of Planar Systems of Differential Equations carried out in GeoGebra software to learn about Introduction to Dynamic Systems. Of a qualitative nature, the research was developed by students of Bachelor's Degree in Mathematics from a Brazilian Federal University who were enrolled in the discipline Introduction to Dynamic Systems. As research conclusions, we presented the following contributions to the learning process: dynamism observed in the development of didactic sequence assignments to enable students' learning based on results investigation and interpretation; joining the theoretical / algebraic - graphic / visual approaches through visualization allowed by GeoGebra.

Keywords: *GeoGebra Software; Dynamic Systems; Bachelor's Degree in Mathematics.*

RESUMO
O presente artigo apresenta uma pesquisa que investigou as contribuições de uma sequência didática relacionada à dinâmica de Sistemas Planares de Equações Diferenciais, utilizando o software GeoGebra, para a aprendizagem de Introdução aos Sistemas Dinâmicos. De natureza qualitativa, a pesquisa foi desenvolvida com alunos do curso de Bacharelado em Matemática de uma universidade federal brasileira, matriculados na disciplina Introdução aos Sistemas Dinâmicos. Como conclusões da pesquisa, apresentamos as seguintes contribuições para a aprendizagem: o dinamismo presente no desenvolvimento das atividades da sequência didática, possibilitando a aprendizagem dos alunos a partir da investigação / interpretação de resultados; a complementação entre as abordagens teórico-algébrica e gráfico-visual, por meio da visualização proporcionada pelo GeoGebra.

Palavras-chave: *Software GeoGebra; Sistemas Dinâmicos; Bacharelado em Matemática.*

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Introduction

Differential Equations stand out for a teaching system that, overall, emphasizes the involved analytical resolutions and algebraic manipulations that open room for students' difficulty in dealing with previous contents, be them on Basic Mathematics or Differential and Integral Calculus concepts. Given such teaching perspectives, Differential Equations learning was impaired and, consequently, their applications in contextualized situations-problems are hardly prioritized in classrooms.

Several studies (OLIVEIRA, IGLIORI, 2013; LOPES, REIS, 2022a, 2022b; REIS, ARAÚJO, 2023) have suggested a qualitative Differential Equations approach to mitigate the aforementioned difficulties. This approach must be contextualized through situations-problems associated with students' future action field; moreover, they must provide a balanced approach among analytical, graphic and numerical treatments, based on using computer resources to help teaching. It is also recommended to insert more conceptual and qualitative questions to help concept learning.

In this perspective, in this article we present research carried out with students of Bachelor's Degree in Mathematics from a Brazilian Federal University which aimed, in general, to present / discuss some contributions of the use of Digital Technologies for Mathematics Education on Higher Education and, specifically, to investigate the contributions from a didactic sequence related to the dynamics of Planar Systems of Differential Equations conducted in GeoGebra software to learn about Introduction to Dynamic Systems – an elective discipline in the discipline matrix of the referred course.

However, before introducing the didactic sequence and analyzing their contributions to the learning process, we provide an overview of the history of Differential Equations up to their fundamental contribution to the development of the study and research field nowadays known as Dynamic Systems.

1. From Differential Equations to Dynamic Systems

The origins of the Differential Equations theory dates back to the beginnings of the development of Infinitesimal Calculus. Isaac Newton (1643-1727) was the first person to write Differential Equations and Gottfried Wilhelm Leibniz (1646-1716) introduced some of its basic ideas. He was the first one to write down the relation $\int t dt = \frac{1}{2}t^2 + C$, wherein C is a real constant.

It was the first time the words "integral" and "differential" dt were approached as in modern Differential Calculus, besides having the solution for the Differential Equation $x'(t) = t$.

An interesting questioning lies on the fact that there is this Mathematics fields and on how it has triggered the interest of great mathematicians such as Leonhard Euler (1707-1783), besides Newton, Leibniz, among others. A likely answer to such a question is that several laws of Physics or, on a broader sense, several laws that describe the world around us, can be written as relations between a function and its derivatives, in other words, as equality concerning a function and its derivatives, which is a Differential Equation. Newton (1746) classified Differential Equations into 3 types:

$$x' = f(x) \qquad x' = f(t, x) \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Newton wrote $x(t) = \sum_{n=0}^{\infty} a_n t^n$ to solve them and, without carrying about convergence, he replaced the series in the Differential Equation and found the a_n values. Because the term a_0 remained undetermined, he got to the conclusion that there were endless solutions to them; however, it was only well understood in the second half of the 18th century.

On the other hand, Leibniz implicitly used the method of variables separation to resolve a Differential Equation; this method was later formalized by Johann Bernoulli (1667-1748), who made important contributions to advancements in Integral Calculus, mainly to the development of a method to resolve Differential Equations by using principles of mechanics to model natural phenomena and find solutions. His oldest brother and rival, Jacob Bernoulli (1655-1705), in his turn, resolved the isochrone problem, also known as constant descent curve.

Nevertheless, in early 18th century, the main techniques to resolve Differential Equations had already been created: Separation of Variables, Expansion in Power Series, Change of Variables and Order Reduction. However, despite all advancements in it, there was not a general theory to resolve Differential Equations. Euler, who developed his mathematical studies along with Johann Bernoulli, sought to improve the understanding about the role and structure of functions by developing a procedure to the solution of several types of Differential Equations. He also accounts for having better understood the elementary functions, among them, trigonometric, exponential and logarithmic ones.

Overall, for approximately 200 years, one had to formulate a specific Differential Equation to each problem and, after that, the resolution to it would be sought. This viewpoint has changed overtime, and it got clear that a more flexible approach should be adopted to solve these equations.

A well-known example of the classical mechanics is the simple harmonic pendulum is, which consists of studying the movement of a point particle of mass M, supported by a string of inextensible length L and worthless mass. If $\theta(t)$ is the angle the string makes with vertical at time t, then $\theta(t)$ satisfies the following Differential Equation, wherein g is the physical constant denoting the intensity of a gravitational field: $\theta''(t) = -\frac{g}{L}\, sen(\,\theta(t))$.

In this case, it is much easier to show another solution besides the stationary ones: $\theta_n \colon \Re \to \Re$ given by $\theta_n(t) = n\pi$ to all real t. It is so, because many of them cannot be written in terms of the ones composed by elementary functions such as finite number of arithmetic operations.

Actually, most Differential Equations cannot be resolved through analytical methods, which were called “quadratures” by old scholars; they consist in apllying these operations plus the integration or, as equivalent, taking the primitives.

Airy equation $x'' = tx$ is another relatively simple example that cannot be resolved through “quadratures” (KAPLANSKY, 1976). There is a theory about what Differential Equations can be resolved; it is quite similar to the Galois Theory of Algebraic Equations, applied to Differential Equations.

The great differential in the history of Differential Equations was observed in late 19th century, when French mathematician Jules Henri Poincaré (1854-1912) introduced a radically new viewpoint about the study of Differential Equations by advocating for a qualitative analysis completed by the numerical study of differential equations and of their solutions.

Thus, this qualitative approach of Differential Equations can be taken as boosting tool to build a new study and research field, namely Theory of Dynamic Systems, as stated in the website of the Institute of Pure and Applied Mathematics[5]. The website claims that Theory of Dynamic Systems dates back to the work by Poincaré about Differential Equations, in late 19th century. The justification is that most Differential Equations cannot be resolved through formulas, so Poincaré advocated for a new approach: solutions must be the object of a qualitative analysis based on using geometric and probabilistic tools available, and they must be completed by a numerical study of the Differential Equation.

The use of this qualitative analysis type allowed concluding, for instance, that the asymptotic behavior of the trajectories is simple in any planar dynamic system – described by the autonomous Differential Equation (that does not depend on time) –, is mainly limited to fixed points and limit-cycles.

Poincaré analyzed the problem of the three restricted bodies and showed that trajectories in the surroundings of certain orbits, called homoclinic orbits, are extremely complex. This result can be seen as the initial milestone of the field known as chaotic Dynamic Systems (BROUWDER, 1983).

In late 20th century, great mathematicians such as George David Birkhoff (1884-1944), Dmitri Anosov (1936-2014), Yakov Sinai (1935-), among others, have proven the strength of this idea by contributing to a great advancement in the Theory of Dynamic Systems.

2. Digital Technologies and GeoGebra software in Higher Education

Research on Digital Technologies have been gaining strength in the recent scene of Mathematics Education in Higher Education (REIS, ESTEVES, 2020), mainly those related to teaching and learning processes focused on Differential and Integral Calculus and Differential Equations, as we will justify / explain below. However, it is important highlighting that other Mathematics fields in Higher Education such as Analytical Geometry (SANTOS, 2011), Linear Algebra (CHIARI, 2015) and Euclidian Geometry (OLIVEIRA, LIMA, 2018) have also been approached in research about using Digital Technologies to teach concepts and properties linked to such mathematical knowledge fields.

Thus, in order to build elements of a theoretical-bibliographic reference capable of covering the analysis of our research data and, at the same time, of contrasting the conclusions resulting from it, we will highlight some research carried out from the perspective of investigating contributions from the use of Digital Technologies and GeoGebra software in teaching and learning processes, mainly of Differential and Integral Calculus and, later on, of Differential Equations, to develop a differentiated research that has approached the Real Analysis teaching process.

Gonçalves and Reis (2013) presented research that approached the applications of derivatives in Calculus I teaching based on investigative activities carried out in GeoGebra software. The researchers advocated for the potential of using education tools in classroom and highlighted that such tools can

[5] https://impa.br/pesquisa/sistemas-dinamicos-e-teoria-ergodica/

provide students with dynamic knowledge and the possibility of "visualizing" a problem that, many times, would not be possible without using these tools, thus:

> The use of software allows mathematical concepts to be explored through non-statistical analysis that can be manipulated and provide a different perception about Mathematics. Computer environments are prone to the conduction of investigative activity, because students can be motivated to explore situations, to form their own thinking and to investigate. (GONÇALVES, REIS, 2013, p. 424)

In their research conclusion, Gonçalves and Reis (2013) highlighted the following contributions from investigative activities based on using the GeoGebra software: to re-signify students' knowledge about derivatives' applications, to create a differentiated learning environment to complete classroom assignments; to form a "new" mathematics' teacher for elementary and high school, and for Higher Education, based on reflections about the relevance of carrying out didactic activities in mathematical software.

Martins Júnior and Reis (2020) presented research that has approached teaching and learning in discipline Calculus I, with regards to visualization aspects, by investigating some contributions from exploratory activities linked to graphics of derivative functions – the study comprised teachers who used the GeoGebra software. With respect to the visualization allowed by computer using, the researchers have stated that:

> The visualization process has been quite used in research, mainly at Mathematical Education scope, since it has elements necessary for the teaching and learning process applied to mathematical contents at all levels. Although visualization is important for teaching and learning process, it still represents a secondary subject in comparison to many mathematical aspects, such as algebraic and geometric processes, but its use has become an opportunity to develop research on Mathematical Education. (MARTINS JÚNIOR, REIS, 2020, p. 77187)

As research conclusion, Martins Júnior and Reis (2020) pointed out the following contributions from exploratory activities based on using GeoGebra software: support to the process to show the main definitions used for graphics with derivative functions; support to the work by teachers in exploring intuition and the visualization provided by GeoGebra software, by providing students with the best understanding of approached contents; facilitating the confrontation of the algebraic aspect to the visual aspect to open room for using it, either in teachers education or in students learning; optimizing classroom development by providing teachers with the possibility to easily show the association between abstract and more concrete aspects for students' learning.

On the one hand, research on Digital Technologies and GeoGebra software using – having the teaching and learning processes applied to Differential and Integral Calculus as scenario – has been significantly increasing, including the Calculus of Several Variables (REIS, COMETTI, SANTOS, 2019).

On the other hand, most research about Differential Equations have focused their investigations on contributions from a teaching process that prioritizes the exploration of contextualized problem-situations or the analysis of physical phenomena (DULLIUS, 2009; BARROS FILHO, 2012); or, yet, of situations that use Mathematical Modeling from different approaches and perspectives (FECCHIO, 2011; ARAÚJO, 2020). However, some of these researches approach the use of Digital Technologies as didactic-pedagogical possibility to teaching and learning processes applied to Differential Equations, as highlighted below.

Dullius (2009) investigated the main students' difficulties to learn Differential Equations and leaned over getting answers to the following questions: How can one work with technological resources to mitigate such difficulties? What are the advantages and disadvantages of using such Differential Equations teaching resources? As a way of collecting data, the researcher interviewed Mathematics and Physics professors and, according to answers given by them, she concluded that:

> With respect to the use of technological resources in the teaching and learning process applied to Differential Equations, professors have taken computer as a very important tool to build graphics and to interpret parameters, even with animation; but they have highlighted that students need to know how to critically use it, to analyze the introduced solutions and to have in mind that the computer does not make decisions, students are the ones who must make them. (DULLIUS, 2009, p. 93)

As research conclusion, Dullius (2009) highlighted that Differential Equations' teaching focus on analytical resolution methods and on the rare use of technological resources. Students get confused about the meaning of terms in Differential Equations and about their association with the phenomenon they correspond to and symbolize. Students are reluctant in adopting a more qualitative treatment to Differential Equations. Technological resources allowed graphic visualization and calculus preparation. However, several students did not mention any positive contribution, because such resources did not collaborate with analytical resolutions.

Barros Filho (2012) sought to investigate how problem resolution based on using Digital technologies could help the significant learning of Ordinary Differential Equations' application in Sciences' problem-situations by using software Maple. After finishing and analyzing activities planned for carrying out the research, the researcher made an important conclusion concerning students' attitude towards the use of Digital Technologies to teach Differential Equations:

> Students faced a hard time because they were not used to the proposition that demands an active attitude, which leads to continuous work throughout the whole activity. Working with software Maple demanded them to use skills they had never explored before in other disciplines of the exact sciences field. (BARROS FILHO, 2012, p. 161-162)

As research conclusions, Barros Filho (2012) highlighted that, with respect to Differential Equations teaching: only few studies, at the time of his research, have used Digital Technologies in the teaching and learning process applied to Differential Equations. Students sought to give some meaning to solutions through numerical and graphic treatment, but only when they tried to conceptualize them in a different way.

Finally, it is essential pinpointing the scarcity of research focused on investigating the use of Digital Technologies related to the teaching and learning process applied to mathematical-content disciplines approached in Bachelor's Degree in Mathematics, mainly Dynamic Systems – the focus of the current study. One of the few exceptions to these studies is the research by Oliveira and Reis (2017), who investigated contributions from GeoGebra software using in teaching and learning processes applied to Riemann Integral in discipline Real Analysis, which is offered in Bachelor's Degree in Mathematics. Given the challenges of using Digital Technologies in Mathematics, at Higher Education scope, the researchers stated that:

> The use of some types of software may point out the process to break up with the teaching of some concepts almost exclusively approached by algebraic and symbolic notions; thus, it impairs the visualization and experiencing of activities. Technologies must aim, among other things, at the construction of concepts, mathematics' doing, and at the investigation and signification of numerical solutions. The dynamics of using software can encourage students to assess, experience and look for new solutions related to a given problem. (OLIVEIRA; REIS, 2017, p. 419)

As research conclusions, Oliveira and Reis (2017) highlighted that exploratory activities based on using the GeoGebra software have presented the following contributions: potentiation of intuitive aspects concerning the construction of some concepts resulting from activity explorations; thus, it also attributes some possibilities to rigorously deal with concepts underlying Riemann's Integral in Real Analysis teaching; the possibility of entangling intuition to rigor processes in the two-way road between Calculus and Analysis; the possibility of effectively using different software – despite their limitations – to build concepts related to Riemann's Integral and to allow technical advancements.

Now, we start introducing the research methodology, by detailing their procedures and the involved context.

3. Methodologically detailing the research and its context

The Bachelor's Degree in Mathematics at the Federal University of Ouro Preto (UFOP) lasts 8 semesters and aims at training bachelors either for scientific research or for teaching at Higher Education level, from the perspective of study continuity at Masters and PhD level in Pure Mathematics, Applied Mathematics fields, or at neighbor fields. In total, 10 students join the referred course on a yearly basis, and its discipline matrix holds a set of mandatory and elective disciplines.

The present research was carried out with 5 students enrolled in the elective discipline “Introduction to Dynamic Systems”, which was taught by the 1st author of this article, in the 2st semester of 2021. The workload of the referred discipline totals 90 hours and its amendment comprises the following contents: Existence and uniqueness of solutions; Dependence of solutions concerns initial conditions and parameters; Systems of Linear Equations; Systems of Nonlinear Equations; Poincaré-Bendixson Theorem; Stability in the Liapounov sense; Applications.

It is important highlighting that, by mid-March 2020, UFOP officially adopted the remote teaching system due to the Covid-19 pandemic; this system was kept in place up to early 2022. Thus, classes were taught through videoconference, at Google Meet, and communication with students took place during these classes and through assignments and forums made available at Moodle Platform, which is a virtual teaching and learning environment that had already been adopted by the university few years ago. This interactive environment allowed intermediating the communication with students, creating evaluative activities, creating discussion forums, posting media contents such as videos and links, among other resources.

Accordingly, and from the qualitative research approach, a didactic sequence related to Planar Differential Equation Systems dynamics was planned, developed and evaluated in GeoGebra software – this sequence will be properly detailed below. It is important highlighting that, based on Zabala (1998, p. 18), didactic sequence is “a set of ordered, structured and articulated activities to achieve certain educational goals that have a beginning, and an end, known by both teachers and students”.

Our didactic planning, in particular, was planned from the perspective of goals designed for discipline “Introduction to Dynamic Systems”. It was developed in 2 hours, for 3 days of this discipline’s classes, and assessed by students through evaluation questionnaire. Recordings of synchronous activities carried out in Google Meet, as well as access to asynchronous activities posted in Platform Moodle, were used for data collection purposes.

4. Describing the didactic sequence

We explored the dynamics of Planar Systems (or bi-dimensional) of linear Differential Equations with constant coefficients in the present didactic sequence:

$$x' = ax + by$$

$$y' = cx + dy$$

wherein a, b, c and d are real constants and x' and y' denote the derivatives of functions $x = x(t)$ and $y = y(t)$, respectively. The system above can be re-written into matrix format type $X' = AX$, wherein $X(t) = (x(t), y(t))$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The matrix described above is associated with the Planar System of Differential Equations.

The main aim of the didactic sequence was to allow students to perceive and conclude about the dynamics of Planar Systems of Differential Equations by using the free access GeoGebra software, based on Linear Algebra concepts and on the dynamics proportional to GeoGebra.

The dynamics of $X' = AX$ form systems strongly depends on exponential matrix A, which we have denoted through $exp(A)$. The obtainment of $exp(A)$ can be achieved from its Jordan Form that, according to Doering and Lopes (2018) and Hirsch, Smale and Devaney (2012), is related to the concept of conjugated matrices (or similar).

It is said that two squared and ordered matrices (A and B) are conjugated, in case there is an invertible matrix P, so that $B = P^{-1}AP$. The importance of this concept to the exponential calculation of matrices is related to the fact that if $B = P^{-1}AP$, then, $exp(B) = P^{-1}exp(A)P$. Accordingly, in case the calculation of $exp(A)$, is simple, then, the calculation of $exp(B)$ is reduced to a product of matrices.

The canonical form of Jordan Real states that any square matrix A is conjugated to a certain matrix J (the so-called Form of Jordan of matrix A), which is given by blocks of square matrices throughout their principal diagonal. Furthermore, each block is gotten from their eigenvalues, be them real differences, repeated reals or complexes (as for this last case, the real and imaginary parts to get a real matrix).

As our interest was to explore bi-dimensional planar systems, we worked with order-2 square matrices. With respect to this specific case, it is relatively simple to get the Canonical Forms of Jordan set for matrix associated with the system, as well as to determine $exp(J)$.

In case the planar system is subjected to an initial condition of form $X(t_0) = X_0 = (x_0, y_0)$, thus, solution is given by $X(t) = exp(tA)X_0 = P^{-1}exp(tJ)P$. This last inequality tells us that assessing the dynamic of system $X' = AX$ is the same as assessing the dynamics of system $Y' = JY$, when change in variables $Y = P^{-1}X$ is introduced.

Subsequently, we will describe the set of activities that comprised the didactic sequence developed in each of the 3 days of classes, with emphasis on those that were carried out by students based on the synchronous form (they were guided by the discipline's professor), as well as on those that were recommended to be carried out by students in an asynchronous way.

Activity 1 (synchronous) was carried out at day 1, as well as the proposition for Activity 2 (asynchronous).

Activity 1: Revisiting Linear Algebra concepts

Construction script in GeoGebra:

a) Build two vectors v_1 and v_2 in the entry box, which is independent, by using command (<Point>). For example, if $v_1 = (1,1)$ and $v_2 = (-1,3)$, we have v_1 = Vector ((1,1)) and v_2 = Vector ((–1,3));

b) Type in the entry box $P = \{\{x(V_1), x(V_2)\}, \{y(V_1), y(V_2)\}\}$ and press "Enter". Thus, GeoGebra will build matrix P;

c) Press option “Viewport 2” in menu “Display”;

d) Press “Viewport 2”, subsequently, go to the entry box and type in P^(-1)*v_1 (press “Enter”) and, then, P^(-1)*v_2 (press “Enter”). Accordingly, GeoGebra will build points A = (1,0) and B = (0,1);

e) Type in e_1 = Vector(A) and e_2 = Vector(B);

f) Repeat the activity to several values of v_1 and v_2 at your own choice.

Figure 1 depicts the result found through the script of the described construction.

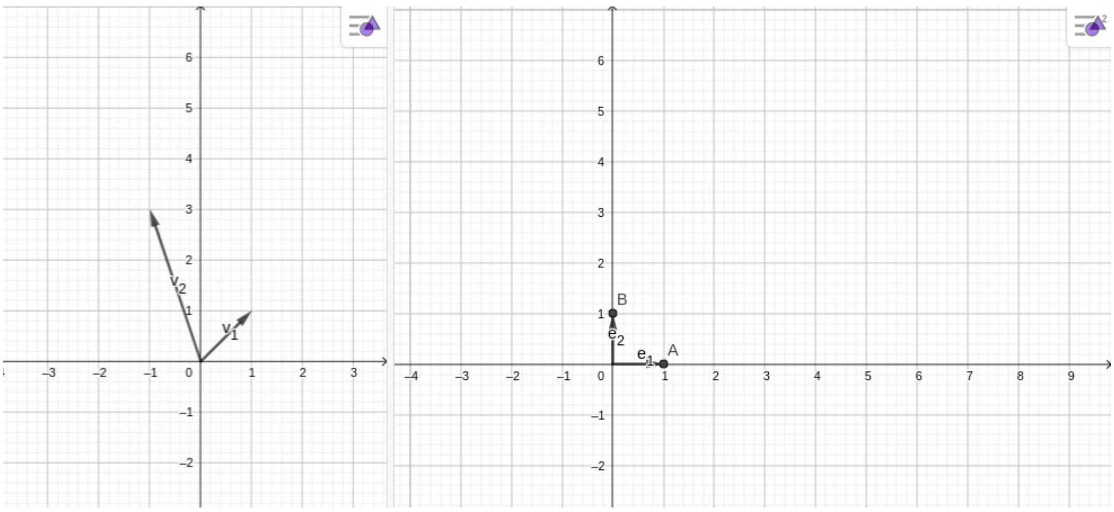

FIGURE 1: Illustration of Activity 1
SOURCE: Research Collection

To reflect:

a) Would you be able to set some conjecture based on constructions carried out in GeoGebra?

b) Would you be able to show the conjecture set in the previous item?

c) Would the result be valuable to any invertible matrices of order n?

Expected answers:

a) For any invertible matrix $P = [v_1\ v_2]$, it is worth identifying $P^{-1}v_i = e_i$, so i = 1.2

b) As $Pe_i = v_i$, then, $P^{-1}v_i = e_i$

c) Yes, because, if $P = [v_1\ v_2\ \dots\ v_n]$, then $Pe_i = v_i$ and the same argument of item b is taken into account.

This activity aimed at re-signifying the sense of linear transformation and some plane properties, because students had already attended the discipline Linear Algebra. The presented script, however, could be used in Introduction to Linear Algebra or Analytical Geometry classes, for example.

Activity 2: Finding the Jordan Form

Students were asked to find the Jordan Form of 6 different matrices, based on the following statement: given matrix A, find its Jordan Form J and a matrix $P = [v_1 v_2]$, wherein v_1 and v_2 are columns of P, invertible, such as $A = PJP^{-1}$:

$$a)\ A = \begin{pmatrix} -\frac{7}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{5}{4} \end{pmatrix}$$

$$b)\ A = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{7}{4} \end{pmatrix}$$

$$c)\ A = \begin{pmatrix} -\frac{5}{4} & -\frac{3}{4} \\ -\frac{9}{4} & \frac{1}{4} \end{pmatrix}$$

$$d)\ A = \begin{pmatrix} -\frac{11}{4} & -\frac{1}{4} \\ \frac{9}{4} & -\frac{5}{4} \end{pmatrix}$$

$$e)\ A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{5}{2} & -\frac{3}{2} \end{pmatrix}$$

$$f)\ A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}$$

Expected answers:

a) $J = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$, $v_1 = (1,1)$ e $v_2 = (-1,3)$ are eigenvectors associated with $\lambda_1 = -2$ e $\lambda_2 = -1$, respectively.

b) $J = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $v_1 = (1,1)$ e $v_2 = (-1,3)$ are eigenvectors associated with $\lambda_1 = 2$ e $\lambda_2 = 1$, respectively.

c) $J = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, $v_1 = (1,1)$ e $v_2 = (-1,3)$ are eigenvectors associated with $\lambda_1 = 2$ e $\lambda_2 = 1$, respectively.

d) $J = \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix}$, $v_1 = (1,1)$ is a generalized eigenvector and $v_2 = (-1,3)$ is a eigenvector associated with $\lambda_1 = -2$.

e) $J = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $v_1 = (1,1)$ e $v_2 = (-1,3)$ are real eigenvectors that form a basis for the real space. Complex eigenvalues are given by $\lambda_1 = a + i\,b$ e $\lambda_2 = a - ib$, wherein $a = -1$ and $b = 1$.

f) $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this case, we have complex eigenvalues given $\lambda_1 = -i$ e $\lambda_2 = i$. A pair of linearly independentes eigenvectors is given $v_1 = (1,1)$ e $v_2 = (-1,3)$.

Activity 3 (synchronous) and Activity 4 (asynchronous) were carried out on day 2; they are described below.

Activity 3: Discussing Activity 2

Activity 2 was conducted by students on an asynchronous form, we assessed J matrices found by students; we corrected each item from *a)* to *f)*, and we discussed doubts presented in their resolution.

Activity 4: Seeing the trajectories of Linear Systems

The aim of this activity was to explore 2 linear systems through constant coefficients:

$$X' = AX, \text{ subjected to the initial condition } X(t_0) = X_0,$$

$$Y' = JY, \text{ subjected to the initial condition } Y(t_0) = Y_0,$$

wherein $Y_0 = P^{-1}X_0$.

For the last system, J is the Jordan form of matrix A and P is a invertible matrix such that $A = PJP^{-1}$.

We elaborated a construction in GeoGebra to explore the trajectories of the involved systems for each one of the matrices presented in items from *a)* to *f)* in Activity 2. In order to focus on concepts, rather than on the construction elaborated in GeoGebra, we made the option for providing students with links of the created applets, as shown in the frame below.

Items	**Access Link**
a), *b)* and *c)*	https://www.geogebra.org/m/fkuevftu
d)	https://www.geogebra.org/classic/kqkmtah3
e) and *f)*	https://www.geogebra.org/m/cjsbxyrs

Frame 1. Links of access to the items

We present in the note below[6] the construction script of the apllet accessed in the 1st link described in Frame 1.

Figure 2 depicts the initial screen, which is presented when the 1st link is accessed.

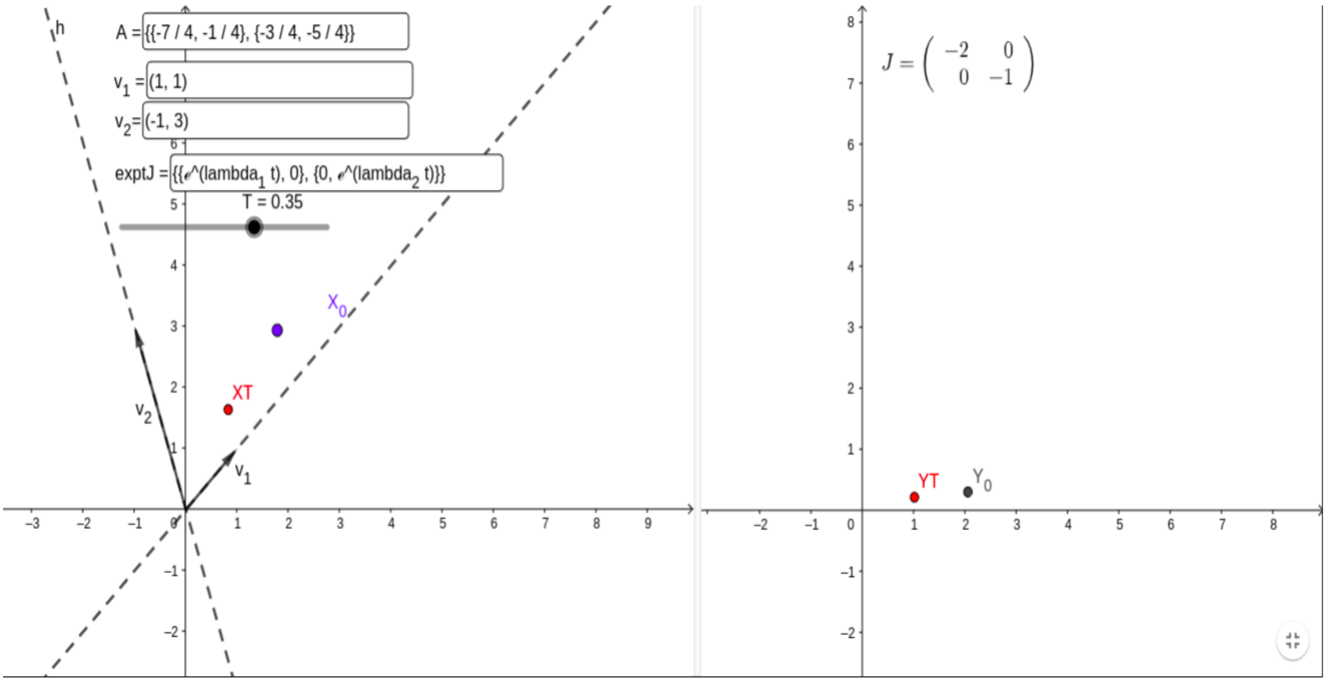

FIGURE 2: Initial screen to access the 1st link
SOURCE: Research Collection

We have requested students to take the following steps in each one of the links:

I) Put X_0 in several points of the plane, move the scroll T and observe what happens in points *x(T)* = *XT* and *Y(T)* = *YT*, when module T gets arbitrarily big;

II) In particular, put X_0 over the line generated by v_1 and v_2 (in the figure – dashed line);

III) At the end of each stage, disable the trail and change the zoom of viewing windows to rule out the created curve to go on with the analysis.

Based on the standard, the matrix worked out in the 1st link of Table 1 is given by item *a)*. When students performed step I in this matrix, one can find a figure similar to that presented in Figure 3.

[6] https://docs.google.com/document/d/1X54CE4Sh_Ps0BCUfHI4q7RKxiAtPt2M1RWmY1Hb_PSw/edit?usp=sharing

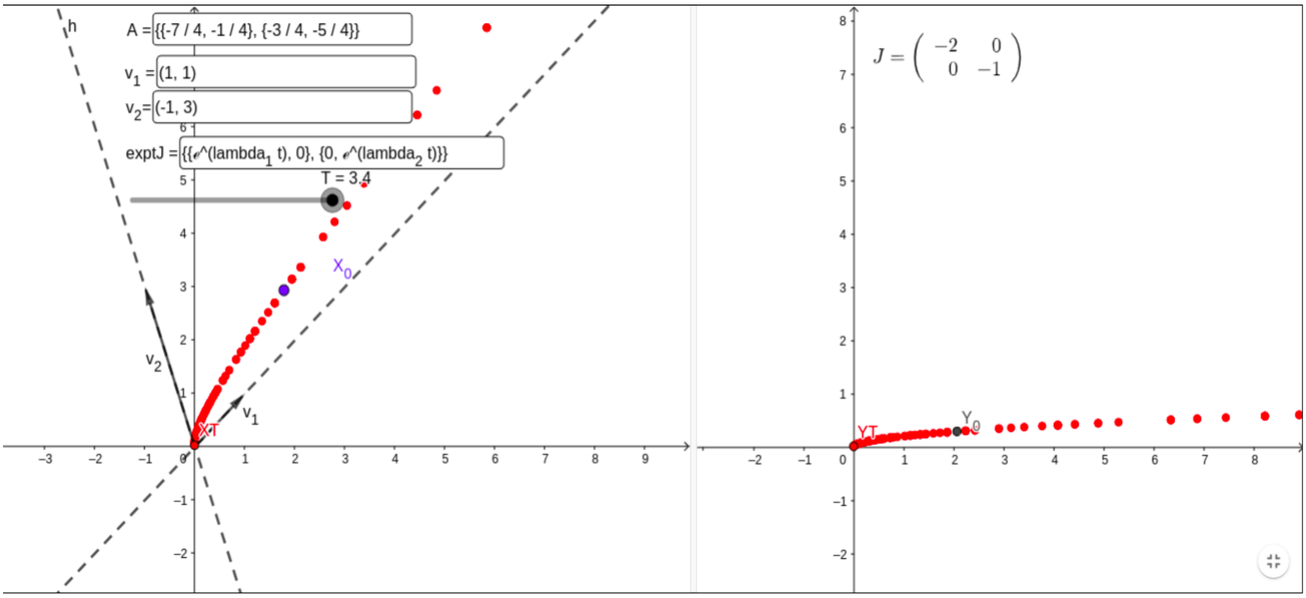

FIGURE 3: Image found by moving the scroll
SOURCE: Research Collection

One finds Figure 4 by taking step II and by putting X_0 throughout the line generated by v_1.

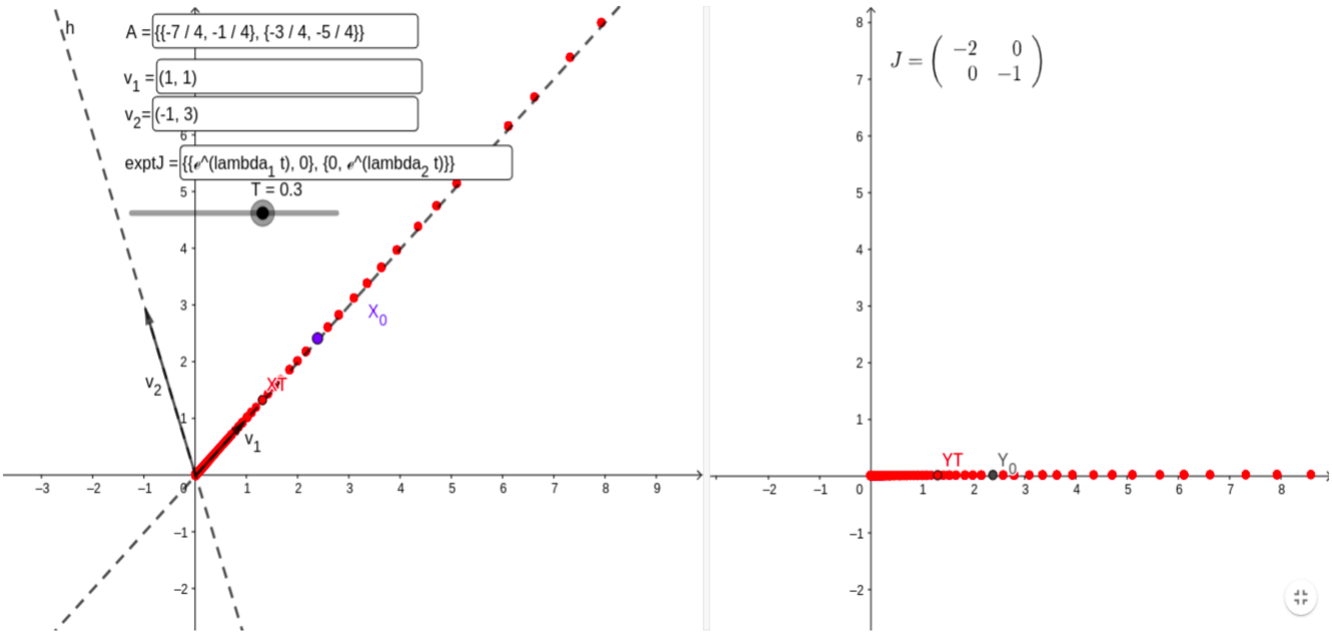

FIGURE 4: Behavior of trajectories when X_0belongs to the line generated by eigenvector $v_1$found by moving the scrool
SOURCE: Research Collection

After taking the requested steps, students could change Matrix A, in the corresponding box, as shown in Figure 5, to insert matrices in items *b)* and *c)*. Accordingly, they could analyze the behavior of trajectories by taking the requested steps.

FIGURE 5: Entry box for matrix A
SOURCE: Research Collection

For reflection:

I) What is the behavior of solutions $X(t) = XT$ e $Y(t) = YT$ when the absolute value of T is taken arbitrarily large in each one of the matrices? Describe the observed similarities and/or differences;

II) Do solutions $X(t) = XT$ present the same behavior for what matrices? Similarly, do solutions $Y(t) = YT$ have similar behavior to what matrices *J*? Compare the answers: Do the recorded result go against the approached/learned content?

Expected answers to each one of the matrices presented in items from a) to f):

a) In both cases, when T gets arbitrarily large, solutions tend to the origin. The lines generated by the eigenvectors $v_2 = (-1,3)$ and e_2 are asymptotes of *X(t)* and *Y(t)* respectivaly. Both of them correspond to eigenvalue – 1. When the absolute value of T gets arbitrarily large and T is negative, the abolute value of the solutions tends to the infinite.
b) In both cases, when $|T|$ approaches to infinity and T is negative, solutions approache to the origin, and $X(t)$ asymptotes the line generated by the eigenvector v_1=(1,1) and $Y(t)$ asymptotes e_1. Both of them correspond to the eigenvalue 1. When T gets arbitrarily large, the abolute value of solutions approaches to infinite.
c) In both cases, when $|T| \to \infty$, the absolute value of solutions goes to the infinite, with $X(t)$ asymptoting eigenvector v_2 and $Y(t)$ asymptoting eigenvector e_2 for negative values of t (both correspond to eigenvalue 1); as well as with $X(t)$ asymptoting eigenvector v_1 and $Y(t)$ asymptoting eigenvector e_1 for positive values of t (both correspond to eigenvalue 1).
d) In both cases, when T gets arbitrarily large, solutions approaches to the origin, with $X(t)$ asymptoting to generalized eigenvector $v\_2 = (-1,3)$ and $Y(t)$ asymptoting to e_2. When $|T| \to \infty$ and T<0, the absolute value of solutions approaches to the infinite.
e) In both cases, when T gets arbitrarily big, solutions approaches to the origin. When $|T| \to \infty$ and T is negative, the abosolute value of the solutions goes to the infinite.
f) If T varies, $X(t)$ and $Y(t)$ get limited, and the trace of $X(t)$ is an ellipse and the trace of $Y(t)$ is a circle centered at the origin and radius equal to the $|Y_0|$.

It is worth observing that the matrix worked out in item *d)* presents eigenvalue $\lambda\_1 = -2$ of multiplicity 1. Based on this situation, there is only one linearly independent eigenvector, which we have chosen to be $v\_2 = (-1,3)$. It is necessary to take the generalized eigenvector $v\_1 = (1,1)$ to get a basis to the cartesian plane. Students can dynamically observe that the trajectories approache to the origin when $T \to \infty$ by moving the scroll T. Figure 6 depicts the trajectory when the initial condition X_0 is in the fourth quadrant generated by v_1 and v_2.

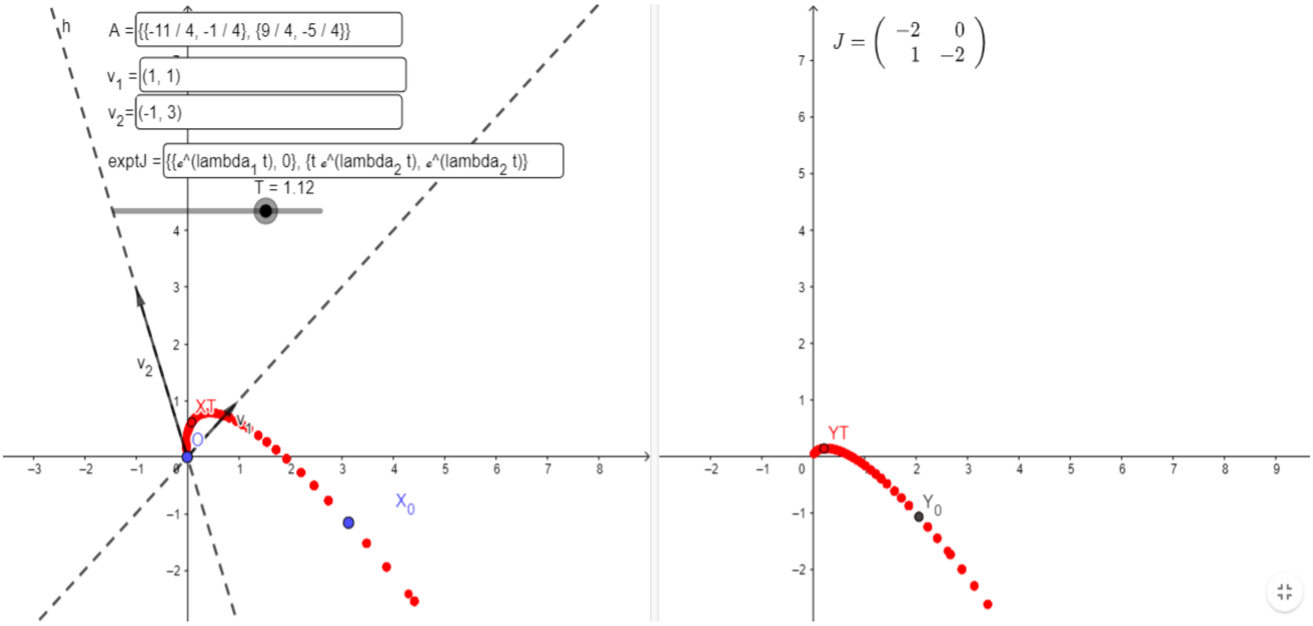

FIGURE 6: Behavior of trajectories when one has eigenvalue of multiplicity 1
SOURCE: Research Collection

On the other hand items *e)* and *f)* work the complex case. In item *e)*, we have trajectories given by spirals that tend to the origin when $T \to \infty$. Figure 7 depicts this fact when X_0 is in the first quadrant generated by the real eigenvalues $v_1 = (1,1)$ and $v_2 = (-1,3)$, which are associated with complex eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$, respectively.

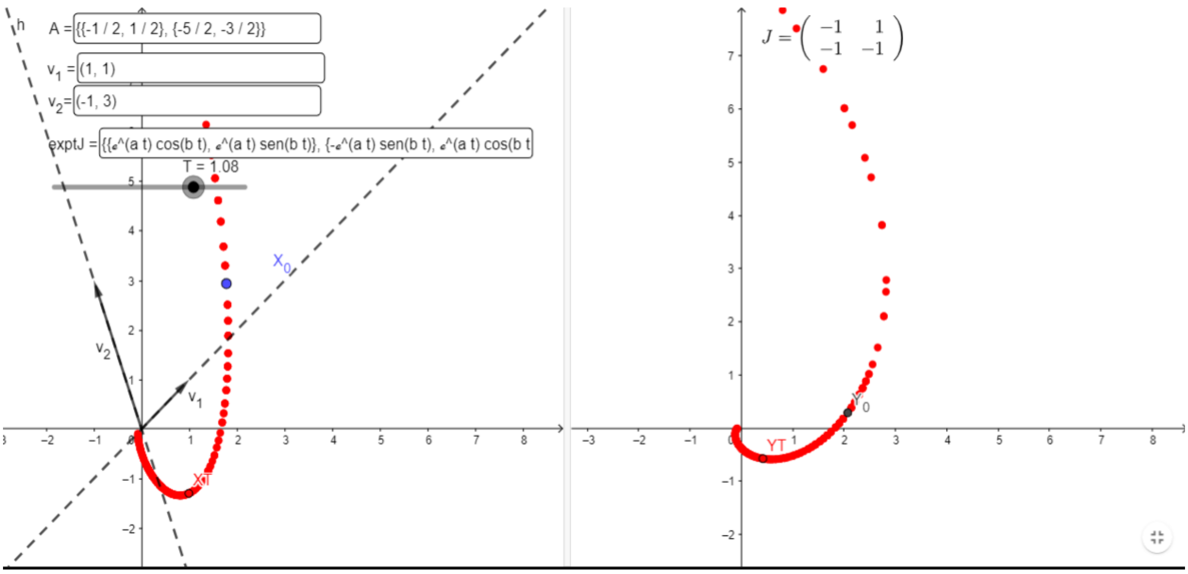

FIGURE 7: Behavior of trajectories when there is a complex eigenvalue with negative real part
SOURCE: Research Collection

As for item *f*), one finds the periodical trajectories that take place when the complex eigenvalue has null real part. Accordingly, students should drive the matrix in the corresponding entry box:

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}$$

Figure 8 depicts the trajectories found in this case.

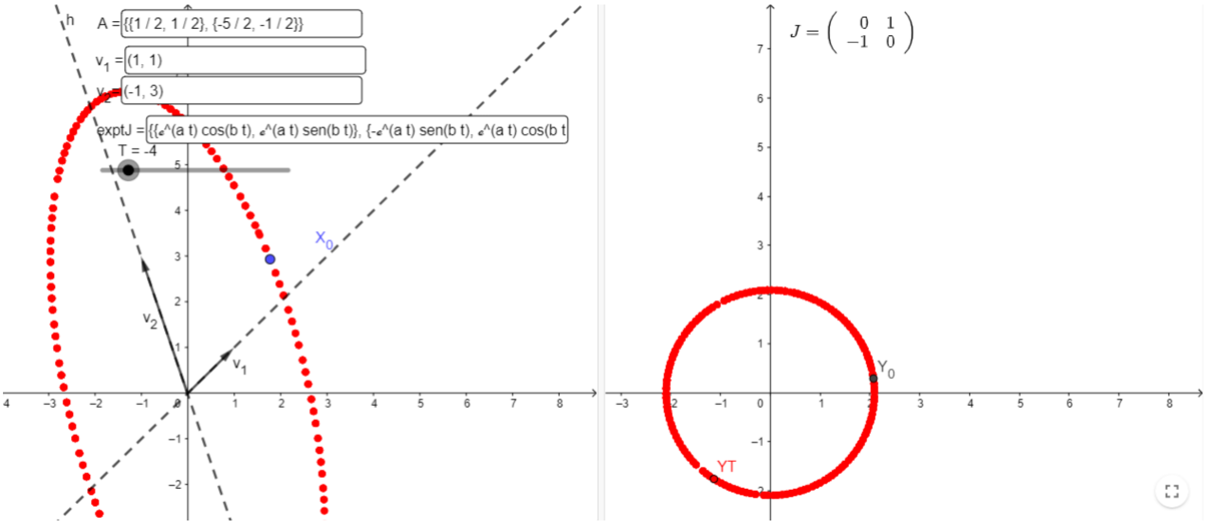

FIGURE 8: Behavior of trajectories when one has a complex eigenvalue with null real part
SOURCE: Research Collection

It is worth highlighting that, due to the aforementioned described activities, it was expected for students to be able to infer, based on the visualization provided by GeoGebra, that the dynamics of the system is set by the Jordan Form J associated with the matrix of the linear system. In other words, it is not necessary to explicitly get solutions in order to understand the dynamics of a planar linear system, but only to know the Jordan Form associated with the matrix of the linear system. Finally, Activities 5 and 6 (asynchronous) were carried out on day 3; these activities were proposed after day 2, as well as Activity 7 (synchronous) – they were described below.

Activity 5: Understanding the importance of conjugation

Consider the matrices:

$$A = \begin{pmatrix} -\frac{7}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{5}{4} \end{pmatrix} \text{ e } B = \begin{pmatrix} -5 & 1 \\ -12 & 2 \end{pmatrix}.$$

As already worked out in Activity 2, A has 2 linearly independent eigenvectors $v_1 = (1,1)$ and $v_2 = (-1,3)$. We ask:

1) Assess whether B has $w_1 = (-1,-3)$ and $w_2 = (1,4)$ as linearly independent eigenvectors, and find the Jordan Form of B;
2) Is there some similarity between the Jordan Forms of A and B?
3) Conclude that A and B are conjugated matrices.
4) Use X_A to denote the solution of $X' = AX, X(t_0) = X_0$ and X_B to denote solution of $X' = BX, X(t_0) = Y_0$. Assess whether $Y_0 = T^{-1} X_0$, wherein $T = \begin{pmatrix} -1 & 0 \\ -13 & 4 \end{pmatrix}$, then $X_B(t) = T^{-1} X_A(t)$ for all real t.

It is worth observing that the Jordan Forms of A and B are the same. Accordingly, it is seen that $A = PJP^{-1}$ and $B = QJQ^{-1}$, wherein P and Q are invertible matrices. It is possible including that A and B are conjugated because $A = PJP^{-1}$shows that $J = P^{-1}AP$; therefore, $B = QP^{-1}APQ^{-1} = (QP^{-1})A(QP^{-1})^{-1}$. It was expected that students would get to set the association between solutions in systems $X' = AX, X(t_0) = X_0$ and $X' = BX, X(t_0) = Y_0 = T^{-1}X_0$ in item 4. In this case, matrix T is the same product QP^{-1}. Students could, for example, resolve the 2 proposed systems and, subsequently, check whether the product $T^{-1}X_A(t)$ meets solution $X_B\ (t)$.

Activity 6: Reflecting about solutions of Linear Systems

If J is the Jordan Form of matrix A, one must take into account systems $X' = AX$ and $Y' = JY$, subjected to initial conditions $X(t_0) = X_0$ and $Y(t_0) = P^{-1}X_0$, respectively; wherein P is an invertible matrix, so $A = PJP^{-1}$. Can you set some association between $X(t)$ and $Y(t)$ to all real t?

Activity 7: Matching the ideas and understanding the dynamic behavior of Planar Systems

Initially, we have discussed with students the resolution of enquires made in Activities 5 and 6, in order to get to know their perception and to trigger a discussion. Activity 5 was carried out by resolving the systems right through the eigenvalues of the involved matrices. Activity 6 was performed by applying the Existence and Uniqueness Theorem of Ordinary Differential Equations.

Subsequently, we present the following question: How many solutions the systems $X' = AX, X(t_0) = X_0$ and $Y' = JY, Y(t_0) = Y_0 = P^{-1}X_0$ have?

It is worth highlighting that, if necessary, students could see that if $Z(t) = P^{-1} X(t)$, then $Z'(t) = P^{-1} AX(t) = JP^{-1} X(t) = JY$, in other words, Z is $Z'(t) = JZ$ and $Z(t_0) = P^{-1} X(t_0) = Y_0$. Based on the Existence and Uniqueness Theorem of Ordinary Differential Equations, one finds that $Y(t) = Z(t) = P^{-1} X(t)$, which is the association asked for in Activity 5.

After the discussions and the presentation of doubts, we proposed to students the following questions:

I) What does identity $Y(t) = P^{-1} X(t)$ tell us in terms of system trajectories?

II) Matrices A and B have the same Jordan Form J. Given $X_0 \in \mathbb{R}^2$, set $Z_0 = QP^{-1}X_0$, wherein P and Q are invertible matrices, so $A = PJP^{-1}$ and $B = QJQ^{-1}$. If we take into consideration systems $X' = AX$ and $Z' = BZ$ that are respectively subjected to initial conditions $X(t_0) = X_0$ and $Z(t_0) = Z_0$, what is the existing association between the respective solutions?

As for question I, our idea was to lead students to the conclusion that matrix P^{-1} makes a linear change on the plane and changes solution X into solution Y. This statement is depicted in Activity 4, through GeoGebra links made available in Table 1. With respect to question II), students were invited to understand something more general: the dynamic behavior of solutions set for 2 linear systems is the same, as long as the corresponding system has the same Jordan Form, as justified by the previous question.

After concluding the didactic sequence, we asked students who filled out the evaluation questionnaire that was developed in Google Form and made available in Moodle Platform.

5. Contributions to learning Introduction to Dynamic Systems

Based on the analysis applied to recordings of classes in Google Meet, of the proposed activities in Moodle Platform and of answers to the evaluation questionnaire, it was possible listing some contributions from the described didactic sequence by using the GeoGebra software to learn Introduction to Dynamic Systems.

Initially, we highlight that we sought to get to know few students' experiences and some perceptions by 5 students participating in the research in the 1st part of the evaluation questionnaire; they were herein identified only by A1, A2, A3, A4 and A5.

Overall, all students have stated to have had a good performance in the discipline Introduction to Ordinary Differential Equations, which was previously attended. One of the students linked this fact to the educational practice of its professor:

> *My performance was good. I liked this discipline much; the professor always brought historical data, whenever possible, and showed some modeling. (A3 – Questionnaire, 2021)*

With respect to difficulties in teaching and learning processes of Ordinary Differential Equations experienced by students, lack of examples and applications were pointed out by them, as well as the notation and use of some Ordinary Differential Equations resolution methods.

From the perspective of a Mathematics undergraduate student, who can become a professor in the future, we asked students, as future Mathematics teachers, to assess the importance of carrying out activities based on the use of Digital Technologies to teach mathematical contents. All of them were unanimous in pointing out the "extreme importance" of using Digital Technologies to teach Mathematics, in general. However, we highlight that most students have justified such an importance based on the visualization possibility – only one of them had approached the didactic aspects of visualization in its answer:

Digital technologies are not just an excellent didactic resource if they are well explored, as shown by the activities carried out in classes of the discipline, but also by an undeniable part of the current social and academic life of people. Accordingly, these are tools we used in our daily lives; the idea is to use them the best way possible in order to benefit from them, to enhance the teaching and learning process. We can make the class more dynamic, allow the best visualization of geometric concepts / objects, facilitate the communication between distant people; they have several possibilities. (A2 – Questionnaire, 2021)

It is interesting highlighting that only one student had not yet carried out activities based on using Digital Technologies in other previously attended disciplines. The other 4 students had already had the opportunity to perform didactic activities in GeoGebra software, mainly in discipline Differential and Integral Calculus. One of them stated to have discussed texts about teaching technologies in disciplines that it had attended in Mathematics Bachelor's Degree courses, such as Teaching Practices and Supervised Internship.

In the 2nd part of the evaluation questionnaire, we gave students the opportunity to assess the developed didactic sequence by focusing on its contributions to learn the approached Introduction to Dynamic Systems contents.

According to the students, the possibility of applying the assessed concepts in "regular" classes stood out as great contribution from the didactic sequence, because it not just allowed completing the assessed theory, but also the emergence of students' attitude, which is different from that we often observe in classroom environment, when we determine/show mathematical results, as stated by one of the students:

They have contributed very much. The sequences approached a content that was seen in classroom notes, but, as much as I had carefully read and discussed in the classroom, I forgot it a few weeks later. The activities allowed fixing the theory, by detailing the elements and it allowed us to set an association between solutions, rather than just accepting that result. Besides, due to the discussion we saw the importance and the use of what we are studying. (A2 – Questionnaire, 2021)

We observed that the possibility highlighted by the student of "not just accepting the result", but to establish associations between the obtained solutions in different activities of the didactic sequence, revitalizes the perspective addressed by Dullius (2009) about the importance of critically using technologies by students who, after all, must "make decisions", mainly when it comes to the interpretation and analysis of solutions to problems that are proposed to them, in order to learn several mathematical knowledge.

Another contribution highlighted by students was the discussion about concepts and solutions that, so far, were only approached in an algebraic form. Based on the activities carried out in GeoGebra, they could be graphically understood, as we have observed in the answers given by another student:

I had already seen the contents in algebraic form, then, the part of taking the test at home was not new. The part of discussion in the meetings and the use of GeoGebra have much contributed to the understanding. (A4 – Questionnaire, 2021)

With respect to contents, according to which the didactic sequence assessed in GeoGebra has contributed for students to be able to re-signify concepts and properties, solutions for Planar Systems of Differential Equations were mentioned by almost all students, as we have highlighted in the answer of one of them:

> *GeoGebra helped the understating about how a plan of phase is given and how eigenvalues interfere with the behavior of solutions. And it is necessary showing how it is similar to solutions for X(t) and Y(t). (A4 – Questionnaire, 2021)*

However, another great highlight lied on the importance that Jordan Form gained due to its applications in the resolution of dynamic systems, since its traditional approach comprises applications in Linear Algebra, itself – this discipline is assessed in the first years of college graduation, as pointed out by another student:

> *I evaluate as very positive contribution, because it allowed me to quite effectively understand the content that had been previously taught, mainly about the importance of the Jordan Form. It made me quite satisfied, because the Jordan Form is something that I studied for several semesters and seeing it in the resolution of dynamic systems made me excited with the theory. (A5 – Questionnaire, 2021)*

Some students also highlighted the possibility of visualizing the concepts, properties and applications by pinpointing how the didactic sequence, based on GeoGebra using, has contributed to re-signify the assessed contents. However, at this time, they made a straight association with dynamic system contents, as observed in the following answer:

> *GeoGebra allowed visualizing solutions of systems by using the matrix we had chosen and it enabled obtaining images and provided a much better visualization of the behavior of solutions as time went on; it is much harder analyzing this behavior as figures (static). (A2 – Questionnaire, 2021)*

We have observed that such an answer reinforces the pedagogical potential of using educational tools in classroom made by Gonçalves and Reis (2013), who highlighted the potential of such tools to provide a dynamic knowledge, with the possibility of visualizing problems and of exploring mathematical concepts through non-static constructions. As pointed out by the researchers, it can "provide a different perception of Mathematics".

Another student, in its turn, highlighted the contribution of visualization, however, it mentioned the Jordan Form applications. It made us recall that one of our goals, with Activity 4, was to actually have students inferring the connections between the dynamics of a planar linear system and the Jordan Form, exactly through visualization provided by GeoGebra, as we have already highlighted when the didactic sequence was described:

> *Visualization through GeoGebra has contributed a lot, mainly to Jordan Forms and matrices exponential. Jordan Forms are one of the contents seen in Linear Algebra and it was the first time I saw an application. (A3 – Questionnaire, 2021)*

At this point, it is worth pointing out that visualization has been one of the research focus in mathematical education. On the other hand, based on the aims of the present study, we highlight that we understand visualization based on the following terms established by Arcavi (2003):

> Visualization is the ability, the process and the product of the creation, interpretation, use of reflection about figures, images, diagrams, in our minds, in paper or with technological tools, in order to describe and communicate information, to think about and develop previously unknown ideas and advanced understandings. (ARCAVI, 2003, p. 217)

Based on this description, we have noticed a range of aspects linked to visualization, as well as the reason why it needs to be taken into account in teaching and learning processes applied to Mathematics. As for our research, in particular, if we take into consideration the use of a technological tool (in this case, GeoGebra software), we believe that, even without seeking a theoretical reference for visualization, students have highlighted its relevance for “thinking about and developing ideas”; mainly when it comes to understanding concepts, properties and solutions of dynamic systems explored in the didactic sequence.

It is also important pointing out that, in compliance with Martins Júnior and Reis (2020), our research shines light on the importance of valuing the visualization elements provided by software using in teaching and learning processes, mainly in the Higher Education context, to create and explore ideas that, certainly, “can become allies to the understand of mathematical contents”.

Finally, by being questioned about eventual difficulties in elaborating the didactic sequence, be it in interpreting activities or in implementing it through GeoGebra, students were unanimous in highlighting that few difficulties only came up at the time to interpret some didactic sequence activities and that the integrated use of the technologies available has contributed to their mitigation, as highlighted by one student:

> *Just a little. During the classes, no, because we made everything together, professors have followed-up, we had the step-by-step about what to do and afterwards it was written in Moodle. I got distracted sometimes, the internet fell during one of the activities, but I found myself afterwards. However, I do not know if I have properly interpreted what was to be answered in all asynchronous activities in Moodle. (A2 – Questionnaire, 2021)*

Accordingly, it is important making a comparison to the conclusion by Barros Filho (2012), concerning the attitude of students participating in their research, given the use of Digital Technologies in Differential Equations teaching, when they found several difficulties “since they are not used to a proposition that demands active attitude, that generates continuous work throughout the whole activity”. Obviously, from their research to the present time, one decade has passed. Maybe, even due to the need of using Digital Technologies imposed by the pandemic, most Higher Education students nowadays got to develop the necessary skills to use such technologies, even if we do not have elements to state the occurrence of ultimate changes regarding students’ attitude in knowledge construction, given the possibilities these technologies bring up to the educational scene in Higher Education.

Final Considerations

As Higher Education Mathematics professors, we understand that we must always be concerned with acknowledging several processes to build mathematical knowledge and how they can and have been mobilized, based on the teaching context, always taking into account the disciplines taugh, their amendments, the didactic materials available and, mainly, the role of these disciplines in the mathematical formation of our students.

In this perspective, with respect to Differential Equations teaching / Dynamic Systems in Mathematics courses, it is important taking into account, as shown by the researches discussed here, the contributions of a qualitative approach to balance analytical, graphic and numerical treatments for Differential Equations learning. As shown in our research, the contributions of didactic sequences related to the dynamics of Planar Systems of Differential Equations, based on the GeoGebra software, for Introduction to Dynamic System learning, mainly focus on:

- Re-signification of knowledge by students in relation to the concepts of Dynamic Systems and also in relation to the applications of concepts of Linear Algebra;
- Creation of a learning environment provided by the way the didactic sequence was developed through activities to complete classroom assignments;
- Dynamics observed in the development of didactic sequence activities, and it allowed students to learn based on results' investigation/interpretation;
- Combination between the theoretical-algebraic and graphic-visual approaches through the visualization provided by GeoGebra.

Finally, it is necessary recalling that the use of Digital Technologies in Mathematics teaching in Higher Education remains configured as a great challenge, mainly in Mathematics Bachelor's Degree, and it is a fruitful scenario for the priority given to build advanced mathematical knowledge. Nevertheless, it is also necessary mentioning Oliveira and Reis (2017), by warning that the use of dynamic software can contribute to "break up with the teaching process applied to some concepts worked out almost exclusively based on algebraic and symbolic notions".

Our research has shown that the visualization provided by GeoGebra allowed students to experience the development of didactic sequence activities, and it helps the solid and significant construction of concepts. We are bold enough to state that they could "make Mathematics", as it must be expected for a future mathematics professor.

Accordingly, we conclude by pointing towards the relevance of carrying out new research focused on approaching the use of Digital Equations in teaching and learning processes applied to disciplines of Mathematics Bachelor's Degree courses, even from the perspective of creating a formative cycle of Mathematic professors who value education practices that have contributed to the mathematical training of their students, but that would also reason about processes to build mathematical knowledge and about its mobilization in the education scenario.

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