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History, similarities, and differences between Parabola and Catenary: a study supported by GeoGebra[1]

História, semelhanças e diferenças entre a Parábola e a Catenária: um estudo com apoio do GeoGebra

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ABSTRACT

The studies on the parabola in textbooks is something common, different from the catenary curve, even though both have certain similarities. The catenary, for example, was the subject of great discussions within the History of Mathematics and the evolution of Differential Calculus, but the discussion about it is still limited. The objective of this work is to carry out a study of these curves, presenting their similarities and differences with the contribution of the GeoGebra software. The methodology used in this study is basic qualitative research. The results show the comparison of these curves in GeoGebra and the extent to which there is similarity between them.

Keywords: *Analytical Geometry; Differential Calculus; GeoGebra.*

RESUMO

O estudo da parábola em livros didáticos é algo comum, diferente da curva catenária, embora ambas tenham certas semelhanças. A catenária, por exemplo, foi objeto de grandes discussões dentro da História da Matemática e da evolução do Cálculo Diferencial, mas a discussão sobre ela ainda é limitada. O objetivo deste trabalho é realizar um estudo dessas curvas,

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apresentando suas semelhanças e diferenças com a contribuição do software GeoGebra. A metodologia usada neste estudo é a pesquisa qualitativa básica. Os resultados mostram a comparação destas curvas no GeoGebra e até que ponto existe similaridade entre elas.

Palavras-chave: *Geometria Analítica; Cálculo Diferencial; GeoGebra.*

Introduction

The parabola and the catenary, despite their graphic similarity to a certain extent, are mathematical representations derived from different types of algebraic expressions. However, the parabola is given greater attention by textbooks. In the school context, it is common to choose the parabola instead of the catenary for modeling curves, such as the representation of suspension bridges or suspended wires, as the parabola has a simpler equation to handle, and does not require knowledge of advanced calculus or natural logarithms. However, both curves have their particularities and importance.

The parabola, with its reflective property, is capable of radiating light or sound, which allows its use in telescopes, satellite dishes, lighthouses, and reflectors, in addition to enabling the field of Architecture to design environments with appropriate acoustic characteristics for auditoriums, theaters or churches (Eves, 2011; Sousa, Alves & Aires, 2023; Sousa et al., 2023).

The catenary and its respective equation can be considered as one of the most important solutions among the challenging problems in the History of Calculus (Talavera, 2008). The equation generated by the catenary curve together with the development of Differential Calculus had a strong influence on the development of hyperbolic functions.

GeoGebra has great potential to demonstrate the difference between these curves clearly, given the strong properties of working with algebra and geometry in an integrated way, enabling dynamic demonstrations with mathematical precision (Alves, 2019; 2020).

The objective of this work is to carry out a study of the parabola and the catenary, presenting their similarities and differences with the contribution of the GeoGebra software. The methodology of this study is of a qualitative nature, being a basic research, in which we seek to broaden the look on the subject and its discussion.

1. Parabola

Menaecmus, (350 BC), was the first to deal with conic sections, sectioning cones with planes perpendicular to the generatrix. Apollonius of Perga (225 BC) brought the study of the curves of conic sections from double and straight conical surfaces, as we use today. Although conics have been known since antiquity, their study gained notable relevance in the 17th century, based on the works of Gérard Desargues (1593-1661), Blaise Pascal (1623-1662), Johannes Kepler (1571-1630) and Galileo Galilei (1564-1642) (Boyer, 2012; Eves, 2011).

The most famous applications of conics are due to Galileo (1564-1642), who concluded that the trajectory of a cannonball describes a parabola, as well as Kepler (1571-1630) and Newton (1643-1727) who found through their research that the orbits of the planets are elliptical. Galileo's theories were confirmed years later by Newton (1643-1727), based on the Law of Universal Gravitation. According to Boyer (2012), such discoveries made it possible to conjecture a relationship between conics and nature, such as the orbit of planets and some comets in the solar system, which made the study of these curves go beyond mathematics, becoming of interest to other sciences, such as Astronomy and Physics.

In the case of the parabola, Lima (2014, p. 115) says "let d be a line and F be a point outside it. In the plane determined by d and F, it is called parabola of focus F and directrix d to the set of points equidistant from d and F" (Figure 1):

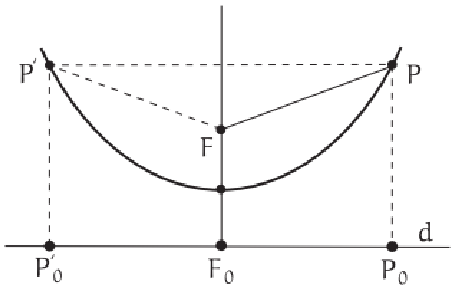

FIGURE 1: Geometric representation of the parabola
FONTE: Lima (2014, p. 115).

The point P belongs to the parabola with focus F and directrix d, since the distance from point P to F is the same distance between point P and P_0. That is, $d(P,F) = d(P,P_0)$, with the segment PP_0 perpendicular to the directrix d and the perpendicular FF_0 lowered from the focus on the directrix, it is configured in an axis of symmetry (Lima, 2014). The essential elements of a parabola are the focus (F), the directrix (d), the vertex (V), the parameter (p), which represents the distance from the focus to the directrix, and the line VF, which is the axis of symmetry. The deduction of the equation of a parabola with focus F and directrix d, with $p > 0$ representing the distance from F to d is (Figure 2):

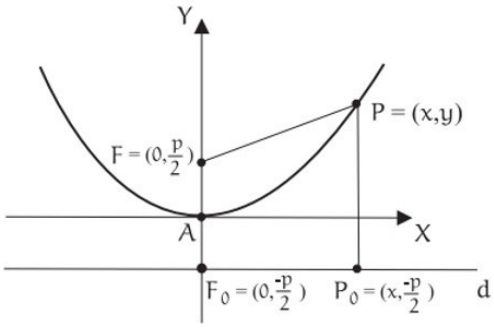

FIGURE 2: Deduction of the parabola equation
FONTE: Lima (2014, p. 115).

A system of axes is taken in which the vertex of the parabola is the origin of the system and the vertical axis is the straight line FF_0, the parabola's symmetry axis. Note that the point F has coordinates $F = (0,\ \frac{p}{2})$ and the equation of the directrix d is $y = -\frac{p}{2}$. If the point $P = (x, y)$ belongs to the parabola, then we have that $y \geq 0$. Since the vertical axis is the axis of symmetry and if $P = (x, y)$ belongs to the parabola, then $P' = (-x, y)$ also belongs.

Thus, $P = (x, y)$ being a generic point of the parabola. We have that the distance from P to the directrix d is $y + p/2$, while the distance from P to the focus F is $\sqrt{x^2 + (y - p/2)^2}$. And since P is a point belonging to the parabola, we have that $y + p/2 = \sqrt{x^2 + (y - p/2)^2}$, where squaring both sides and developing the expression, we have:

$$\left(y + \frac{p}{2}\right)^2 = x^2 + (y - p/2)^2$$

$$y^2 + py + \frac{p^2}{4} = x^2 + y^2 - py + \frac{p^2}{4}$$

and by reducing like terms, we get the expression:

$$x^2 = 2\text{py or y} = \frac{x^2}{2p}$$

which is the canonical equation of the parabola with vertex at the origin and axis of symmetry FF_0.

The parabola can also be represented as the graph of a quadratic function, with the explicit equation $y = ax^2 + bx + c$, with $a \neq 0$, or written as a function of the coordinates of its vertex – the canonical form – which brings it through the expression $f(x) = a \cdot (x - h)^2 + k$, with $a \neq 0$. There are other ways of representing the parabola, as a conic section, with polar coordinates, as a locus, but we will restrict ourselves to these representations. The most famous property of the parabola is related to focus, which is its reflective property. This property states that incident rays parallel to the axis are reflected to the focus (Figure 3):

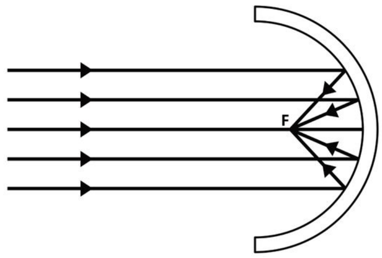

FIGURE 3: Reflecting property of the parabola
SOURCE: Shutterstock (2023, copyright free)

Something similar happens with sound waves. Consequently, this property is widely used by satellite dishes, which have the shape of a paraboloid of revolution, obtained by rotating a parabola (Figure 4):

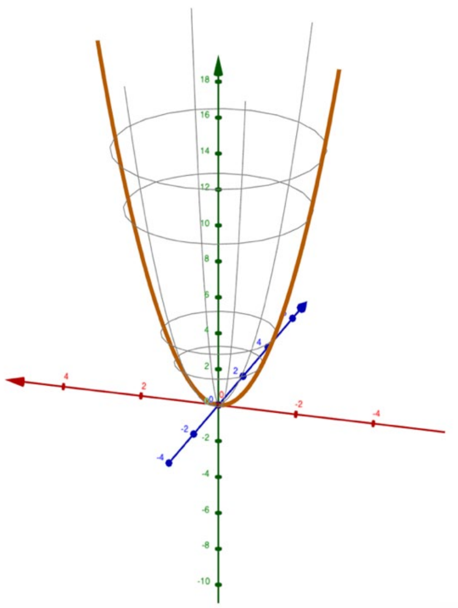

FIGURE 4: Paraboloid of revolution (or parabolic surface)
SOURCE: Elaborated by the authors (2023)

It is noteworthy that the "contrary" property is also valid. Rays emitted by the focus are scattered in directions parallel to the axis. Some instruments are built to work based on this idea, such as flashlights, spotlights, and car headlights.

2. Catenary

Catenary (from Latin, *catena*) is the name of the curve formed by a flexible wire/current, of constant density throughout its length, which is suspended only by its two ends, being subjected only to the force of gravity (Yates, 1974; Maor, 2004; Lima & Miranda, 2021). Thus, the catenary, in a strict sense, is not a curve, but a family of curves, each of which is determined by the coordinates of its extremes (x_0, y_0), (x_1, y_1) and by its length L.

The first studies on the catenary started with Galileo Galilei, who tried to describe such a curve in an analytical way. However, Galileo mistakenly conjectured that the catenary would be the approximation of a parabola, in analogy to the trajectory of a projectile (Yates, 1974; Talavera, 2008; Mendes, 2017; Lima & Miranda, 2021). However, the fact that the parabola has the form of an inextensible string, subjected to uniformly distributed vertical loading, was already a conclusion known by Beeckman in the year 1615 and, after Galileo's mistake, found again by Huygens in 1646 (Pauletti, 2002). Huygens investigated the geometry of the catenary, considering it a curve assumed by a perfectly flexible and inextensible chain, with uniform linear density, hanging from two hooks not situated on the same vertical, proving with arguments from physics that Galileo's conjecture was wrong, without show, however, the analytical expression of the curve (Eves, 2011).

Also, according to Pauletti (2002), in the year 1690, Jakob Bernoulli proposed a challenge to the scientists of the time, inviting them to a contest in search of the shape of the catenary. After a year, Johann Bernoulli, Leibniz, and Huygens solved the problem. There was great rivalry among the contestants, making it difficult to actually attribute authorship of the discovery. While Huygens' solution was based on some axioms and theorems of Geometry, Leibniz and Johann Bernoulli used Calculus, which at the time was a recent creation, as shown in Figure 5:

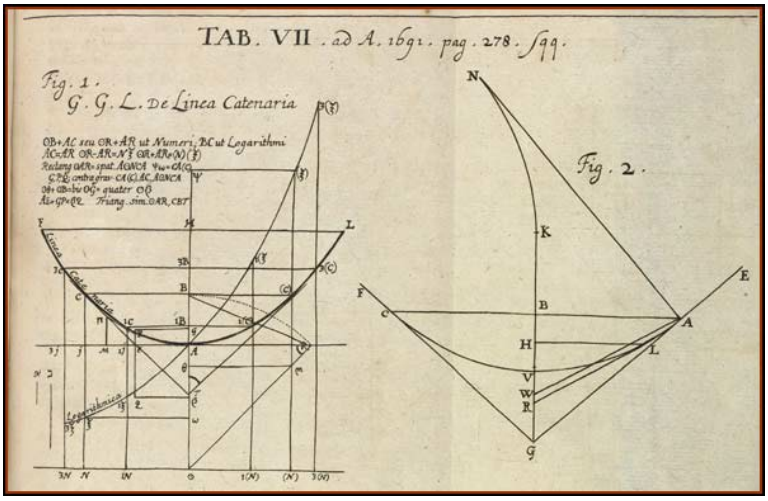

FIGURE 5: Leibniz and Johann Bernoulli's solution (1691)
SOURCE: Maor (2004)

About this discovery, Maor (2004) explains that the catenary is a curve such that its equation in modern notation is:

$$y = \frac{e^{ax} + e^{-ax}}{2a}$$

where a is a constant and depends on the parameters current, which are its linear density and the voltage at which it is held, as Simmons (1987, p. 611) shows (Figure 6):

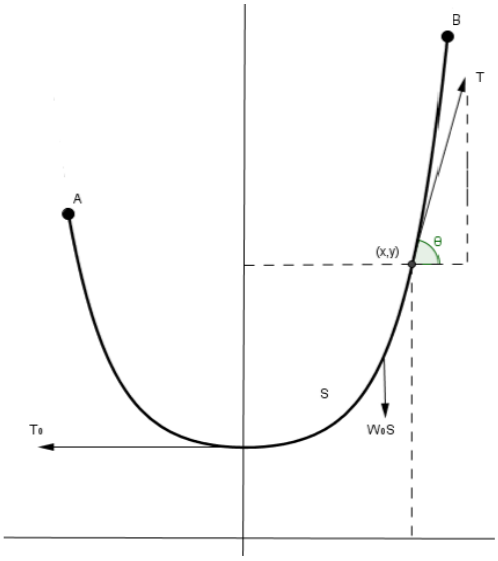

FIGURE 6: The catenary in Simmons' work
SOURCE: Simmons (1987, p. 611)

However, the hyperbolic equation that provides the definition of the catenary curve was created years later, in 1757, by the Italian mathematician Vincenzo Riccati (1707 – 1775). Jesuit and Mathematics professor, Ricatti dedicated himself to the development of differential equations, infinite series, quadratures and hyperbolic functions (Eves, 2011). Briefly, we can observe this curve in electricity wires hanging from poles, in architectural works, among other situations.

Seen as a function of the hyperbolic cosine, the catenary can be defined by a curve generated from a flexible cable, of constant density, hanging between two extremes, under the action of its own weight (gravity), in which its minimum point is $(0, a)$, with $a > 0$, with an equation equal to:

$$y = a.\cosh\left(\frac{x}{a}\right)$$

and its graphical representation is similar to the sketch shown in Figure 7:

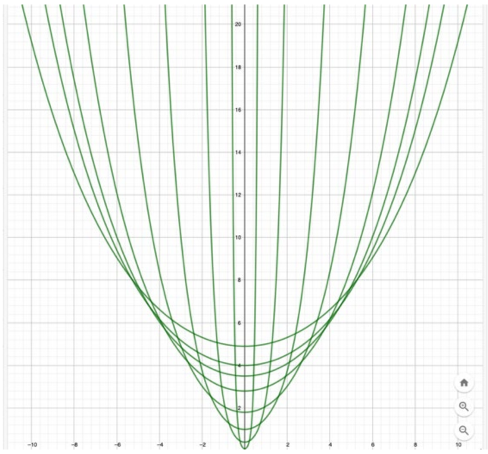

FIGURE 7: Catenary curve from the hyperbolic cosine function sketch in GeoGebra
SOURCE: Elaborated by the authors (2023)

Visually, the most explicit way to differentiate a catenary from a parabola is through their respective equations (Barbosa, 2013). In the case of the catenary, its equation is given by the hyperbolic function and its exponential equivalent, that is:

$$f(x) = a.cosx\left(\frac{x}{a}\right) = \frac{a}{2}.\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

where a is a constant determined from gravity and the material of the string. We can achieve catenary differentiation when the part of the curve between its lowest point and a point (x, y) is under the action of three forces, which are the voltage at the lowest point, the variable voltage T_0 at the point (x, y), acting in the tangent direction, given the flexibility of the wire and a downward-pointing weight force, equivalent to the weight of the wire between the lowest point and a given point (x, y).

Based on the work of Simmons (1987) and supported by other readings, such as Swokowski (1994) and Leithold (1994), we can arrive at the catenary equation by looking again at Figure 4 and the following demonstration:

Data s as the length of an arc between a given point and a variable point (x, y) and the measure W_0 the linear density of the yarn. When we equate the horizontal member $T = T_0$ and T vertical to the weight of the wire, we have:

$$Tcos\theta = T_0 \text{ and } Tsen\theta = W_0 s$$

The tangent of θ can be obtained from the quotient of the two expressions, in which the variable T is also eliminated:

$$tg\theta = \frac{W_0 s}{T_0}$$

or, equivalently, $\frac{dy}{dx} = as$ where $a = \frac{W_0}{T_0}$.

Differentiating the expression with respect to x and eliminating the variable s, we get expression (I):

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} - a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

which is the differential equation of the catenary. From a process of successive integrations and an auxiliary variable $v = \frac{dy}{dx}$, we can determine the differential equation of the catenary. By substituting v in expression (I), we have:

$$\frac{dv}{dx} = a\sqrt{1 + v^2}$$

and by separating the variables and integrating both members, we find the expression (II):

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int a\, dx$$

which results in:

$$\left(\sqrt{1 + v^2} + v\right) = ax + c_1$$

with $x = 0$, it implies that $v = 0$ and $c_1 = 0$. Then, the expression can be rewritten as:

$$\left(\sqrt{1 + v^2} + v\right) = ax$$

When solving the equation in v, we get:

$$\frac{dy}{dx} = v = \frac{1}{2}(e^{ax} + e^{-ax})$$

Integrating both sides of the equality:

$$y = \frac{1}{2a}(e^{ax} + e^{-ax}) + c_2$$

The constant c_2 can be set to zero, depending on the position of the y-axis. Thus, considering the definition of hyperbolic cosine, we can rewrite the expression as:

$$y = a\cosh\left(\frac{x}{a}\right), a > 0$$

This algebraic demonstration is important for understanding the catenary curve and its peculiarities, serving as a basis for studies on this topic. In fact, Yates (1974), on the function of the hyperbolic cosine, explains that it plays a dominant role in electrical communication circuits. “For example, the engineer prefers the convenient hyperbolic form to the exponential form of solutions to certain types of transmission problems” (p. 117).

However, we emphasize that, at the beginning of this discussion, we commented on Galileo's mistake in his attempt to prove the identification of the catenary with the parabola. From this analytical expression, such a comparison becomes more viable. Observing the explicit formula of the catenary, we can reflect on Galileo's mistake and to what extent it is reasonable to confuse these curves. We even emphasize that Galileo had his reasons for identifying the catenary with the parabola. Developing $cosh(x)$ in a Taylor series, we find:

$$\begin{cases} e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \\ e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \ldots \end{cases}$$

where the hyperbolic cosine is:

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} = 1 + \frac{x^2}{2} + O(i)$$

which shows us that the catenary equation corresponds to that of a parabola plus a fourth order term, that is, the error between the parabola and the catenary of the order of $\left(\frac{1}{2}\right)^4$. Therefore, close to the origin, both equations will be very similar.

3. Building these curves in GeoGebra

- *Parabola*

For the construction of the parabola from a quadratic equation of the type $f(x) = ax^2 + bx + c$, we create three sliders a, b and c that represent the parameters of the function. By typing in the input bar, the expression *y=a*x^2+b*x+c*, we get a parabola, as shown in Figure 8:

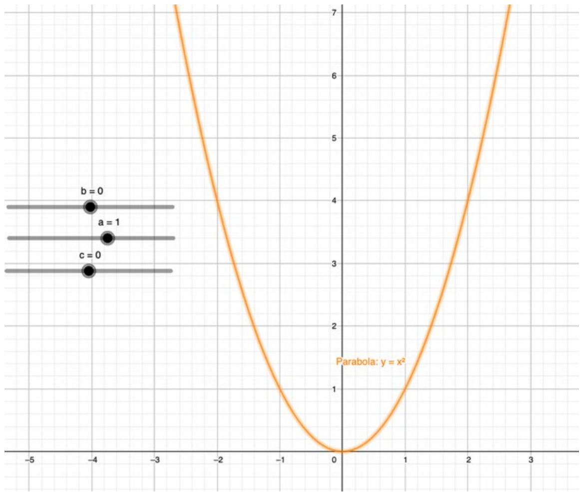

FIGURE 8: Construction of the parabola in GeoGebra
SOURCE: Elaborated by the authors (2023)

By manipulating the sliders a, b and c, we can observe the behavior of the parabola, such as the position and opening of its concavity, its translation on the y-axis, among other details.

- *Catenary*

For the construction of the catenary, we carry out a process similar to that of the parabola, inserting the equation $y = a.\cosh\left(\frac{x}{a}\right)$, which in the input window must be typed in the form *g(x)=a*cosh((x - b) / a)*. When we enter the parameter a in this equation, we are automatically adopting the same slider a used for the parabola. After entering the expression, we should see the graph as in Figure 9:

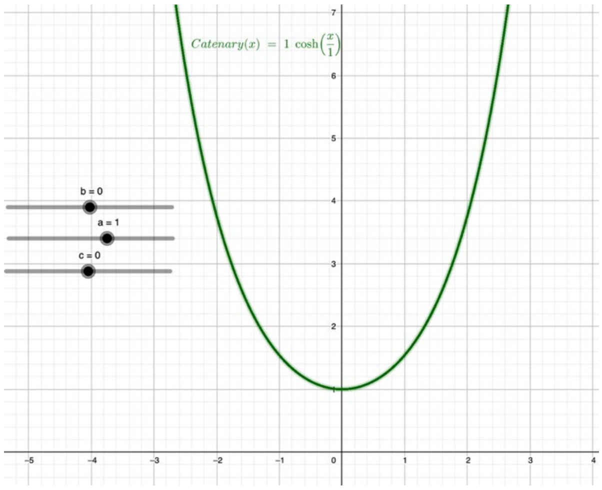

FIGURE 9: Construction of the catenary in GeoGebra
SOURCE: Elaborated by the authors (2023)

When we manipulate the sliders a and b we have, respectively, a change in the opening of the catenary curve and its horizontal translation.

3.1 Comparison between the curves

If we look at both curves together, with the same values for parameters a and b and parameter $c = 0$ (since this only relates to the parabola), we can see that the curves, despite having points in common, are different, as shown in Figure 10:

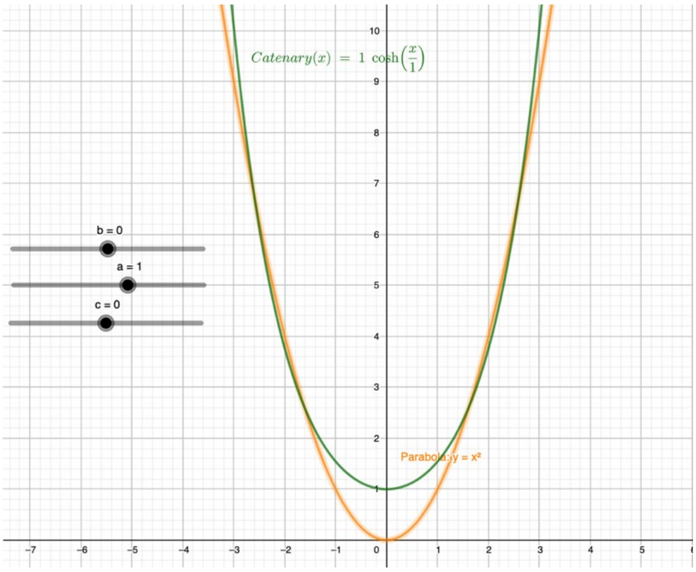

FIGURE 10: Comparison between the parabola and the catenary
SOURCE: Elaborated by the authors (2023)

From the movement of parameter a, we can observe the families of parabolas and catenaries, showing the main differences between these curves. For this, we use the “Show trace” function in GeoGebra, as illustrated in Figures 11(a) in orange, and 11(b) in green:

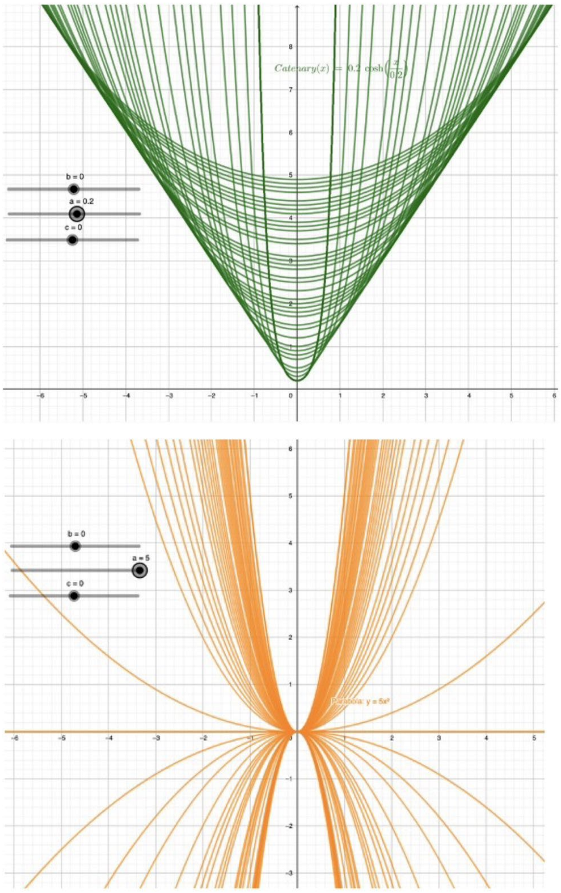

FIGURE 11a AND 11b: Parabola and catenary families built in GeoGebra.
SOURCE: Elaborated by the authors (2023)

When analyzing the graphs of the parabola and the catenary in an overlapping way, we can guess the reasons why the ancient mathematicians, at first, pointed out that the parabola was the curve that deformed under its own weight (Mata, 2003). We can observe in real situations how the curves resemble each other and contain intersection points. We have an example illustrated in Figure 12, in which we show the proximity of both curves compared to the vault of a church, for example:

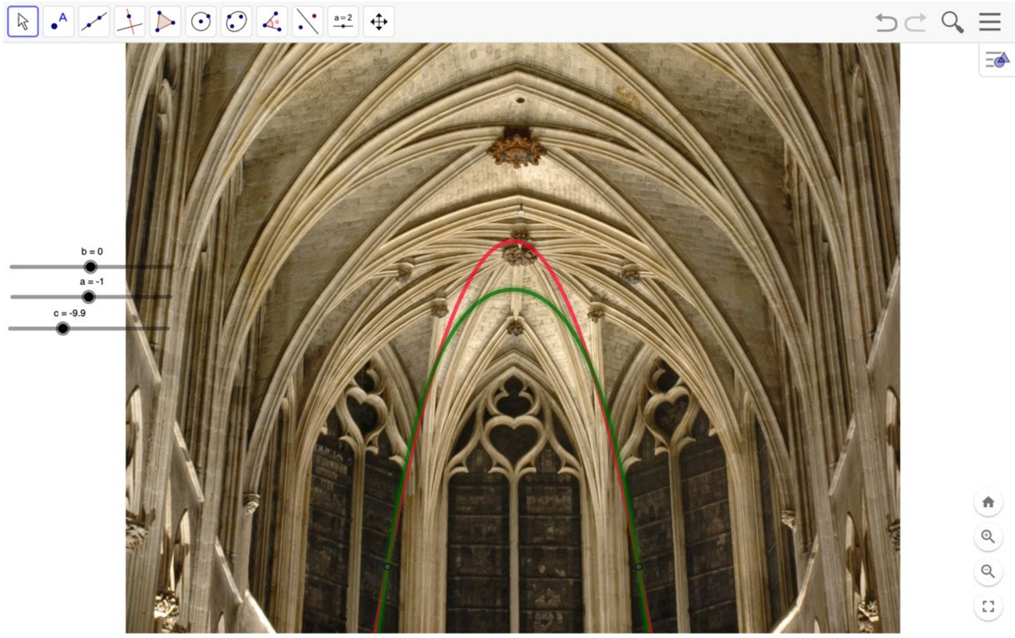

FIGURE 12: Parabola and catenary as a vault of a church built in GeoGebra.
SOURCE: Elaborated by the authors (2023).

In both constructions, we can observe a certain proximity between their vertices at some points, which may have been the cause of the misunderstanding of mathematicians in the past, such as Galileo (Talavera, 2008). However, the use of GeoGebra in this case helps us to visualize the differences and similarities between both curves. We can still observe the differences between the curves in the 3D plane, from the parameterization of their equations and projection in the three-dimensional plane.

For the parametrization of the parabola, we have that in the λ curve of the Cartesian equation $(x-a)^2 = k(y-b)$, which implies $y = 1/k(x-a)^2 + b$, with vertex at point $V(a,b)$ and focal line parallel to the y-axis. By establishing an independent variable t being $x-a$, the variable y can be expressed by:

$$y = \frac{1}{k}t^2 + b$$

Thus, the parabola λ has parametric equations:

$$\begin{cases} x = t + a \\ y = \dfrac{1}{k}t^2 + b \end{cases}$$

with $t \in \mathbb{R}$. In GeoGebra, we can parameterize and construct the parabolic surface, first creating a slider $\alpha = 360°$, which defines the angle of rotation of the curve. And then, from the command *Surface(<Curve>, <Angle>, <Line>)*, we insert in the input field, respectively, the previously constructed parabola curve, as in the previous steps, the angle, and the reference axis for the rotation. For this construction we use the command in the following structure:

Surface(Parabola, α, Y-axis)

typed in the input bar, which generated the surface illustrated in Figure 13:

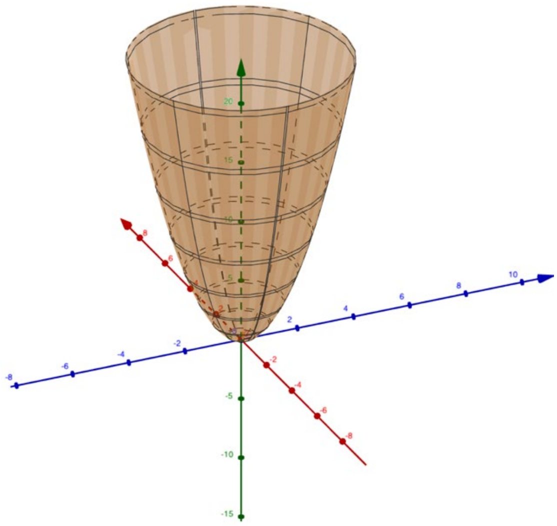

FIGURE 13: Parameterization of the Parabola/Surface in 3D
SOURCE: Elaborated by the authors (2023)

Regarding the parameterization of the catenary, we have that from its equation $y = a\cosh(x/a)$, we can establish an immediate parameterization, considering the variable t as a parameter:

$$\begin{cases} x = t + a \\ y = a.\cosh\left(\dfrac{t}{a}\right) \end{cases}$$

But there is also the possibility of using the initial definition of the hyperbolic cosine and:

$$\cosh\left(\frac{t}{a}\right) = \frac{e^{\frac{t}{a}} + e^{-\frac{t}{a}}}{2}$$

and eliminating the exponential terms using logarithms, considering that:

$$ln\, t = \frac{x}{a} \Rightarrow -ln\, t = -\frac{x}{a} \Rightarrow -\frac{x}{a} = ln\, \frac{1}{t}$$

Under these conditions, we have that $x = a.ln\, t$ and:

$$\frac{2y}{a} = e^{\ln t} + e^{-\ln t} = t + \frac{1}{t}$$

This implies the parameterization:

$$\begin{cases} x = a.\ln t \\ y = \dfrac{a}{2}\left(t + \dfrac{1}{t}\right) \end{cases}$$

with $t > 0$, to guarantee the existence of the logarithm. In the GeoGebra environment, the parameterization and construction of the catenary surface can be done with a protocol similar to that of the parabola. Thus, we create a slider $\beta = 360°$, which defines the rotation angle of the curve, followed by the previously used command *Surface(<Curve>, <Angle>, <Straight>)*, in which we insert, respectively, the catenary curve already constructed, the angle and reference axis for the rotation. For this construction we use the command in the following structure:

Surface(Parabola, β, Y-axis)

typed in the input bar. Such execution shows us the surface of Figure 14:

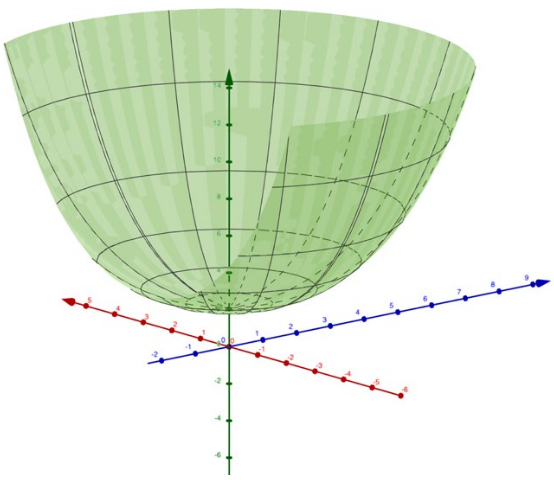

FIGURE 14: Parameterization of the catenary/Surface in 3D
SOURCE: Elaborated by the authors (2023)

In addition, the comparison between the two surfaces can be shown clearly with the contribution of the GeoGebra 3D window, as shown in Figures 15 and 16:

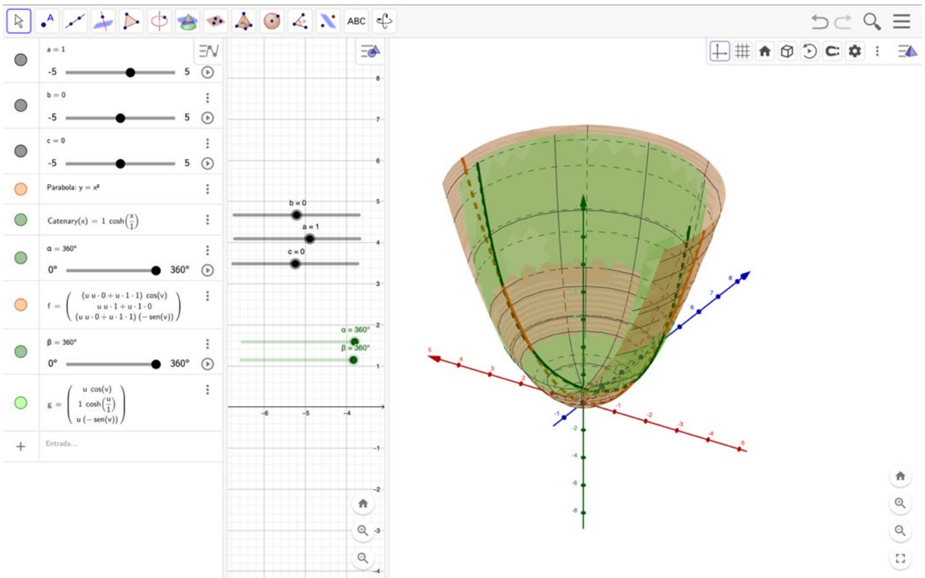

FIGURE 15: Comparison between the parabola and the catenary, with a=1
SOURCE: Elaborated by the authors (2023)

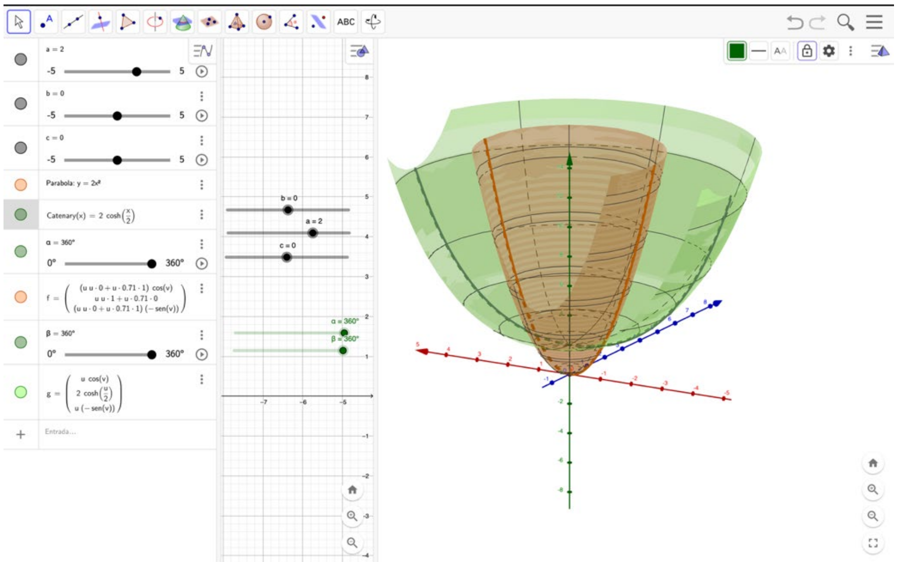

FIGURE 16: Comparison between the parabola and the catenary, with a=2
SOURCE: Elaborated by the authors (2023)

In this way, we kept the sliders $b = 0$ and $c = 0$, moving only the slider a, where we can see that the closer to zero the value of a is, the closer the curves are. The proof of the mathematical formulas of the parabola and the catenary, as shown from the Taylor series, coincide in their first three terms, differing only from the fourth term, which makes their graphs similar, but not the same, for small values of x, making their differentiation more explicit as the x values increase, which can be evidenced with the GeoGebra 3D window, as shown.

4. Final considerations

We rely on works on the History of Mathematics in order to understand how the studies of these curves occurred in the past, as well as their mathematical structure, usability, and possible applications. So, we understand, in fact, that the parabola, for its understanding and simpler algebraic/analytical and geometric demonstrations, has been more widely discussed over the years, while the catenary for a long time was an intriguing subject to ancient mathematicians.

It is known that, with the absence of significant internal efforts, a cable tends, in static equilibrium, to behave like a current and, consequently, its static configuration tends to conform to a catenary curve. However, it was verified that some articles that adopt finite differences use the parabola as a static configuration.

The construction and discussion of this work shows us how, at certain points, it is not just a simple mistake to confuse these parabola and catenary curves. It's something that can compromise entire architectural structures, for example. In addition, we reinforce the need to mathematically demonstrate this subject, taking due care so that the mathematics and physics implicit in the theme are interpreted correctly. Thus, the use of GeoGebra provided visual and algebraic subsidy, being a different approach for understanding the theme.

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