https://doi.org/10.23925/2237-9657.2024.v13i3p027-044

The Rubik's cube and GeoGebra: a visual exploration of permutation groups

O cubo mágico e o GeoGebra: uma exploração visual de grupos de permutação

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ABSTRACT

This article aims to use the construction of the Rubik's Cube in GeoGebra as a primary tool for visualization and manipulation in the teaching of permutation groups in Abstract Algebra. We bring a brief discussion about the concept of group, aspects of the Rubik's cube, the Rubik's group as a group of permutations and possibilities for its exploration in GeoGebra. Based on this study, we recognize the potential to delve into permutation groups in Abstract Algebra through a visual interface that associates their properties with a tangible and manipulable object. Additionally, there is the potential for simulating their movements using Dynamic Geometry software, such as GeoGebra.

Keywords: *Rubik's cube; Group Theory; Permutation Groups; GeoGebra.*

RESUMO

Este artigo tem como objetivo utilizar a construção do Cubo Mágico no GeoGebra como ferramenta primária para visualização e manipulação no ensino de grupos de permutação em Álgebra Abstrata. Apresentamos uma breve discussão sobre o conceito de grupo, aspectos do

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Cubo Mágico, o grupo de Rubik como um grupo de permutações e as possibilidades de sua exploração com o GeoGebra. Com base nesse estudo, reconhecemos o potencial de aprofundar-se nos grupos de permutação na Álgebra Abstrata por meio de uma interface visual que associa suas propriedades a um objeto tangível e manipulável. Além disso, há a possibilidade de simular seus movimentos usando software de Geometria Dinâmica, como o GeoGebra.

Palavras-chave: *Cubo de Rubik; Teoria dos Grupos; Grupos de Permutação; GeoGebra.*

Introduction

In 1974, Ernö Rubik, a professor from Budapest, Hungary, introduced his fascinating invention called the Rubik's Cube. This mathematical toy became a part of popular culture, inspiring competitions, and captivating both children and geniuses who enjoyed the mental challenge of solving it (Carter, 2009). The problem posed by this game-like toy involves starting from a position where the faces display their smaller squares in different colors and performing a sequence of movements on the cube's axes so that all six faces show a single color (Joyner, 1997; 2008; Chen, 2004).

It is common to observe that students, even those enrolled in undergraduate mathematics courses, may face difficulties when attempting to solve the Rubik's Cube problem in a practical and objective manner. Additionally, there may be a lack of clear understanding regarding the mathematical properties involved in the necessary movements. According to Vágová and Kmetová (2018, p. 1054), "the importance of using mental images and visual processing in mathematics has been increasingly recognized". The authors also note that since the early '90s, the debate about the real existence of a mental image in the human brain has taken on a new form, especially with the availability of neuroimaging techniques.

This article aims to use the construction of the Rubik's Cube in GeoGebra as a primary tool for visualization and manipulation in teaching permutation groups in Abstract Algebra. The software allows visualizing and manipulating the Rubik's Cube, improving the understanding of permutation groups through interactive visual learning.

GeoGebra is a powerful tool for illustrating mathematical concepts (Sousa; Alves; Aires, 2023), including the relationship between the Rubik's Cube and permutation groups, providing a visual and interactive representation of the mathematical concepts involved.

This work is an initial section of ongoing doctoral research, which seeks different ways of approaching Abstract Algebra and the algebraic structures of finite groups outside of traditional molds, in an attempt to explore visualization and increase the assimilation of their concepts.

Before discussing some aspects of the Rubik's Cube construction, let's briefly revisit the concept of a group in Abstract Algebra.

1. A brief description of the group concept

A non-empty set G, equipped with an operation $*$, is termed a group if, and only if, for every pair of elements (a, b) in G, a unique $c \in G$ can be associated, where $c = a * b$ (Gonçalves, 1995; Wussing, 1984; Davvaz, 2021). Equivalently, it is understood that there exists a function:

$$*: G \times G \to G$$

$$(a, b) \longmapsto a *$$

and for the pair $(G,*)$ to be considered a group, the following properties must be satisfied:

(i) (Associative) $a * (b * c) = (a * b) * c$, for any $a, b, c \in G$.

(ii) (Existence of identity element) There exists $e \in G$, such that $a * e = e * a = a$, for any $a \in G$.

(iii) (Existence of inverses) For each $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$. Additionally, if the property:

(iv) $a * b = b * a, \forall\, a, b \in G$ (commutative) is identified and satisfied, we say that the group is *abelian*.

In the case of finite groups, the number of elements in G is called the order of the group, and the operation table $*$ is called the group table. It allows us to analyze all possibilities and arrangements among its elements.

These group properties will be visualized and manipulated within GeoGebra to provide an intuitive understanding of the permutations within the Rubik's Cube.

2. Aspects of Rubik's cube

For the understanding of the mathematics involved in the Rubik's Cube, we rely on some important works over the years on the subject, such as Warusfel (1981), Bandelow (1982), Chen (2004), Joyner (1997; 2008), and Carter (2009). Here, we explore the most classic case of the cube, which is its 3×3 structure.

The Rubik's Cube has a different color for each of its faces. The colors of the cube's faces may vary depending on its place of manufacture. The pieces of the Rubik's Cube (the small colored cubes) can be permuted in various ways through possible movements. Each configuration of the Rubik's Cube can be represented as a specific permutation of the pieces, where the goal when solving the cube is to reach a specific configuration, such as having all faces with a single color (Warusfel, 1981). We have an example of the cube in Figure 1:

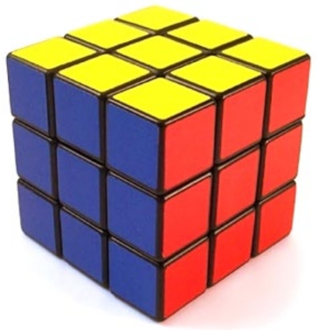

FIGURE 1: Rubik's cube.
FONTE: Free access.

However, it is worth noting that we do not focus on color for understanding its movements, but rather on its position. We can use the initial letters of the following terms to refer to the faces of the cube and the possibilities of movements (Table 1):

Right = R	Front = F	Up = U
Left = L	Back = B	Down = D

Table 1. Possibilities of movements in the Rubik's cube.

The Rubik's Cube was constructed in GeoGebra by representing each of its six faces as sets of smaller square tiles. Each face is comprised of 3×3 squares, matching the layout of the traditional 3×3 Rubik's Cube. This construction allows for individual manipulation of each face.

Each movement of the cube, represented by R (Right), L (Left), U (Up), D (Down), F (Front), and B (Back), is constructed as a rotation of a specific layer of the cube in GeoGebra. For instance, an R movement corresponds to a 90-degree clockwise rotation of the rightmost layer of the cube. This rotation can be visualized by selecting the group of squares that form the right face and applying a transformation that rotates these elements around the central axis of that face.

Interactive tools such as buttons were incorporated into the GeoGebra interface to facilitate the manipulation of each face of the cube. For example, pressing the R button rotates the right face 90 degrees clockwise. Similarly, an R^{-1} button is used for a counterclockwise rotation, and an R^2 button allows a half-turn (180 degrees). This setup enables users to visualize and perform sequences of movements directly in the software.

Each face rotation in GeoGebra is mapped to a permutation of the smaller cubes, represented as cycles. For example, a 90-degree clockwise rotation of the right face (R) corresponds to a permutation involving the elements of that face. These permutations are visualized dynamically, allowing users to understand how the group structure of the Rubik's Cube evolves with each movement.

To better understand the dynamics of each of the smaller facets, Joyner (1997, p. 69) provides us with a diagram (Figure 2):

```
                    +--------------+
                    |  1    2    3 |
                    |  4    u    5 |
                    |  6    7    8 |
     +--------------+--------------+--------------+--------------+
     |  9   10   11 | 17   18   19 | 25   26   27 | 33   34   35 |
     | 12    l   13 | 20    f   21 | 28    r   29 | 36    b   37 |
     | 14   15   16 | 22   23   24 | 30   31   32 | 38   39   40 |
     +--------------+--------------+--------------+--------------+
                    | 41   42   43 |
                    | 44    d   45 |
                    | 46   47   48 |
                    +--------------+
```

FIGURE 2: Diagram of each of the larger and smaller faces of a 3 × 3 Rubik's Cube.
FONTE: Joyner (1997, p. 69).

This diagram allows us to visualize the layout of the 3×3 Rubik's Cube model, highlighting the orientation of each face as F, B, L, R, U, D on the central small cubes, as shown in Table 1. Thus, it is possible to check the generators corresponding to the six faces of the cube (Joyner, 1997), which, in turn, can be written in disjoint cycle notation as organized in Table 2:

F	(17, 19, 24, 22) (18, 21, 23, 20) (6, 25, 43, 16) (7, 28, 42, 13) (8, 30, 41,11)
B	(33, 35, 40, 38) (34, 37, 39, 36) (3, 9, 46, 32) (2, 12, 47, 29) (1, 14, 48, 27).
L	(9, 11, 16, 14) (10, 13, 15, 12)(1, 17, 41, 40) (4, 20, 44, 37) (6, 22, 46, 35)
R	(25, 27, 32, 30) (26, 29, 31, 28) (3, 38, 43, 19) (5, 36, 45, 21) (8, 33, 48, 24)
U	(1, 3, 8, 6) (2, 5, 7, 4) (9, 33, 25, 17) (10, 34, 26, 18) (11, 35, 27, 19)
D	(41, 43, 48, 46) (42, 45, 47, 44) (14, 22, 30, 38) (15, 23, 31, 39) (16, 24, 32, 40)

Table 2. Disjoint cycle generated from 3 × 3 Rubik's cube.

The mechanism allows each face the freedom of rotation from 0° to 360° around the axis that fixes the central small cube to the internal mechanism, both clockwise and counterclockwise, always taking the face in front of the manipulator as the reference. Each rotation of the cube makes a 90° movement, and by performing the same movement four times, completing a total of 360°, the cube returns to the initial position, making it a neutral operation (Joyner, 2008; Carter, 2009).

Note that when we rotate a face, the color of the central square on the face always shows the same color. Therefore, we identify each face by the color of its center. Thus, we use the six letters to designate the six faces, as well as various pieces and positions. For example, the four central parts of the edges corresponding to the U face will be UR, UF, UL, and UB, while the four parts of the vertices corresponding to the U face will be URF, UFL, ULB, and UBR. Note that UR and RU are the same piece. The colors of the vertices are ordered clockwise. Thus, URF, RFU, and FUR denote the same part.

The names of the faces are also used to refer to quarter-turn movements clockwise, where the face is oriented towards the manipulator. An example of this is: R involves a 90° clockwise rotation of the face to the right. The half-turn of the R face, whether clockwise or counterclockwise, does not matter; we denote it as R^2 because it corresponds to two 90° clockwise rotations. The counterclockwise rotation can be denoted as R^{-1} or R' for the sake of mathematical convention.

A sequence of movements is written from left to right. For example, RU means that the R movement is applied first, followed by U. The context allows us to distinguish whether a sequence of two or three letters corresponds to a sequence of movements or to a piece.

GeoGebra enables the visualization of movement patterns, making it possible to identify and explore cycles within the permutations. By observing sequences like ($R\ U\ R^{-1}\ U^{-1}$), students can see the transformation of the cube's state and grasp the concept of the order of a sequence of movements. Such visual feedback reinforces the understanding of permutation cycles and their effects on the structure of the cube.

3. Rubik's group

As seen earlier, we designate as R, L, F, B, U, and D the clockwise movements of the right, left, front, back, up, and down faces, respectively. According to Chen (2004), the set of possible movements of the Rubik's Cube can be transformed into a group, which we can denote as $(G,*)$. Two movements can be considered equal if they result in the same configuration for the cube; for example, rotating a face 180° clockwise would be the same as rotating 180° counterclockwise.

Thus, the group operation can be defined as follows: if M_1 and M_2 are two movements, then $M_1 * M_2$ is the movement where you first execute movement M_1 and then M_2. We call the set of all allowed movements in shuffling the cube the Rubik's Group or Group R. In this way, let's verify the existence of the three conditions that satisfy the group definition for the Rubik's Cube:

1st condition: Associativity. When we perform any sequence, for example, of 3 generic movements X, Y, Z, we observe that: $X(Y\,Z) = (XY)Z$. Manipulating the cube, the proof of this condition is evident, and we consider it trivial.

2nd condition: Identity element. The existence of the identity element is verified by the movement of doing nothing on any of the 6 faces. With this, we ensure the identity I of any of the six faces of the cube or of any sequence of movements.

3rd condition: Inverse element. The inverse element in the Rubik's Group R means undoing the sequence of one or more movements performed. Here, there are two important considerations to be made: whether the movement was of only one face or a sequence of movements. Let X be a generic movement, a 90 degrees clockwise rotation of one of the 6 faces; then its inverse element will be the movement, also of 90 degrees, of the same face, but counterclockwise. This inverse element can be denoted as X^{-1}. If a sequence of movements is executed, then the inverse element will be the execution of the inverses of the movements but in the reverse order.

Using GeoGebra, we visually demonstrate the associative property through sequences of movements, the identity element as the 'do-nothing' movement, and the inverse elements by reversing sequences.

The inverse element in a group is unique, which is why, in the case of the Rubik's Cube, it is necessary for it to be well-defined to avoid duplicity when seeking the inverse element of a movement or a combination of movements. For this reason, we must be clear that performing only one movement is a particular case of performing a sequence of movements, ensuring the uniqueness of the identity element within the Rubik's Group R.

Despite the commutative property holding true between movements on opposite faces, the same does not apply to movements on adjacent faces. Movements UR and RU are not equal, as shown in Figure 3:

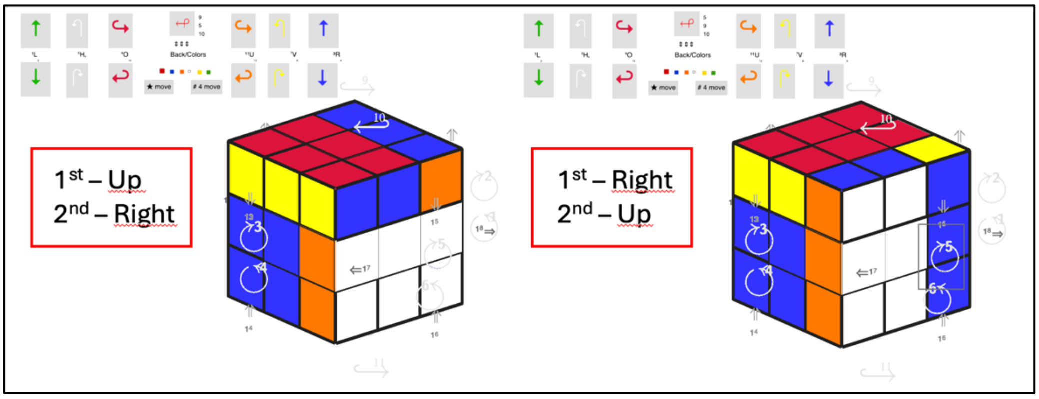

FIGURE 3: UR and RU sequences and their differences.
FONTE: Prepared by the authors.

This leads us to conclude that the Rubik's Group R is not abelian, making the cube, in particular, a challenging puzzle to solve.

4. Permutation groups, Rubik's group, and GeoGebra

In the study of Group Theory, particularly permutation groups, we can observe that there are relationships or combinations that can be visualized concretely, which facilitates the understanding of the subject (Zazkis et al., 1996).

The possibility of visualizing and manipulating this type of group using software like GeoGebra allows for understanding its relationships and properties, identifying symmetries, and grasping possible applications of the subject (Sousa; Alves; Aires, 2024).

According to Joyner (1997), a permutation group is defined as a set of all products of a finite set of elements within a symmetric group S_X. For the purposes of this study, we focus on how this property can be visualized through the manipulations of the Rubik's Cube in GeoGebra, without delving into the formal proof.

Using GeoGebra, we can observe the properties of permutation groups by performing sequences of movements on the Rubik's Cube. For instance, applying the sequence ($URU'R'$) repeatedly in GeoGebra allows users to visualize how the cube eventually returns to its original state, thereby demonstrating the cyclic nature of permutations. In Figure 4 we have an example:

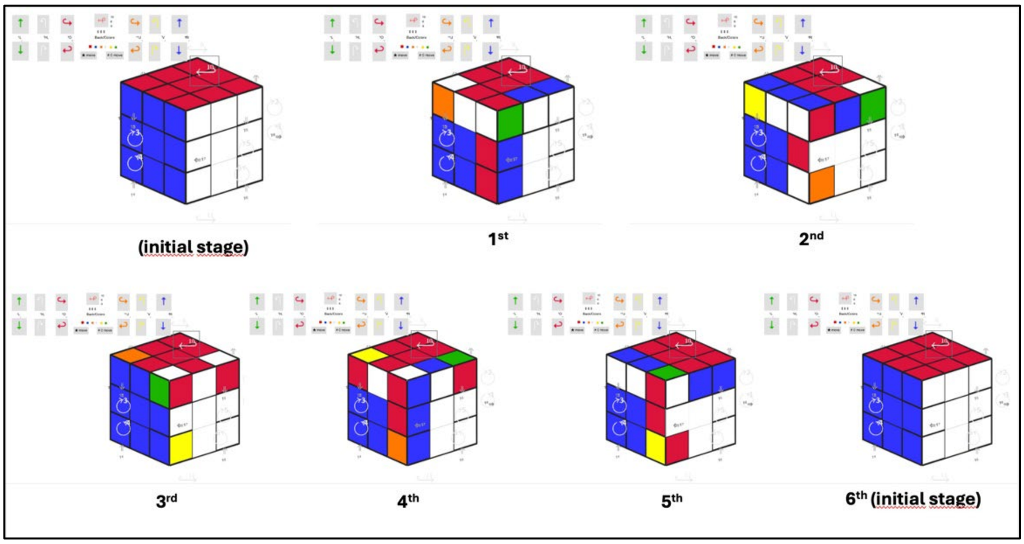

FIGURE 4: $URU'R'$ sequence and the cyclic nature of permutations.
FONTE: Prepared by the authors.

The cube starts in its original configuration. It's important to track specific pieces (especially corner pieces) to observe how they are affected by the sequence of moves. Let's understand the steps of the sequence $URU'R'$ (Table 3):

1st - Move U (Up)	**Meaning**: Rotate the top face of the cube 90 degrees clockwise. **Effect on the Cube**: This move shifts all pieces on the top layer to the right, affecting both corner and edge pieces. It changes their position and orientation on the top layer, setting up the cube for the subsequent moves.
2nd - Move R (Right)	**Meaning**: Rotate the right face of the cube 90 degrees clockwise. **Effect on the Cube**: The right face rotation moves three corner pieces and three edge pieces on the right face to new positions, creating a change in their arrangement. This move will be reversed later in the sequence.
3rd - Move U' (Up inverse)	**Meaning**: Rotate the top face of the cube 90 degrees counterclockwise. **Effect on the Cube**: This move partially reverses the effect of the initial U move, returning the top face to nearly its original orientation. However, since the right face was rotated, the corner and edge pieces are now in new positions that reflect both the U and R rotations.
4th – Move R' (Right inverse)	**Meaning**: Rotate the right face of the cube 90 degrees counterclockwise. **Effect on the Cube**: This move reverses the earlier R rotation, but because the top face was rotated after that R move, the pieces do not return to their initial positions. The right face rotation moves the pieces back in a counterclockwise direction.

Table 3. Steps of the sequence $URU'R'$.

Let's understand the cycle: The sequence $URU'R'$ forms a cycle primarily among the corner pieces. It moves three of the corner pieces around in a clockwise cycle. Repeating the sequence multiple times will eventually return the cube to its original state, illustrating the concept of a permutation cycle (six repetitions are usually needed to return to the initial state).

When the sequence $URU'R'$ is performed, it creates a cycle of three corner pieces. Tracking the movement of these corners across multiple repetitions of the sequence makes it possible to see how they permute cyclically. By the time the sequence is repeated six times, the cube's corner pieces will return to their original positions, while the middle layer pieces might follow different patterns.

This sequence is a practical demonstration of permutation cycles, as it swaps the positions of specific corner pieces while leaving others unchanged. Repeatedly performing this sequence shows how pieces move through smaller cycles, revealing the cyclic nature of permutations within the cube. In GeoGebra, the sequence can be visualized by setting up buttons or tools to execute each rotation (U, R, U', R') sequentially. This interactive visualization helps students understand how the sequence leads to a cyclic permutation of the cube's corner pieces, enhancing the understanding of group properties and cycles.

The permutation group of a finite set X, denoted by S_X, consists of all possible permutations of X. The order of this group is $n!$, where n is the number of elements in X, and $n!$ represents the factorial of n, which is the product of all integers from 1 to n (Garcia & Lequain, 2002). With the notion of Permutation Groups, it is possible to establish isomorphisms between Symmetric Groups and Rubik's Subgroups. This combination of these two concepts allows the analysis and creation of sequences of movements to solve the cube.

Expanding on the concepts related to the symmetry of the Rubik's Cube, we have that the set of all permutations of Z_n is denoted by S_n and is called the symmetric group with n letters (Schültzer, 2005). When solving the Rubik's Cube, users often perform various movements of the following type: make a move, say M_1, then another move M_2, and then make the inverse of the first move, ${M_1}^{-1}$. For example, the sequence $(R^{-1}D^2RB^{-1}U^2B)^2$ consists of such movements. This movement is a twist of two corners: the *URF* corner is rotated once clockwise, and the *BLD* corner is rotated once counterclockwise.

To understand how to calculate the number of movements or sequences of movements of the cube until an identity is obtained, it is necessary to comprehend the concepts of the order of a group, cycles, and cycle product. Given the brevity of the manuscript, we will use Proposition 1 in an abbreviated manner (Garcia & Lequain, 2002):

Proposition 1. Let μ_1, ..., $\mu_n \in S_n$ be disjoint cycles of lengths l_1, ... l_n, respectively. The order of the product $\mu_n \ldots \mu_1$ has an order equal to the least common multiple of (l_1,... l_n).

Example: Let μ = (1 2)(3 4 5) be the disjoint product of a 2-cycle and a 3-cycle. Since the order of (1 2) is 2 and the order of (3 4 5) is 3, it follows that $O(\mu) = lcm(2, 3) = O(\mu) = 6$.

In the case of the Rubik's Cube, consider the following example: Let the sequence $S = F^4$. By repeating the F movement 4 times, we return to the identity, as shown in Figure 5:

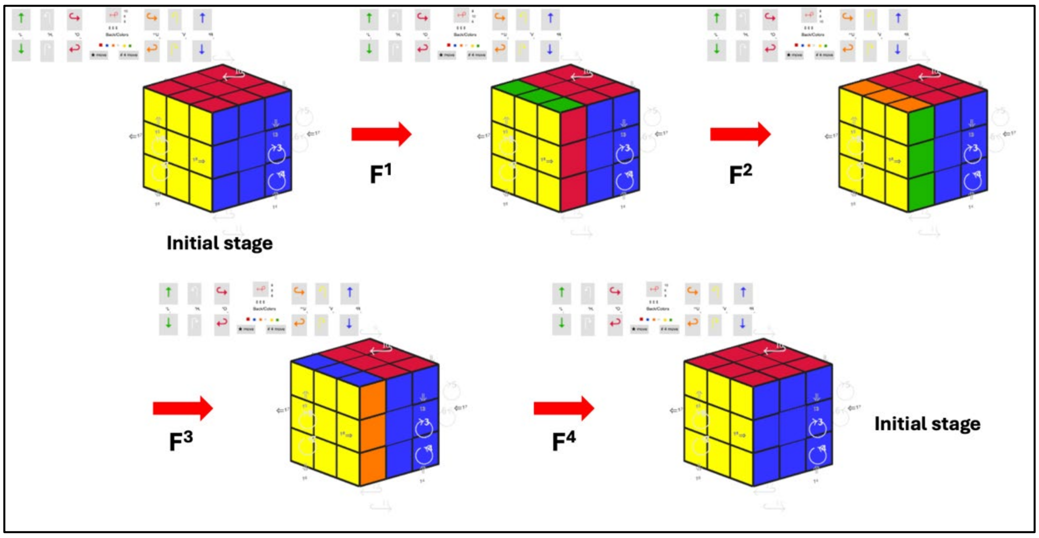

FIGURE 5: Sequence $S = F^4$.
SOURCE: Prepared by the authors.

In this case, the order of S is 4, i.e., $O(S) = 4$. All movements (UU^{-1} $DD^{-1}RR^{-1}LL^{-1}FF^{-1}BB^{-1}$) of the faces, both clockwise and counterclockwise, are of order 4.

It is possible to calculate the order of a sequence of movements, but to arrive at this value, we need to observe on the Rubik's Cube itself which small cubes will be affected by the change in position caused by this sequence of movements. By executing the movement sequence one or more times, we notice that all the smaller cubes that move occupy only certain positions, meaning that during the permutation, they are not all interchanged with each other. Thus, it can be observed that these permutations occur in groups that form a cycle, where only the small cubes belonging to the same cycle are permuted among themselves.

The possibility of exploring the Rubik's Cube in GeoGebra can provide a better understanding of permutation groups and other related topics. In a non-exhaustive manner, we can illustrate with the construction shown in Figure 6, based on a publication in the Spiegel magazine (1981) and on the Rubik's Cube homepage[4] itself:

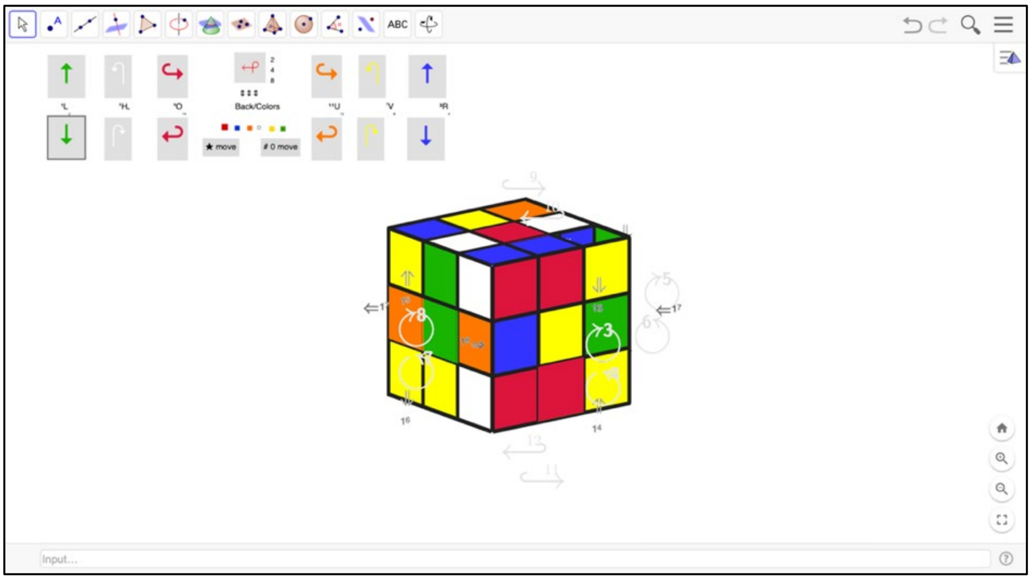

FIGURE 6: Construction of the Rubik's cube in GeoGebra.
SOURCE: Prepared by the authors.

Let's understand this construction: Each face of the Rubik's Cube has buttons in GeoGebra that perform a 90-degree rotation either clockwise or counterclockwise. For example, pressing the F button rotates the front face of the cube clockwise, while F' rotates the same face counterclockwise. These movements change the positions and orientations of the pieces (edges and corners) of the cube.

When a button is clicked, the corresponding layer of the cube moves, simulating a real rotation. This allows the user to see how the pieces are rearranged and understand that each movement is a permutation, swapping the positions of the pieces

Each individual movement performed by a GeoGebra button is a permutation of the cube's pieces, altering their positions and orientations. For example, clicking

[4] Homepage: https://rubiks.com/en-US/

on the R button (right face movement) performs a permutation that swaps the pieces on the right layer and its adjacent layers, modifying their places according to the rotation.

GeoGebra also allows the creation of buttons that execute specific sequences of movements, such as $F\ R\ U\ R'\ U'\ F'$, which is a simple sequence to swap two adjacent edges of the cube. Being able to visualize a complete sequence of movements helps to understand how permutations can form cycles or transpositions, fundamental concepts in permutation groups.

A summary of the construction movements in GeoGebra is presented in Table 4. Each movement can be intuitively understood in the construction, based on the arrows indicating the direction of the movements.

Movement	*Symbol*	*Description*	*Permutation*
Front Clockwise	F	Rotates the front face 90 degrees clockwise	Permutes the pieces on the front face
Front Counterclockwise	F'	Rotates the front face 90 degrees counterclockwise	Reverses the permutation made by F
Right Clockwise	R	Rotates the right face 90 degrees clockwise	Permutes the pieces on the right face
Right Counterclockwise	R'	Rotates the right face 90 degrees counterclockwise	Reverses the permutation made by R
Upper Clockwise	U	Rotates the top face 90 degrees clockwise	Permutes the pieces on the top face
Upper Counterclockwise	U'	Rotates the top face 90 degrees counterclockwise	Reverses the permutation made by U
Left Clockwise	L	Rotates the left face 90 degrees clockwise	Permutes the pieces on the left face
Left Counterclockwise	L'	Rotates the left face 90 degrees counterclockwise	Reverses the permutation made by L
Down Clockwise	D	Rotates the bottom face 90 degrees clockwise	Permutes the pieces on the bottom face
Down Counterclockwise	D'	Rotates the bottom face 90 degrees counterclockwise	Reverses the permutation made by D
Back Clockwise	B	Rotates the back face 90 degrees clockwise	Permutes the pieces on the back face
Back Counterclockwise	B'	Rotates the back face 90 degrees counterclockwise	Reverses the permutation made by B

Table 4. Table of Movements and Functions.

The movements in Table 4 correspond to the buttons implemented in GeoGebra, which allow users to visualize how each rotation affects the cube's structure. In the foreground, the Rubik's Cube is made up of 27 smaller individual cubes, which together form a 3 × 3 × 3 cube. However, in the real situation, we only have 21 moving parts, namely 1 axis system (with 6 single-color fixed center pieces), 8 three-color corner pieces and 12 two-color border pieces. Spatial movement with the software allows for better understanding (Figures 7 and 8):

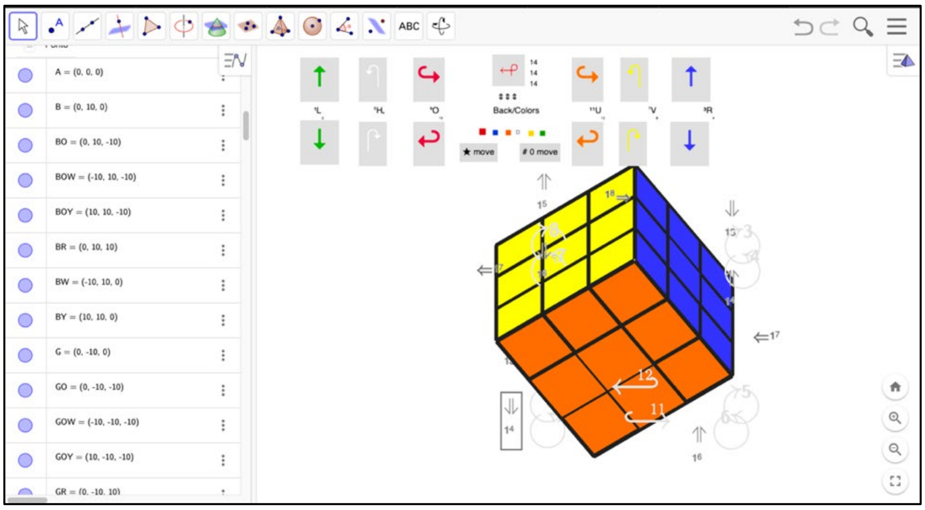

FIGURE 7: Rubik's Cube in GeoGebra.
SOURCE: Prepared by the authors.

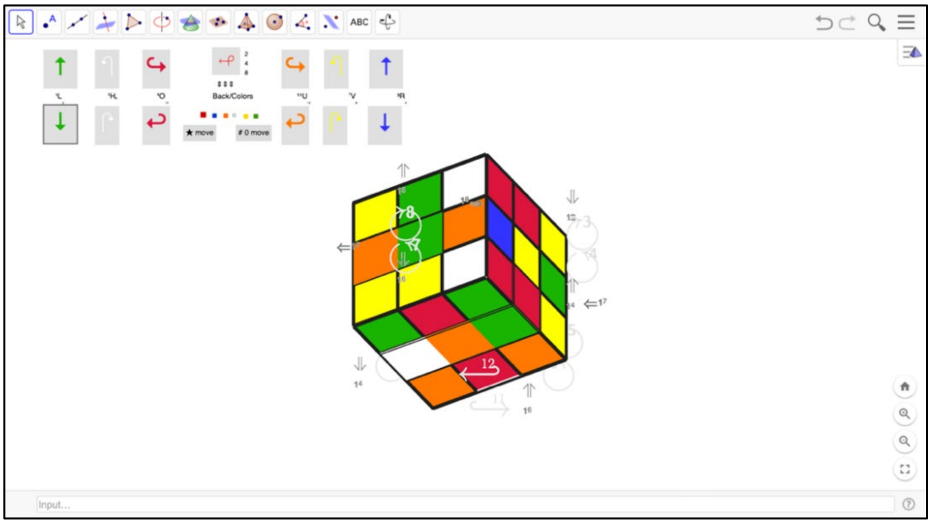

FIGURE 8: Shuffle of the smaller faces of the Rubik's Cube.
SOURCE: Prepared by the authors.

The buttons with arrows are intuitive and indicate the color to be moved and the direction of the movement. The ‘back/colors’ button performs the inverse movements, and the ‘move’ buttons move the corners and centers as many times as they are pressed, following the directions indicated by the arrows.

For mathematical understanding through subgroups of the Rubik’s Group, formed by sequences of steps and movements, and the development of a strategy that uses the least possible memorization, i.e., types of movements to be performed, concepts of commutators and conjugates are employed. This method involves few combinations of sequences, which, in turn, are intuitive. The use of these concepts is commonly found in more sophisticated solving methods, where the goal is to optimize the number of movements required to solve the cube (Bandelow, 1982; Travis, 2007; Joyner, 2008).

A commutator is a sequence of movements that has the net effect of performing a single specific operation on the cube, consisting of three parts: (i) an operation (X), a second operation (Y), and the inverse of the first (X'). The notation for a commutator is $[X, Y] = XYX'Y'$. In a simplified way, a commutator is a way to swap two pieces or sets of pieces without affecting the rest of the cube.

On the other hand, the conjugate is related to using an algorithm X to change the state of a piece, applying X, followed by an operation Y, and finally the inverse of X (X'). The result is a new algorithm that performs the same operation in a different position. Formally, if X is an algorithm (sequence of movements) and Y is another algorithm, then the conjugate of X by Y is given by YXY'. Thus, for solving the Rubik's Cube, conjugates are used to apply algorithms in specific situations, altering the orientation or position of pieces without affecting other parts of the cube.

This model, at first, can be explored in the software or even with the physical object, seeking to understand the mathematical concepts associated with the movements, as presented in Table 5:

Step	**Objective**	**Procedure**
1	Form a cross on one of the faces.	Performed intuitively.
2	Arrange the 4 edge cubes, in the median layer, of the faces adjacent to the one chosen in the 1st step.	$U^{-1}F^{-1}UF$ *or* $UFU^{-1}F^{-1}$
3	Arrange the cubes on the opposite side of the face chosen in the 1st step, forming a cross on all six faces of the Rubik's cube.	$(RU^{-1}R^{-1}U)(F^{-1}UFU^{-1})$ *or* $(U^{-1}F^{-1})(U^{-1}F)(U^{-1}F^{-1})(U^{-1}F)U^{-1}$
4	Position all 8 corner cubes correctly	$R^{-1}(ULU^{-1})R(UL^{-1}U^{-1})$ *or* $(ULU^{-1})R^{-1}(UL^{-1}U^{-1})R$

5	Orient all 8 corner cubes correctly.	$(LD^2\ LF^{-1}D^2F)U(F^{-1}D^2FLD^2L^{-1})U^{-1}$ *or* $(F^{-1}D^2FLD^2L^{-1})U^{-1}(LD^2L^{-1}F^{-1}D^2F)U$

Table 5. Solution strategy with steps, objectives, and necessary sequences.

However, in each stage, it will generally be necessary to repeat each sequence several times until the desired result is achieved, making the method slow and involving many movements. Despite this, its repetition allows for a better understanding of the combinations of movements and logic associated with the executed sequences. The expected final result can be illustrated in the software, as shown in Figure 9:

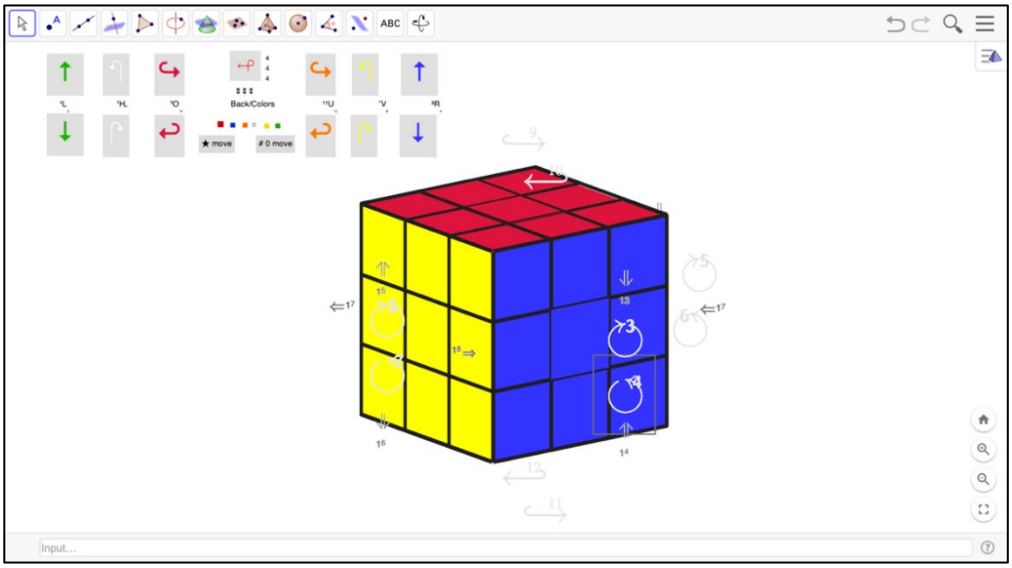

FIGURE 9: Rubik's Cube solved in GeoGebra.
SOURCE: Prepared by the authors.

There are many other ways to bring the cube back to its original position, and these are proposed by various authors, which can be explored using the software, considering its mathematical characteristics. Such strategies can be discussed when exploring more complex concepts, such as permutation groups and the particular case of the Rubik's Group. However, given the brevity of our manuscript, we provide access to the construction as a study proposal for enthusiasts of this toy, available at: https://www.geogebra.org/m/ft4dwxvb.

Final considerations

The mathematics of the Rubik's Cube involves understanding combinatorial ideas such as permutations and counting arguments. Group Theory is crucial for analyzing and comprehending the mathematical properties of the Rubik's Cube, as each combination of movements forms a group, and the study of these groups enables an understanding of their structural properties, such as the ability to solve any configuration using a specific sequence of movements.

GeoGebra provides a dynamic platform for understanding the group structure of the Rubik's Cube, making abstract algebraic concepts tangible and interactive. This tool can play a crucial role in enhancing students' understanding of permutation groups by offering a hands-on approach to exploring the cube's movements.

Permutation groups are applied in the development of efficient solving methods for the Rubik's Cube, such as the Layer-by-Layer Method (or CFOP method - Cross, F2L, OLL, PLL), which utilizes the mathematical principles of permutation groups to simplify the solving process of the Cube. Thus, the relationship between the Rubik's Cube and permutation groups lies in the mathematical representation of the cube's configurations as permutations and the analysis of the groups generated by possible movements.

Visualization plays a significant role in the mathematical understanding of the Rubik's Cube. Some of its contributions include enhancing intuition and spatial understanding, recognizing patterns and symmetries, applying abstract theoretical concepts in a practical manner, and optimizing problem-solving abilities. In the case of the Rubik's Cube, technology can enhance this understanding through virtual simulation, modeling, and graphical representation of interactive features.

The use of GeoGebra can provide visual and algebraic support, offering a differentiated approach to understanding the topic. It is possible to explore graphical representation, movement animations, permutation matrices, and the study of group theory in abstract algebra with the software. The visualization possibilities with the software and the exploration of its interface to comprehend the construction and permutations of cube movements can be an approach for mathematics education instructors, utilizing a tangible object and relating this practical example to real-world applications.

Acknowledgements

The second author is grateful for the encouragement and financial contribution of the National Council for Scientific and Technological Development (CNPq) for the development of this research in Brazil. The third author is grateful for the funding provided by National Funds through FCT – Foundation for Science and Technology, I.P., under the project UIDB/00194/2020 (CIDTFF).

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Enviado:25/01/2024 - Aceito: 26/09/2024