Measuring the invisible: a process among arithmetic, geometry and music

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Abstract

Studies establishing a parallel between music and mathematics abounded in the 16thcentury. Relative to proportions, theorists considered the audible and visible proportions to be analogous. However, upon approaching the corresponding treatises by focusing on how mensuration was performed, we noticed differences in the very notion of quantifying. In this paper we sought to identify the role arithmetic and geometry played in the musical tradition through the notion of quantification. For that purpose we took as point of departure the definition of the musical interval given by the most important theorist of the 16th century, Gioseffo Zarlino, and then sought to identify the core of the debates on music held within the realm of mathematics, as particularly exemplified by the case of Vincenzo Galilei.

Keywords

Mathematics; Music; Geometry; Arithmetic; Measuring

Medindo o invisível: um processo entre a artimética, a geometria e a música

Resumo

Estudos que estabeleciam o paralelo entre a música e a matemática eram abundantes no século XVI. No que diz respeito às proporções, podemos dizer que os teóricos admitiam que as proporções audíveis e visíveis eram análogas. Contudo, quando examinamos os tratados e neles analisamos os processos de mensuração, notamos existir diferenças na própria noção de quantificação. Neste artigo, procuramos identificar os papéis da aritmética e da geometria na tradição musical através da noção de quantificação, tendo por foco a definição do intervalo musical atribuída pelo teórico mais importante do século XVI, Gioseffo Zarlino, e apontar para os debates ligados à música no contexto matemático.

Palavras-chave

Matemática; Música; Geometria; Aritmética; Mensuração

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Introduction

Gioseffo Zarlino (1517–1590) — probably the most famous Italian music theorist of the late Renaissance - provided a number of definitions for the musical interval in his works. In his first treatise, *Le Istitutione Harmoniche*(1558),[1] he defined an interval as one of the musical elements attributed to nature together with the low and the high.[2] In his second treatise, *Dimostratione Harmoniche*,[3] published in 1571, he defined interval as the distance between the low and the high that was expressed by a ratio and stated that such distance could be known by measuring the bodies that produced the intervals.[4]

It is important to note that in *Istitutione,* Zarlino defined the interval not in relation to the sonorous body (*corpo sonoro*), but according to the distinction between common and proper intervals as advocated by Aristides Quintilianus (fl. late 3rd–early 4th century A.D.).[5] Zarlino explains that the common interval, the definition of which implied the notion of magnitude, is a space between two limited ends, but that it is not his focus of interest; rather, he will consider the proper interval, or distance between low and high sounds:

> "[...] the interval, which is attributed to nature, can be named, according to Aristides Quintilianus, in two different ways, either as common or proper. The interval is named common, given that a magnitude is limited by two ends and is therefore an interval; although such definition considers the space between the two extremes, I do not intend to talk about this because it is not under our consideration. The interval is named proper because the distance between the low and the high sounds is called an interval and that is under the musician's consideration."[6]

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[1] Gioseffo Zarlino, *Le Istitutione Harmoniche* (Venezia: Francesco de' Franceschi Senese, 1558).

[2] "[... elementi]. Quelle che si attribuiscono alla natura sono l'acuto, il grave & lo intervalo"; Ibid., 81.

[3] Gioseffo Zarlino, *Dimostratione Harmoniche* (Venezia: Francesco de' Franceschi Senese, 1571).

[4] "[...] ma perche ogni intervalli musicali ha distanza, che si trova tra il suono grave & acuto: la quale senza dubio cade sotto alcuna proportione: però volendo i musici havere la ragione de tale distanza: non hanno ritrovato meglior mezzo, quanto la misura de i nominate corpi dalli quali nascono i suoni."; Ibid., Ragionamenti I, definizione iii, 22.

[5] According to Andrew Barker, the Greek author Quintilianus cannot be earlier than the 1st century A.D. nor later than the 4th; Andrew Barker, "Harmonic and Acoustic Theory," in *Greek Musical Writings* (Cambridge: Cambridge University Press, 1989), 392. Quintilianus wrote the treatise *De musica* (*Perì musike*); see Quintilianus, *De musica libri III*, in *Antique musicae auctores septem: Graece et latine*, ed. Marcus Meibom (Amsterdam, Apud Ludovicus Elzevirium, 1652).

[6] "[...] lo intervallo adunque, il quale si attribuisce alla natura, si chiama in due modi, come vuole Aristide Quintiliano, cioè commune, et proprio. Si dice commune; conciosia che ogni grandezza terminata da certi fini, è detta intervallo; considerando però il spatio, che si ritrova tra l'uno & l'altro estremo; & di questo non

Unlike *Istitutione*, *Dimostratione* aimed at demonstrating a close relationship between the abstract mathematical calculus of musical intervals and their application to music instruments.[7] In *Dimostratione*, Zarlino explains what a sonorous body (*corpo sonoro*) is, namely, "everything that enabled sound production"[8], and then continues:

> "[...] the musician (as you recall sir) proceeds by making and obtaining results of his causes from the whole and the part of the sonorous body, be it a string, or anything else that is often utilized, that is divisible into infinity; then, by understanding and placing the larger number (and not the smaller) of any ratio in such an order, by the whole of the sonorous body, I divide it into so many parts [...][9]

In the quotation above the relationship among ratio, intervals and *corpo sonoro* is explicit and presents the *corpo sonoro* in two different ways: first, the *corpo* is approached as a geometrical object "which is divisible into infinity", and then it is considered as a material object capable of being divided "into so many parts". The ease with which Zarlino mixes both divisions cannot be underestimated, because a real instrument cannot be divided as a geometrical one can. One might infer from such a procedure that Zarlino classified music as a geometric, rather than an arithmetic science, which fact is rather curious.

It is curious, indeed, because up to the 16th century music was mostly understood as an arithmetical science, as transmitted through the works of Anicius Manlius Severinus Boethius (470/5?–524?) and Aristotle (384 -322 B.C.).[10] Zarlino, following both, defined music as a science of the *quadrivium* that was subordinate to arithmetic. In *Istitutione*, chapter 20, Zarlino explains that the secondary, or subordinate, sciences - such as *perspectiva* and music - took their first principles from the principal sciences,

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intendo io parlare, perche é molto lontano dalla nostra consideratione. Si chiama proprio, perche la distanza, che è dal suono grave all'acuto, è detta intervallo, & questo è considerato dal musico."; Zarlino, *Istitutione*, Part II, 15: 81-2.

[7] Zarlino, *Dimostratione*, Proemio.

[8] Ibid., Rag. I, def. iii, 22.

[9] "[...] il musico [...] va facendo e cavando le sue ragioni dal tutto e dale parti fatte del corpo sonoro: sia poi corda, o qual si voglia oltra cosa, che torni al proposito: il qual corpo è divisibile in infinito"; Ibid., Rag. I, def. iii, 55-6; English translation in John E. Keheller, "Zarlino's Dimostrationi Harmoniche and Demonstrative Methodologies in the Sixteenth Century (PhD dissertation, Columbia University, 1993), 102.

[10] Severius N. Boethius, *Institutio Arithmetica* 1,1,8, in *Boethian Number Theory: A Translation of the De Institutione Arithmetica*, ed. & transl. Michael Masi (Amsterdam: Editions Rodopi B.V., 1983); Boethius, *Fundamentals of Music*, transl., intr., & notes Calvin Bower (New Haven: Yale University Press, 1989); Aristotle, *Posterior Ana*lytics I, 13, 79a5-6, in *Great Books of the Western World* Vol. 8. *Aristotle I*, transl. G.R.G. Mure (Chicago: Encyclopaedia Britannica, 1952) vol. 1, 95-137, on 108.

geometry and arithmetic, respectively.[11] He further explains the musical structures based on Aristotle's four causes:

> "[…] the agent, that is, the musician, who is called the efficient cause; the material, which are the strings, also called the material cause; and the form, that is, the proportion, called the formal cause; nevertheless, the last two are intrinsic causes of something, and the agent and the goal are its extrinsic causes."[12]

Nevertheless, contrary to 'the philosopher '(i.e., Aristotle), Zarlino believed that music was more mathematical than natural.[13] Regarding the placement of the subordinated sciences, it is important to consider that the status of, and types of definition for the subordinated sciences during the Middle Ages were attributable to the commentaries produced on Aristotle's works; both music and optics continued to be considered complex mathematical disciplines during the Renaissance.[14]

The Aristotelians distinguished the mathematical from the natural knowledge by advising mathematicians to separate, through thought, the forms found in nature that addressed abstract and immobile beings, in contrast to the natural philosophers, who were to directly concern themselves with the objects of *physis* which were subjected to generation and corruption.[15] However, in *Posterior Analytics,* Aristotle defined the

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[11] Zarlino, *Istitutione*, Part I, 20:30.

[12] Ibid., Part I, 41: 54; English translation in Lucille Corwin, "Le Istitutioni Harmoniche of Gioseffo Zarlino, Part I. A Translation with Introduction" (PhD dissertation, City University of New York, 2008), 486.

[13] Zarlino, *Istitutione*, Part I, 20: 31.

[14] See Fumikazu Saito, *O Telescópio na Magia Natural de Giambattista della Porta* (São Paulo: Livraria da Física; Educ; FAPESP, 2011), 73-172; Alistair C. Crombie, *Science, Optics and Music in Medieval and Early Modern Thought* (London: The Hambledon Press, 1990); David C. Lindberg, *Theories of Vision from Al-Kindi to Kepler* (Chicago: The University of Chicago Press, 1976); David C. Lindberg, "The Science of Optics," in *Science in the Middle Ages*, ed. D.C. Lindberg (Chicago: The University of Chicago Press, 1978), 338-68, on 340; David C. Lindberg, "Optics in Sixteenth-Century Italy," in *Novità Celesti e Crisi del Sapere*, ed. P. Galluzzi (Firenze: Giunti Barbera, 1984), 131-48; Filippo Camerota, "Renaissance Descriptive Geometry: The Codification of Drawing Methods," in *Picture Machines, 1400-1700*, ed. W. Lefèvre (Cambridge, MA: The MIT Press, 2004), 175-208; Filippo Camerota, "Misurare 'per perspectiva': Geometria pratica e *Prospectiva Pingendi*," in *La prospettiva: Fondamenti teorici ed esperienze figurative dall'antichità al mondo moderno*, ed. R. Sinisgalli (Firenze: Edizioni Cadmo, 1998), 293-308; Graziella F. Vescovini, "L'inserimento della 'perspectiva' tra le arti del quadrivio," in *Arts Liberaux et Philosophie au Moyen Age. Actes du IV^e Congrès International de Philosophie Médiévale. Université de Montreal, 27/08-02/09, 1967*, ed. Institut d'études médievales (Paris: J. Vrin, 1969), 969-74; Jean Gagné, "" in *Arts Libéraux et Philosophie au moyen âge. Actes du IV^e Congrès international de Philosophie médiévale. Université de Montreal, 27/08-02/09, 1967*, ed. Institut d'études médiévales (Paris: J. Vrin, 1969), 975-86.

[15] Aristotle, *Physics* II, 2, in *Works of Aristotle*, 259-355, on 270-1.

subordinated sciences - such as music and optics - in both ways.[16] Those definitions led medieval and renaissance scholars to consider the two aspects, i.e., the mathematical and the physical, in different ways, while still maintaining a relationship of subordination between them.

Therefore it is within the aforementioned definitions of subordinated sciences that the mathematical and the natural elements of music have to be understood when addressing Zarlino`s statements. However, one must keep in mind that although both arithmetic and geometry were then mathematic disciplines, they were two independent sciences, and their respective role in music has to be correctly identified.

Therefore, based on new historiographical approaches,[17] in this paper we sought to explore possible relationships among arithmetic, geometry and music. We began by the 16th-century definition of musical interval and then sought to identify the core of Vicenzo Galilei's (1520?-1591) criticism against Zarlino within the realm of mathematics.[18]

Notions of proportion or the part and the whole

The definition of music as an arithmetic science was based on the relationship between the music interval and the notion of ratio. Although 16th-century readers might have been familiar with the Pythagorean doctrine of the musical ratios, attributing a magnitude to musical intervals was not a simple task. Whereas in music proportion can be easily grasped in the quantitative differences in note duration that entails the musical rhythm, proportional ratios cannot be observed in sheet music as rhythm can. It also cannot be seen as in sculpture, in which one sees the proportional relationship between the parts and between the parts and the whole.

[16] Ibid., I; 7; 9; 13. See also: Edward Hussey, "Aristotle and Mathematics," in *Science and Mathematics in Ancient Greek Culture*, ed. C.J. Tuplin, & T.E. Rihll (Oxford: Oxford University Press, 2002), 217-29.
[17] See Carla Bromberg, "The Mathematical Status of Architecture and Music in the Works of Daniele Barbaro", paper presented at SCIENTIAE 2014, University of Warwick, UK; Carla Bromberg, *Vincenzo Galilei contra o Número Sonoro* (São Paulo: Livraria da Física; Educ; FAPESP, 2011); Fumikazu Saito, "History of Mathematics and History of Science: Some Remarks Concerning Contextual Framework", *Educação Matemática Pesquisa* 14/3 (2012): 363-85; Árpád Szabó, *The Beginnings of Greek Mathematic* (Dordrecht: D. Reidel, 1978); see also Keheller.
[18] Bromberg, *Vincenzo Galilei;* see also Keheller.

Here we assume that magnitude enables qualitative and quantitative descriptions of concepts through numbers, that is to say, magnitudes can be reduced to numbers. However, since measuring involves comparison, the magnitude to be measured must be material.

How were authors able to transpose [or transform] an audible record (numerical) into a visual one (geometrical)? What did it mean in the 16th century to express a numerical proportion (an abstract entity) as a string or part of a string (a concrete and material entity)?

In this sense, according to Ptolemy (d. 168 A.D.) both hearing and seeing needed "some method derived from reason, to address the things that they are not naturally capable of judging accurately"[19]. In music history, the most important method to render sounds visible was through an instrument called *kanōn*, or monochord, which was known since ancient times. Through it musical relationships became quantifiable, and special proportions appeared to underlie special intervals or concords. As a function of their mathematical properties, the concords, which are accessible to perception only as qualities, became included within the realm of arithmetic.[20]

As mentioned above, the derivation of music theory from arithmetical proportionality was traditionally transmitted through Boethius' works,[21] and it was by far the strongest music theory during the 16th century. As a result, music – an arithmetic-based science — developed its elements and conceptions according to the notions and operations of arithmetic.

However, also geometry participated in music, particularly concerning the role played by the monochord, which as an instrument could approximate geometry, then a less abstract science than arithmetic, to physics. In this respect, Á. Szabó, when comparing the *Sectio canonis* (a treatise misattributed to Euclid) to Euclid's *Elements*, described two different Greek traditions involved in the division of the *kanōn*. While one tradition placed the whole and its divisions on a same string, the other used more than one string, placing the whole on one string and its parts on another. As a result Szabó identified the origin of the analogy between numbers and segments, as it appears in Euclid's *Elements*,

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[19] Ptolemy, "Harmonics", Book I, 5.10, in Barker, *Greek Musical Writings*, II: 270-391, 278.

[20] David Creese, *The Monochord in Ancient Greek Harmonic Science* (Cambridge: Cambridge University Press, 2010), 4.

[21] However, there also was a second trend, namely, the one of authors who understood music primarily as a harmonic study and thus paired it with astronomy rather than with arithmetic; see, e.g., Martianus Capella, *Liber De Nuptiis Mercurii et Philologiae*, as cited in Boethius, *Boethian Number Theory*, 13.

in the Greek practice of the *kanōn*.[22] According to him, a musical interval could be represented by the term *diastema,* which had two meanings within music theory. On the one hand, it was analogous to a "musical interval" and to the "ratio between numbers which expressed this interval",[23] and on the other, it also meant "line segment" or the "distance between two points". Some authors agreed and others disagreed with Szabó's definitions. Creese and Barker agreed that in the *Elements* term *diastema* was used to indicate 'distance' between points in geometrical constructions.[24] However, Barker concluded that currently one became accustomed "to speak of notes as higher or lower, as if they were placed as points on a vertical continuum", but because "they could be measured and compared like distances along a line, this way of depicting the phenomena, [...] although convenient […] was entirely metaphorical."[25]

As mentioned above, *diastema* meant both the straight line and the numerical ratio of a musical interval. The Greek word *diastema* translates as musical interval. That Barker interprets it in a metaphorical sense might be related to his choosing to interpret the interval as that which is bounded by two sounds with different tensions. This interpretation is based on his finding of the Greek words *oxys* and *barys* in places where he believed high and low could have been used; and since the Greek word for pitch is *tasis*, which literally means tension, the words sharp (*oxys*) and heavy (*barys*) made more sense to him.[26]

The idea of tension connected to intervals is apparent in the work of Aristoxenus of Tarento (fl. 335 B.C.), who said, "an interval is that which is bounded by two sounds having different tensions (i.e., pitches)", to the continue:

> "Thus, according to the basic concept, an interval manifests itself both as a difference in tension […] and as a space capable of taking in those tones that are higher than the lower pitch bounding the interval, and lower

[22] Árpád Szabó, *The Beginnings of Greek Mathematic* (Dordrecht: D. Reidel, 1978), 161-6.

[23] According to Szabó, 107, Euclid frequently used the term *diastema* in the latter sense, as it appears in his third postulate: "It is postulated that a circle can be drawn with any centre and any line segment".

[24] Creese, *Monochord in Ancient Greek*, 32. According to him, word *diastema* occurs thirty times in the *Elements*, exclusively in the geometrical books, specifically I-IV and XI-XII; Andrew Barker, *The Science of Harmonics in Classical Greece* (Cambridge: Cambridge University Press, 2007), 378.

[25] Ibid., 21.

[26] Ibid..

than the higher one [...] A difference in pitch, however, consists in being more or less taut."[27]

As is known, Porphyry (234–305), in his commentary on Ptolemy's theory of harmony, used interval (*diastema*) and numerical ratio (*logos*) as equivalent concepts within the terminology of the Pythagorean musical theory.[28] This use of the terms also appears in *Sectio canonis* and in other works by Porphyry.[29] In his commentary on Ptolemy's harmony, Porphyry explained that "[…] some call a numerical ratio between end points, *diastema*; these could be characterised in terms of their end points as λόγοι, as well as διαστήματα, namely, the fourth would be *epitritos* logos (4:3), the fifth would be *hemiolios* logos (3:2) and so on"[30].

According to Szabó, this quotation shows that the two concepts, musical interval (*diastema*) and numerical ratio (*logos*), were equivalent according to the Pythagoreans, and also because it clearly implies that end points could have functioned as end points of *diastemata* as well as of *horoi*.[31]

It appears that Szabó was attempting to defend the notion that the music interval was more geometrical than arithmetical. In fact, if we consider A and B to be numbers, each one representing a quantity, we may infer a proportion A::B between A and B (Figure 1):

[27] Aristoxenus of Tarento, *Die harmonischen Fragmente des Aristoxenos*, ed. P. Marquard (Berlin: Weidmann,1868), 20 *et seq*. See an English translation in Szabó, 112.
[28] Ibid., 113-4.
[29] Ibid., 114.
[30] Porphyrius, *Kommentar zur Harmonielehre des Ptolemaios*, ed. Ingemar Düring (Göteborg: Elanders Boktryckeri Aktiebolag, 1932), 94; 31 *et seq*., translated by Szabó, 114.
[31] Ibid. Szabó concludes that in addition, since the end points of the straight line were numbers on the *kanōn*, word *diastema* was also used to describe the relationship between two numbers exhibited by the proportional numbers of consonances (12:6, 12:9, 12:8 and so on). Normally, the *horoi* was shown as two numbers on the *kanōn*. *Diastema* in this sense was visible as a length of string in the *kanōn*, that is, the length by which the section of the string that produced the first sound differed from the one that produced the second. However, *diastema* was used only to designate concordant intervals — or consonances, which in turn were designated by the names of the involved strings, because the octave was known as the concord of the *hypate* and *nete*. Therefore, *diastema* may also have been construed as the concord produced by two different lengths of strings rather than a piece of string that did not vibrate; see Ibid.,125.

Figure 1. A and B as numbers

However, we might also consider an interval between two numbers, A and B, represented by a line AB. Accordingly, the relationship between A and B might expressed by its measure, or number (Figure 2):

Figure 2. Relationship between A and B expressed by its measure

In the second case, the geometric distance is identified as arithmetical entity, given that the visual geometrical segment is used to express an arithmetical argument. Nevertheless, it is important to note that this diagrammatic exposition of an arithmetic proportion does not make it a geometrical proposition, because arithmetic recognizes AB as a numerable quantity but not as a distance.

Therefore, it was possible to proceed as Zarlino suggested:

> "And because the differences that are found among low and high vocal and instrumental sounds are not known if not through sounding bodies, musicians, bearing this is mind, chose a string made of metal or another material that produced sound [...] Having the opinion that the

quantity of sound of the string was proportional to the number of its parts that was considered, [once] its length and quantity were known according to the number of its measured parts, they could immediately estimate the distances found between low and high sounds, or vice versa […]"[32]

Zarlino stated that it was possible to divide the string in a way that resulted in intervals equal to those estimated between the low and the high and vice versa. The citation above highlights two aspects. First, there is a correspondence between mathematics and physics, because the thing conceived either geometrically or arithmetically is related to the interval as a physical distance and vice versa. Second, and resulting from the first, it is possible to overlap the quantity that was found through a mathematical method onto the string.

Musical intervals as calculated in the monochord and intervals applied to real instruments

The division of the monochord and the explanations concerning the division of intervals can be found in nearly every music treatise of the Renaissance.[33] The technique and history of the monochord can also be found in a wealth of literature, from Cecil

[32] Zarlino, *Istitutioni*, in Corwin, 314-5; "Et perche le differenze, che si trovano tra le voci & tra i suoni gravi e acuti, non si conoscono, se non col'mezzo de i corpo sonori; però considerando Il Musico tal cosa, elessero una chorda, fatta di metallo o d'altra materia, che rendesse suono[...] esi havendo openione che tanto fusse la quantità del suono della chorda, quanto era Il numero delle parti considerato in essa, conosciuta la sua lunghezza e quantità secondo il numero dell sue parti misurate, subito potevano far giuditio delle distanze che si trovano esser tra gli suoni gravi & gli acuti [...]"; Zarlino, *Istitutione*, Part I, 19: 29.

[33] Adkins lists important authors clustered in groups according to their goals and techniques; just to mention a few: Ramos de Pareia, *Musica practica* (1482), ed. Johannes Wolf (Leipzig: Breitkopf und Härtel, 1901); Franchinno Gaffurio, *Theorica musicae* (1492), ed. facsimile (Roma: Reale Academia d'Italia, 1934); Andreas Ornitopharcus, *Micrologus*, trans. John Dowland (London: T. Adams, 1609); Henricus Grammateus, "Arithmetica applicirt oder gezogen auff die edel Kunst Musica", appendix to *Ayn new kunstlich Buech* (Nürnberg: Stuchs, 1518); Ludovico Fogliano, *Musica theorica*. (Venezia: Antonium et Sabio, 1529); Giovanni M. Lanfranco, *Scintille di musica* (Brescia: Lodovico Brittanico, 1533); Martin Agricola, *Musica instrumentalis deutsch* (Wittenberg: Georgen Thaw, 1545); Heinrich Glareano, *Dodecachordon* (Basel: Petri, 1547); Zarlino, *Istitutione*; Zarlino, *Dimostratione*; Francisco Salinas, *De musica libri septem* (Salamanca, 1577. Facsimile ed. Kassel: Bärenreiter, 1958); Vincenzo Galilei, *Dialogo di Vincentio Galilei nobile fiorentino della musica antica et della moderna*. (Firenze: Giorgio Marescotti, 1581); Wolfgang Figulus, *De musica practica* (Noribergae: Montani, 1565); Andreas Reinhard, *Monochordum* (Leipzig: Valentin, 1604); Abraham Bartolus, *Musica mathematica*, printed as the second part of Theatri Machinarum (Altenburg: Johann Meuschke, 1614); Fabio Colonna, *La sambuca lincea* (Naples: C. Vitale, 1618); Marin Mersenne, *Harmonie Universelle* (Paris: Cramoisy, 1635); Athanasius Kircher, *Musurgia universalis* (Roma: Francesco Corbelletti, 1650); Abdias Trew, *Lycei musici theorico-practici...explicatio tredecim divisionum monochordi.* (Rotenburgi: Jacobi Mollyni, 1635); Lemme Rossi, *Sistema musico* (Perugia: Laurenzi, 1666).

Adkins's pioneer work, a PhD dissertation devoted to its history,[34] to the most recent discussions on the monochord, such as the one published by David Creese in *The Monochord in Ancient Greek Harmonic Science*.[35] Although the manipulation and technique involved in finding intervals in the monochord have been largely addressed, their transposition to real instruments is rarely described in the musical treatises of the 16th century.

As we have mentioned above, Zarlino argued more than once for placing intervals in instruments and also for musicians to divide *corpus sonorus*. As did other authors,[36] he relied on the relationship between interval calculation and monochord divisions. Zarlino divided his *Istitutione* in four parts. In the second one he explains the division of the monochord according to his *senario* ratios, attempted to explain how to '*accomodare*' them in real instruments, and even also how to build an instrument in which such 'harmonies' could be found.[37] Still in this part of *Istitutione*, Zarlino presents an instrument called the *clavicembalo* (Figure 3).

Zarlino did not build the instrument; he commissioned it to the luthier Domenico Pesarese.[38] After providing the illustration, Zarlino explained that placing the intervals was simple, and that anyone who wished to know more about that science was invited to check the "difficulties" demonstrated in his *Dimostratione*.[39]

[34] Diatonic divisions based on superparticular proportions; the second includes the various methods of adding chromatic semitones to a diatonic division, and the third comprises divisions whose notes are determined mathematically in terms of string lengths. See Cecil Adkins, "The Technique of the Monochord," *Acta Musicologica* 39, no. 1/2 (1967): 34-43, on 39.
[35] See note #30.
[36] See note #33.
[37] Zarlino, *Istitutione*, 2. As mentioned in note #33, Adkins divided authors in four groups based on their techniques and goals and placed Zarlino and Galilei in different groups, although both groups tended to present the monochord as a means to aurally demonstrate the intervals of a tuning as a part of the general theoretical discussion. Galilei's group presented one single tuning, whereas Zarlino's group presented a number of them.
[38] Pesarese built the instrument in 1548; Zarlino, *Istitutione*, Part II, 47: 140.
[39] Ibid., Part II, 47: 141.

Figure 3. The *clavicembalo*[40]

Zarlino's illustration corresponded to a type of keyboard instrument, whose scale patterns were calculated with the monochord, which could play in diatonic, chromatic and enharmonic styles. Nevertheless, in *Dimostratione* no instructions are given as how to *accomodare* the intervals. Instead, Zarlino goes back to the division of the monochord and the construction of tetrachords according to those divisions (diatonic, chromatic and enharmonic).

Zarlino was convinced that his abstract rationalization of intervals could be directly applied to instruments: 'Now it is time to use hands, ruler and compass to accommodate the intervals we just talked about and their proportions on the sonorous body'[41].

In his *Dimostratione*, he explains that by placing the larger terms and larger ratio before the smaller terms and smaller ratio, the visual ordering of the natural places of intervals would be preserved, because "[…] the musician proceeds by making and

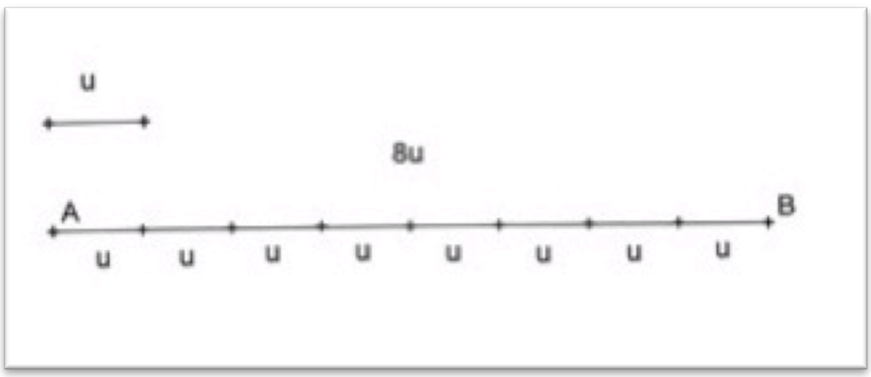
[40] Zarlino, *Istitutione*, Part II, 47: 141.

[41] "[...] hora fa bisogno […] di adoperare le mani, la riga & il compasso accomodando gli intervalli, de i qualli habbiamo parlato, alle proportioni loro sopra il corpo sonoro [...]"; Zarlino, *Dimostratione*, Rag. III, 146.

obtaining results of his causes from the whole and the part of the sonorous body, *be it a string or anything else often utilised*, which is divisible into infinity […]"[42].

In his definition of the *corpo sonoro* given in *Dimostratione*, Zarlino explained that since the interval is a distance between high and low sounds, which are found in proportion, musicians who want to know that distance have no better method of obtaining it than measuring the *corpi* out of which sound is made.[43] However, a number can quantitatively express a magnitude, and since measuring essentially means to compare two different magnitudes, there are instruments that can reduce magnitudes to numbers. For example, if we were to express the size of a segment AB considering a unit *u*, we will have (Figure 4):

Figure 4. A measure and its unit



This was the very core of the criticism Vincenzo Galileu raised against Zarlino's conception, as in last instance, the full issue could be circumscribed to the modes of measuring.[44]

According to Barker, there were in essence two systems of measuring; in one the intervals were understood as gaps between low and high sounds and could be 'imagined' as points strung out along a line that attempted to linearly represent the

[42] "[...] il musico va facendo e cavando le sue ragioni dal tutto e delle parti fatte del corpo sonoro: sia poi corda, o qual si voglia altra cosa, che torni al proposito: il qual corpo è divisibile in infinitio [...]"; ibid., Rag. I, 55; English translation in Keheller, 102.

[43] Zarlino, *Dimostratione*, Rag. I, def .iii, 22.

[44] See James A. Bennett, "The Challenge of Practical Mathematics," in *Science, Culture and Popular Belief in Renaissance Europe*, ed. S. Pumfrey, P.L. Rossi, & M. Slawinski (Manchester: Manchester University Press, 1991), 176-90; James A. Bennett, "Practical Geometry and Operative Knowledge," *Configurations* 6 (1998): 195-222; and James A. Bennett, "Knowing and Doing in the Sixteenth Century: What Were Instruments For?," *British Journal for the History of Science* 36 (2003): 129-50.

intervals in their lengths; the other aimed at representing them by ratios between speeds of movement. In the first case, but not in the second, identifying an auditory unit of measurement was necessary.[45] The common textual witness to the procedure by which the unit involved was accessible to hearing only is a well-known dialogue in Plato's *Republic* between Socrates and Glaucon.[46]

Aristotle described the unit as the smallest audible part called *diesis*, which sometimes equaled a *minor semitone*. This unit, if added as many times as desirable, could construct the different sizes of the musical intervals. His definition, it is important to remember, was based on his understanding of music as an arithmetical subordinated science. Aristotle had explained that while some authors prioritized the mathematical aspects of the study of *harmonia*, others prioritized the audible aspect. The former were called mathematical harmonicists, i.e., the ones who investigate harmonics according to numbers,[47] and were the only ones who possessed the demonstrations of causes.[48]

In the monochord, sometimes the unit was understood as the difference between two segments and sometimes it was the *aliquota*, namely, the minor part that could measure the whole taken as a unit. In *Compendio della Teorica della Musica,*[49] Galilei gave the following definition: "[an] aliquot is a part which is taken many times thus reintegrating the whole"[50].

In geometry the unit was not a number but a magnitude, which was variable. According to Euclid, *Elements* book V, it: "was a *part* of a magnitude, the less of the greater, when it measures the greater". Therefore, 'The greater is a *multiple* of the less

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[45], *Science of Harmonics*, 29-30.

[46] Plato, *Republic*, 531a4-8, transl. Paul Shorey (Cambridge: Harvard University Press, 1972), book 7. According to Barker, *Science of Harmonics,* 23-4, Socrates's metaphors involved adjusting the pitches of an instrument's strings by twisting the tuning pegs until two strings gave notes so nearly identical that they could come no closer without reaching an apparent unison. When that situation was achieved, the unit of measurement had been identified.

[47] Aristotle, *Topics*, 107 a15-16, in *Works of Aristotle*, 143-223, on 151.

[48] This alleged superiority of the mathematical approach to harmonics was challenged by Aristoxenus of Tarentum and Theophrastus. The former made stronger impact, because his writings provoked a counter-attack in the form of a treatise known as *Sectio canonis.* That work contains the first mention ever to the monochord, even though mathematical harmonics pre-existed the monochord; demonstrations of harmonic ratios are evident in writings from the 5th century B.C., but there is no evidence of the monochord's existence before 300 B.C.

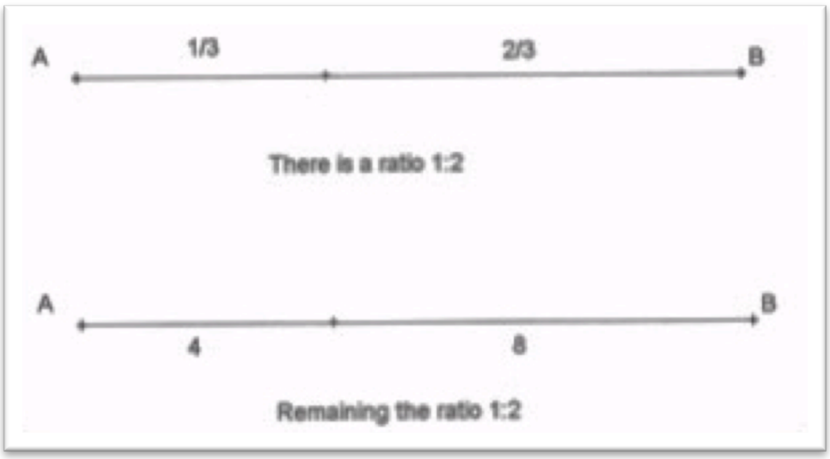
[49] Vincenzo Galilei, *Compendio di Vicentio Galilei della teorica della Musica* (ce.1570), Biblioteca Nazionale Centrale di Firenze, Anteriori di Galileo, vol. 4, fols. 3r-47v).

[50] "Parte aliquota è quella che presa piu volte (aggiunta) rintegra esso numero; et per Il contrario non aliquota è quella che non há tal faculta", Galilei, *Compendio*, 10r.

when it is measured by the less; [and] a *ratio* is a sort of relationship with respect to size between two magnitudes of the same kind"[51].

Furthermore, it is necessary to consider that in arithmetic the unit is indivisible per definition, which is not the case in geometry, whose segments can be infinitely divided. When one considers the studies on perspective and surveying elaborated in the 15th and 16th centuries, it becomes clear that the notion of quantifying in arithmetic was quite different from the one in geometry (Figure 5).[52]

Figure 5. Proportion and ratio



As is known, arithmetic and geometry were two different fields of knowledge. Arithmetic was the science concerned with discrete, and geometry with continuous magnitudes. In practical terms, regarding measuring, arithmetic notions and operations could not be applied to geometry, although a number of mathematicians, such as

[51] Euclid, *Elementi*, V, in *Euclide tutte le opere: testo greco a fonte*, ed. Fabio Acerbi (Milano: Bompiani, 2007), 975-6.

[52] As Zarlino's understood, "I principii proprii della geometria sono (per darvi un esempio) questi: si può condurre uma línea da um punto all'altro, il continuo è divisibile in infinito: et altri simili. Ma quelli dell'arithmetica sono: il numero è moltitudine ordinata di unità, le parti del numero non si coniungono ad um termine commune: i numeri procedono oltra l'unità in infinito: e gli altri; Et quelli della musica sono: l'intervallo è habitudine dei spazii del suono grave e dell'acuto e altri simili, come presto vedrete. Et queste si chiamano principii proprii;" Gioseffo Zarlno, *Sopplimenti musicali*, *libro IV* (Venetia: Francesco de' Franceschi, 1588), 140. From the 16th century on, it became more typical to attribute a number to the magnitude. It is mostly assumed that such attribution derived from the influence of the algebra in Arabic treatises, however we argue that it was originated in the abacus schools; see Antoni Malet, "Renaissance Notions of Number and Magnitude," *Historia Mathematica* 33 (2006): 63-81.

Niccolò Tartaglia (1499–1557) and others - called mathematical practitioners - transgressed this rule.[53]

Associating a number to a magnitude was a common practice among mathematical practitioners.[54] Between the 16th and 17th centuries the development of new experimental and mathematical methods demanded constructing new instruments,[55] and several mathematicians published treatises on this subject. However, many such treatises were primarily aimed at instructing the public in how to construct different types of instruments and in the different ways of measuring distances. It is in such treatises that a constant association between number and magnitude is found. The mathematical instruments allowed associating number and magnitudes, because the instruments themselves incorporated the unit of measurement and as such, the magnitude - which was geometric - could be expressed by a number or an arithmetic ratio.[56]

Mensuration understood in such terms had three important properties. The first was accessibility, that is, the need for a measurement instrument. In this regard, the human body was considered the oldest instrument: the feet and the fingers were widely used to measure distances, width and length. The second property of measurement was its adequacy to the intended purpose. Consistency was the third property, which ensured stable and reliable measurements. Those properties were mentioned in many treatises devoted to studies of astronomy, land surveying and navigation (Figure 6).[57]

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[53] The separation between arithmetic and geometry was mentioned by many authors in the 16th and 17th centuries, such as John Dee (1527–160[8]) and Egnatio Danti (1536–1586; see John Dee, *The Mathematical Preface of the Elements of Geometrie of Euclid of Megara* (1570) [facsimile New York: Science History Publications, 1975); Egnatio Danti, *Le scienze matematiche ridotte in tavole* (Bologna: appresso Compagnia della Stampa, 1577); see also Malet.

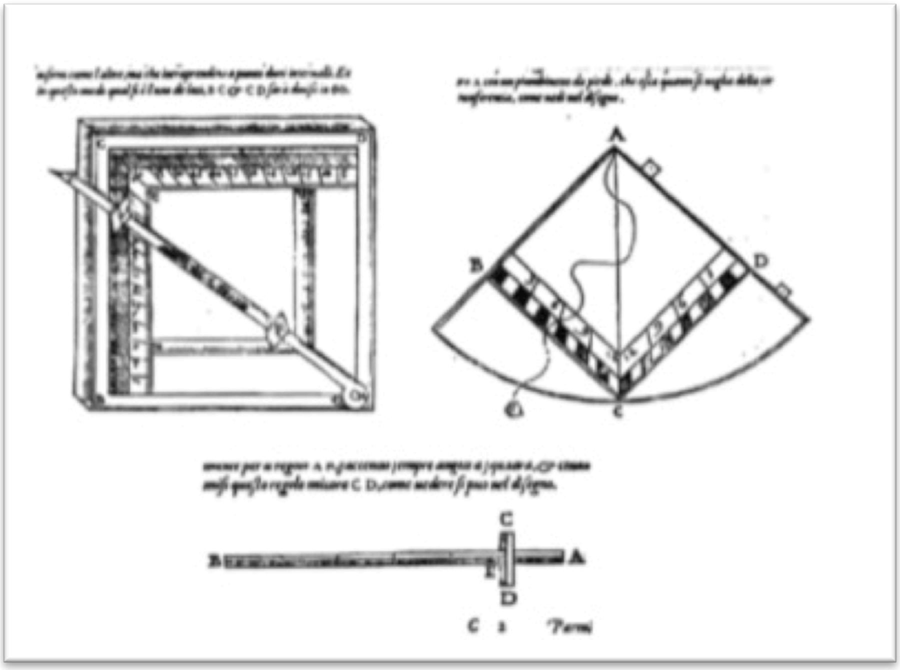
[54] On mathematical practitioners, see Eva G. Taylor, *The Mathematical Practitioners of Tudor & Stuart England* **(**Cambridge: Cambridge University Press, 1954); Hester Higton, "Does Using an Instrument Make You Mathematical? Mathematical Practitioner of the 17th Century," *Endeavour* 25 (2001): 18-22.

[55] Albert van Helden, "The Birth of the Modern Scientific Instrument, 1550-1770," in *The Uses of Science in the Age of Newton*, ed. J.G. Burke (Berkeley: University of California Press, 1983), 49-84; Deborah J. Warner, "What is a Scientific Instrument, When Did It Become One, and Why?" *British Journal for the History of Science* 23 (1990): 83-93.

[56] See Malet; Fumikazu Saito, & Marisa S. Dias, *Articulação de Entes Matemáticos na Construção e Utilização de Instrumento de Medida do Século XVI* (Natal: SBHM, 2011); and Fumikazu Saito, "Instrumentos Matemáticos dos Séculos XVI e XVII na Articulação entre História, Ensino e Aprendizagem de Matemática," *Rematec* 9, no. 16 (2014): 25-47.

[57] Cosimo Bartoli, *Cosimo Bartoli Gentil'huomo, et accademico Fiorentino, Del modo di misurare le distantie, le superficie, i corpi, le piante, le province, le prospettive, & tutte le altre cose terrene....* (Venetia: Francesco de' Franceschi Sanese, 1564).

Figure 6. Mathematical instruments; from left to right: *quadrante*, *squadro baculo*, p.10.[58]



It is important to remember that the aforementioned mathematical instruments were intended to quantitatively represent the distance between two or more points. As concerns music, this procedure demanded more than simply deciding which type of phenomena were to be measured. As a fact, it was necessary to know what had to be measured.

The words space and distance appeared in the definitions of intervals that entailed the notion of measuring. However, the difference between measuring an audible phenomenon and a visual one could only be inferred from the context itself.

Juan Bermudo's (1510–ca.1565) *Declaracion de instrumentos* (1549), a treatise that includes descriptions of tuning instruments, describes distance primarily as an audible phenomenon. Bermudo defines the distance between the strings of the *vihuela* stating that they were consonances: "The same distance will be found between the strings of the vihuela, beginning from the inferior to the superior string, except for the distance between the fourth and the third string, which is a *ditono* [a major third, made of four semitones]"[59]. Bermudo, however, knew that the measures necessary for constructing

[58] Ibid., 3; 8; 10.

[59] "Esta mesma distancia hallareys em toda la vihuela desde una cuerda inferior a la otra mas cercana superior: excepto desde la quarta cuerda ala tercera, que ay un ditono, que es tercera mayor: la qual

instruments were different from the measures of intervals: "instrumental music demands two types of measures, one to build the instruments and another to tune and play them"[60].

Silvestro Ganassi, a Venetian musician who played the recorder and wrote two important treatises on tuning viols and flutes,[61] explains that although many proportions were used to place the frets on the viol, the order of the proportions should not be driven by theory alone, but by actually placing the frets on the viols. While this might be seen as a simple statement on the difference between theory and practice, it actually unfolds as a brand new explanation. According to Ganassi, ordering the proportions was only possible when one took the sizes of the viols into account, because viols, being made by different makers, had different sizes and parts.[62]

Despite the different approaches towards measuring, Bermudo's and Ganassi`s ideas echo Galilei's criticism of Zarlino. Galilei clearly noted that Zarlino's writings on experimenting with musical instruments were merely rhetorical. He observed that Zarlino's textual content did not correspond to the demonstrations,[63] because on the one hand, Zarlino relied on information originated by ancient authors to ponder the quantities and qualities of sound,[64] and on the other hand, he assumed that his conclusions on experiments with strings would also apply to other instruments.[65] Galilei summarizes: '[Zarlino] distracted himself with Euclid's mathematics, Aristotle's philosophy and reading books, finding definitions, predicaments, and other *gentilezze*"[66].

Galilei knew the difference between 'talking about doing' and 'actually doing' things. Zarlino had described in his works the construction of instruments and tuning systems, but proved to be unaware of the properties of instruments. Finally, upon attempting to quantify the physical elements of music, Zarlino was driven by

consonancia tiene quatro semitonos."; Juan Bermudo, *El ilbro llamado declaracion de instrumentos musicales* (Osuna: Juan de Leon ,1555), f. xxviij.

[60] "[...] para la musica instrumental es menester dobladas medidas: una para hazer los dichos instrumentos, y otra para tanerlos."; Ibid., f. iiii.

[61] Silvestro Ganassi, *Lettione Seconda pur della prattica di sonare il violone d'arco da tasti .... Stampata per Lautore proprio* (Venezia, 1543).

[62] Cecil Adkins, "The Theory and Practice of the Monochord" (PhD Dissertation, The State University of Iowa, 1963), 414-6.

[63] Vincenzo Galilei, *Critica fatta di Vicentio Galilei intorno ai supplementi musicali di Gioseffo Zarlino. Anteriori di Galileo*, Biblioteca Centrale Nazionale di Firenze, Anteriori di Galileo, vol. 5, fols. 3r-58r, on 47r.

[64] Ibid., V: f. 40r.

[65] Ibid., V:.ff. 42v-43v.

[66] Ibid., V: f. 32r .

conclusions found in text documents only.[67] Contrariwise, Galilei showed that the mathematical ratios proposed by Zarlino did not have acoustic correspondence.

Conclusion

It is traditionally assumed that instruments are mediators between theory and practice, resulting in either theory mediated by instrument and practice or vice versa. As we attempted to show here, historically instruments did not mediate between theory and practice or between practice and theory, because they should not be reduced to the function of applying theory or materializing practice.[68]

The instrument - as we saw with Galilei and Zarlino - brought up new questions that challenged both the theoretical and the practical knowledge. Accommodating music intervals in real instruments could not be achieved solely through mathematical calculation of ratios. Therefore, the instrument fostered a new type of knowledge likely to change the traditional relationship between theory and practice (and vice-versa).

The traditional historiography on the struggle between abstract mathematical objects and real, physical ones fails to fill in the gap between the theoretical and practical aspects of the mathematical-physical relationship. First, because it subjects the instrument to either theory or practice, and second because it neglects the distinction between arithmetical and geometrical elements. We argue that the theory-instrument-practice relationship should be reconfigured in such a way that neither theory nor experiment is given an epistemological or historical priority. The reason is that in the process of knowledge construction, the instrument is often the first to appear, giving rise to theoretical and experimental questions. In the specific case of the musical instruments, on the one hand luthiers built them even without any knowledge of musical theories and, on the other, the discussions on the monochord were twofold. While approaching the theoretical order, we should bear in mind that mathematical harmonics existed even

[67] Galilei, *Dialogo di Vincentio Galilei;* and Galilei, *Critica fatta di Vicentio Galilei*, V: ff. 3r-58r.

[68] Fumikazu Saito, "Revelando Processos Naturais por Meio de Instrumentos e Outros Aparatos Científicos," in *História da Ciência: Tópicos Atuais 3*, ed. M.H.R. Beltran, F. Saito, & L.S.P. Trindade (São Paulo: Livraria da Física, 2014), 95-115; Fumikazu Saito, "Algumas Considerações Historiográficas para a História dos Instrumentos e Aparatos Científicos: O Telescópio na Magia Natural," in *Centenário Simão Mathias: Documentos, Métodos e Identidade da História da Ciência*, ed. A.M. Alfonso-Goldfarb et al. (São Paulo: PUC-SP, 2009), 103-21; and Fumikazu Saito, "Instrumentos e o 'Saber-Fazer' Matemático no Século XVI", *Revista Tecnologia e Sociedade,* 18 (2013): 101-12.

before the monochord,[69] and relative to the experimental order that divisions on the monochord could not simply be applied to any other musical instrument.

Therefore, the instrument should not be analyzed as a mediator. The instrument promotes new knowledge in the sense that it should be seen within a different conceptual framework [because it belongs to a different conceptual framework]. In the case of Zarlino, theory was expanded to provide practice a theoretical background; nevertheless it failed to demonstrate the theoretical musical system in real instruments. In consequence, Zarlino attempted to establish rules for constructing new instruments (polychordo, clavicembalo) convinced that his system would work in those cases.

While analyzing the mathematical-musical controversies between Zarlino's and Galilei's works, we realized that while the latter's approach was, indeed, different from the former's, it was not opposed to it, as it is argued in the literature, which sees Zarlino as a man of theory and Galilei as a man of experimentation. As we have shown here Galilei did not start from practice to only then move on to theoretical considerations, but an analysis of his conceptual framework clearly shows that it remained mathematical. However, different to Zarlino's, Galilei's mathematical approach was geometrical and practical rather than arithmetical and speculative. As a result, one should not conclude that Galilei was solely concerned with questions related to experiments or of a purely natural order, as the current literature holds.

Regarding experimentation with instruments, as we showed, Zarlino attempted to demonstrate his theoretical system by constructing instruments able to play it. Such attempts were proven flawed by Galilei. And he did that on theoretical grounds and reasoning. As he himself explained in *Critica* (his last work) and before that in his first treatise, the *Compendio* (c. 1570) - in which he had exposed all the mathematical definitions of the musical elements and procedures - his realm of work was mathematics.

Nevertheless, his awareness of the speculative mathematical foundation of mathematics did not prevent Galilei from promoting in practice a tuning system that he knew was alien to the theoretical framework. And he found the solution in practical mathematics. Since the tuning system (temperament) adopted by Galilei could not be included in the arithmetical musical tradition that grounded Zarlino's views, Galilei developed a mathematical argumentation based on geometry. That fact notwithstanding, Galilei did not approach the intervals by studying a vibrational string,

[69] Creese, *Monochord in Ancient Greek*, ch. 2.

as an acoustical body, although he knew the sound relations (e.g., that an octave is produced when one divides the string by its half, etc.), he analyzed them based on the relationships between the length, or parts of the length, of the strings as they appear to geometry. Galilei's discussion of the nature of intervals, such as the diapason and unissono, in his later dialogues should not be detached from this context.

The role arithmetic and geometry played in music is, as we believe, the central point in the debate between Zarlino and Galilei. As appears in their works, especially Zarlino's *Dimostrationi* and *Supplementi* and Galilei's *Discorso* and *Critica,* these authors advocated different notions of music as a science. Zarlino understood it as a mathematical science subordinated to arithmetic, being more mathematical than natural and without taking sound into account sound. In turn, Galilei understood music as mathematical and closer to natural science, because sound ought not to be eliminated from musical considerations. The latter assertion, naturally, is very far from stating that the point of departure of Galilei's analysis was acoustical experiments.

Hence, we hope to have demonstrated that while discussing the role theory and practice played in the mathematical sciences in the 16th century, it is extremely important to go beyond the formal and current definitions that distinguish the theoretical from the practical knowledge. The role the instrument played in the 16th century cannot be understood based on modern philosophical conceptions. Analyses of historical documents are important because only they enable us to grasp the different configurations established by historical figures and let us comprehend the construction of knowledge from an epistemological perspective. Therefore, the role of the instrument must be approached in its specificity, rather than as generic mediator between two types of knowledge. Similarly, Galilei should not be considered an empiricist, since he developed his criteria on clearly mathematical grounds, just as Zarlino should not be considered a mere theorist, in the light of his practical and empirical considerations.