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International Journal for the History of Science

A preliminary survey on the emergence of an arithmetical theory of ratios

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ABSTRACT

This article considers competing traditions of theories of ratio and the process which led to the emergence of an arithmetical theory of ratio. Such a complex process, already began in the Ancient Greece, developed throughout the Middle Ages until the Renaissance and received contributions from the Latin and Arabic traditions culminating with the confluence of such traditions, and consequently with its acceleration during the Renaissance. In this context, it must be regarded the Latin translation from the Arabic of Euclid by Campanus of Novara in the thirteenth century, a singular contribution to the structural indefiniteness and to the tension in the history of theories of ratio, inasmuch as Campanus gave to definition 5 of book V concerning proportionality of ratios an arithmetical meaning inserting the concept "denominatio" for ratio not presented in the original Euclid. In theoretical music context, the division of the tone is also not to be overlooked insofar as it molded indirectly the conception of ratio throughout the history of the discussions involving the arithmetization of ratio.

Keywords:

History of mathematics, Theories of ratio, Arithmetization, Late Middle Ages

RESUMO

Esta pesquisa preliminar discute alguns dados sobre o complexo processo associado ao desenvolvimento de teorias sobre as razões e a aritmetização das razões na Idade Media tardia e o Renascimento, apresentando algumas evidências a favor da coexistência de tradições aritméticas e geométricas no tratamento das razões nessa época. A complexidade em questão deve-se a fatores imensuráveis que polarizaram por vezes o uso das razões na tradição clássica, por vezes na tradição aritmética, num processo que se estendeu até o século XVI, quando a teoria aritmética se consolidou, até se tornar a dominante.

Palavras chave:

História da matemática, Teorias de razões, Aritmetização, Medievo tardio

A preliminary survey on the emergence of an arithmetical theory of ratios

Introduction

This article discusses some historical settings that led to the development of the theories of ratio in the late Middle Ages and to the emergence of an arithmetical theory of ratio in the medieval Latin tradition. The expression "arithmetization of the theory of ratio" is used in this context to characterize the development undergone by ratio mainly in the late Middle Ages and the Renaissance, when this mathematical concept lost its geometric character to assume a semantically distinct yet structurally similar arithmetic one. For instance, ratio lost the meaning of a comparison between two magnitudes of the same nature in order to be identified with number. Ratio became defined by a division and was now identified with the quotient of two magnitudes. The compounding of ratios turned into a multiplication of ratios, and proportions between ratios became an equality of numbers.

Many authors have studied medieval ratio theory, and their analyses have increased considerably our understanding of medieval mathematics. Among them, it is worth to mention Sylla, Busard, Evans, Folkerts, Hoyrup, Lorch and North. Murdoch has provided a general survey about the medieval theory of ratios, as well as a study concerning the introduction of "denomination" into discourse.[1] Grant has focused on Oresme and his idea of fractional exponents, while Molland has concentrated on Bradwardine[2].

In this context, it is important to take into account the Latin translation of Euclid's *Elements* from the Arabic, by Campanus of Novara in the 13th century, an important contribution to indefiniteness in the history of theories of ratio, inasmuch as he gave to definition 5 in Book V, dealing with proportionality of ratios, an arithmetical meaning, for instance, by inserting the concept "*denominatio*", which was not contained in the original text. The medieval conception of ratio had been inherited from both the classical Greek geometrical tradition and the later Greek arithmetical tradition, but Campanus, in his translation, did not distinguish the two Greek traditions, and substituted "denomination" by ratio, which was probably equivalent to treating ratios as if they were fractions.

Campanus' Latin translation of Euclid's *Elements* is generally regarded as the main source for 14th century ratio theory, especially as presented by Bradwardine and Oresme. This theory used an arithmetical vocabulary that did not derive from the geometrical ratio theory expounded in Euclid's Book V, but rather from a number of different sources[3]. Oresme used the term "denomination" and represented the ratio of ratios, "*proportio proportionium*"[4], with ratios in exponents, a procedure that allowed for the division of an arbitrary ratio by an arbitrary number, and indirectly conferred to ratio a continuous feature.

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[1] J. E. Murdoch, "The Medieval Language of Proportions", in *Scientific Change: Historical Studies in the Intellectual, Social and Technical Conditions for Scientific Discovery and Technical Invention, from Antiquity to the Present*, ed. A. C. Crombie (London: Heinemann, 1963), 237-71.

[2] E. Grant, E., "The Mathematical Theory of Proportionality of Nicole Oresme (ca. 1320-1382)" (PhD dissertation, University of Wisconsin, 1957); "Nicole Oresme and his *De proportionibus proportionum*", *Isis* 51 (1960): 293-314.

[3] Murdoch, "Medieval Language".

[4] In the Middle Ages, a different terminology for ratio dominated, in which this mathematical concept was usually translated in Latin as *proportio*, instead of *ratio*; this was reconsidered since the beginning of the 16th century, with the new translations of the Greek classics, however without immediately displacing its medieval usage. For more detailed discussions about the changes in the terminology of ratio from classical and medieval times to the early modern period, see Wilbur R. Knorr, "On the Term Ratio in Early Mathematics", in *Ratio: VII Colloquio internazionale del lessico intellettuale europeo*, Roma, 9-11 gennaio 1992, ed. M. Fattori & M. L. Bianchi (Firenze: Olschki, 1994).

Although medieval mathematicians referred to Book V, which contains a theory of ratios regarding magnitudes, the definition for the equality of ratios given in Campanus' edition of Euclid is not the Eudoxian definition 5 in Book V, but one in terms of denominations of ratios, which does not appear in Heath's edition[5]. Unlike Book *V* of the *Elements*, Book VII, in its original form, contains the arithmetical treatment of ratios, which is not applicable to continuous magnitudes, and thus nor to the treatment of incommensurables. Basically, the arithmetical theory of ratio manifested in Campanus' version of Euclid's *Elements*, equipped with the medieval arithmetical ratio term "*denominatio*", provided the foundation for the late medieval understanding of ratios in mathematical contexts.

A crucial question is why Campanus used arithmetical definitions in his translation, when editions containing the original definitions in Book V were available[6], a fact that increases the tension between different theories of ratios at that time, leading to an attempt of demarcation between such theories, and eventually to the study of the emergence of an "arithmetical theory of ratios" within the *arithmos* tradition of Euclid.

The complex process of arithmetization of the theories of ratio began in Ancient Greece, developed throughout the Middle Ages until the Renaissance, received contributions from the Latin and Arabic traditions, to culminate with the confluence of these traditions – consequently, attended with a great acceleration of this process during the Renaissance.

Up to the Renaissance, the use of ratios did not have a well-demarcated structure, but sometimes presented arithmetical features, sometimes geometrical features or a combination of both. Such structural differences, which kept up with the concepts of ratio and proportion since Antiquity, corresponded to underlying theoretical treatises not only on mathematics, but also on near disciplines like theoretical music.

Two traditions of theories of ratio

In order to comprehend different theories of ratios, it is important to contextualize the idea of compounding ratios, which is crucial for the understanding of the process of arithmetization. This idea is not explicitly defined in proposition 23, Book VI of Euclid's *Elements*, which says that equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.[7] In order to demonstrate it, Euclid needed to compound two ratios: BC:CG and DC:CE, which he adapted to proportional ratios K:L and L:M, respectively, having L in common, before carrying out the operation. Thus, compounding ratios using this classical Greek method consisted in taking ratios of the type *a:b* with *b:c* to produce *a:c*, which then allows for the repetition of this process with *c:d* and so on. This is to say, given a sequence of ratios to be compounded, the second term of a ratio should equal the first term of the subsequent ratio. Therefore, to compound the ratios *a:b* with *c:d*, it was necessary to find a magnitude *e* so that *c:d* would be proportional to *b:e*, and the resulting compounded ratio would be *a:e*.

The idea of compounding is relevant for the emergence of different structures underlying theories of ratios, an argument corroborated by the fact that Szabo also made use of this concept while raising questions in his attempt to show that the pre-Eudoxan theory of proportions developed initially as an inheritance from the Pythagorean theory of music[8] Compounding ratios had strong musical affinities, for it is structurally similar to composing contiguous intervals with the monochord. I use here the expression "compose" to express the process in a musical context in which 2 musical intervals are taken, so that

[5] Euclid, *The Thirteen Books of Euclid*, ed. & trans. T. L. Heath (New York: Dover, 1956).
[6] Murdoch, "Medieval Language", 240.
[7] Heath, 247.
[8] A. Szabo, *The Beginnings of Greek Mathematics* (Budapest: Akademiai Kiado, 1978).

the highest note of the first is equal to the lowest note of the second, so as to produce a new interval whose lowest note is the lowest note of the first, and its highest note is the highest note of the second. Such operation can be applied recursively.

In the composition of intervals with the monochord, it is necessary to start a given interval always from the point one reached in the previous one, which corresponds mathematically to the common terms between subsequent ratios in the operation of compounding mentioned above. Thus, compounding ratios in mathematics corresponds to the composition of musical intervals in music, and vice versa. There is no mathematical reason to define the operation of compounding ratios in such a way, and possibly one would not define it so, unless one first observed its musical meaning, whereby one understands what is otherwise a purely mathematical phenomenon, as the composing of contiguous intervals.

Another point important for the characterization of the different theories of ratio, is that the identification of ratios with fractions relates to the notion of incommensurable magnitudes. If ratios are generally identified with fractions, then ratios between a side and the hypotenuse of an isosceles right angled triangle become inexpressible. The solution for this situation is either to use approximations or to accept perfect decimal fractions into the domain of numbers. The latter appeared sometime in the late 16th century. However, such situation concerning irrational ratios lacked consensus among medieval, Renaissance and early modern mathematicians, a fact that has been highlighted by recent historians.

As it was already mentioned, the medieval concept of ratio was a heritage from both the Greek classic geometric and the late Greek arithmetical tradition. The former derived from definition 5 in Book V of Euclid's *Elements*[9], while the latter seems to appear for the first time in the transversal problem of Menelaus (c. 70-130), who compounded ratios without the constraints mentioned before, namely as a multiplication, and then with Theon of Alexandria (c.335-405)[10], who inserted interpolations in definition 5 of *Elements* Book VI[11], distorting the original Euclidean sense of compounding ratios, approaching the idea of compounding to the multiplication.

Up to the Renaissance, the treatment of ratios had no clear and well-defined structure. Some traditions had mainly arithmetical features, others geometrical and musical ones, whereas still others incorporated both these tendencies. Sylla discusses the confusion over the geometrical and the arithmetical traditions of ratios, showing how both "strangely mingled" within the context of compounding and multiplying[12]. She categorizes two traditions within the Greek and medieval treatment of ratios, one associated with theoretical mathematics, music, and physics, particularly found in Bradwardine's *De proportionibus*; and another associated with practical calculations using ratios and with astronomy[13]. She argues that, "These two traditions may not encompass all ancient and medieval concepts of ratio. Neither were these traditions always separate - in fact, they were often strangely mingled. Nevertheless, they represent two poles of the ways in which ratios and the operations on ratios could be treated"[14].

Drake further suggests that the medieval theory of proportion made use of an elaborate vocabulary that was not derived from Euclid as we know from the

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[9] Heath, 120.
[10] E. Grosholz, "Some Uses of Proportion in Newton's Principia, Book I: A Case Study in Applied Mathematics", *Studies in History and Philosophy of Science* 18 (1987): 208-20.
[11] Heath, 189
[12] E. Sylla, "Compounding Ratios: Bradwardine, Oresme, and the First Edition of Newton's *Principia*", in *Transformation and Traditions in the Sciences: Essays in Honor of I.B. Cohen*, ed. E. Mendelsohn (Cambridge [Mass]: Cambridge University Press, 1984), 11-43.
[13] Ibid, 11.
[14] Ibid, 17.

Elements[15].Though Bradwardine refers to Euclid's Book V, which contains the theory of ratios and proportionality of magnitudes in general, one such reference is, and the definition given in the *Treatise*[16] for the equality of ratios is not the Eudoxian definition in Book V, but a definition in terms of "denominations" of ratios.

This term, although not in Euclid, had made its appearance among the definitions of Book VII in the standard medieval edition of Euclid, the one by Campanus. Book VII begins the special treatment of numerical ratios and proportionality apart from the general theory of magnitudes given in Book V. It is the theory of proportion embodied in Book VII, as embellished in the medieval version and supplemented by ancient arithmetical ratio vocabulary, that lies at the basis of Bradwardine's *De proportionibus* [17]. Thus, Drake suggests that, "Campanus' Book VII embodied a complete definitional apparatus and terminology for the theory of proportion characteristic of medieval mathematics and was given its definitional base without any conceptual need for references to *Book V*"[18].

The notion of "denominations" has been addressed by several scholars; for example, Murdoch discussed it in the case of Campanus[19]. However, there still seems to be some disagreement about what it actually means. Strictly speaking, it comes from rhetoric, where it is connected with metonymy, i.e. the substitution of an attribute for the thing named, for example, "crown" for "king". A more mathematical meaning is that of the unit of a quantity, for example, "*metre"* is the denomination of the quantity 5 meters. On the other hand, scholars seem undecided whether "denomination" in medieval mathematics referred to a ratio as a fraction reduced to its lowest terms, or to the quotient of that fraction. If the former, then this is little more than a simplified Boethian ratio terminology, where 2:4 is *dupla* 1:2. If the latter, then decimalization occurred earlier than what has been thought. Interestingly, Molland, who has discussed it in the context of Bradwardine, has noticed that:

> "Richard of Wallingford, who followed Campanus closely, came near to identifying a ratio with its denomination.[20] Nicole Oresme exhibited the denominations of rational ratios by numbers or numbers and fractions, but he also took into account irrational ratios, and there the matter was not so simple"[21].

There seems to be an important source of misunderstanding here. From Euclid's Book VII, we have unit fractions such as 1/3 and ratios of integers such as 1:3. It seems that medieval mathematicians misread or re-interpreted Euclid in taking the latter as being rational numbers such as 1/3. Further, this may be a source for the new arithmetical theory of ratios, which Sylla discusses[22].

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[15] S. Drake, "Medieval Ratio Theory vs. Compound Indices in the Origin of Bradwardine's Rule", *Isis* 64 (1973): 67-77.
[16] H. L. Crosby, *Thomas Bradwardine His Tractatus de Proportionibus: Its Significance for the Development of Mathematical Physics* (Madison: The University of Wisconsin Press, 1955).
[17] Ibid, 67-8.
[18] Ibid, 70.
[19] J. E. Murdoch, 'The Medieval Euclid: Salient Aspects of the Translations of the 'Elements' by Adelard of Bath and Campanus of Novara*", Revue de Synthese* 89 (1968): 67-94.
[20] A. G. Molland, "The Geometrical Background to the 'Merton School'", *British Journal for the History of Science* 4 (1968): 108-25.
[21] Ibid.
[22] See note 12.

Campanus and Book V of the *Elements*

In general, historians consider that:

> "General acquaintance with Euclid in Europe was encouraged especially by a version of the *Elements* made in the 1290s by Campanus of Novara; it was also the first to be printed, in 1482. However, Campanus had elaborated upon the translation made by Adelard of Bath in the 12th century, so that the text was more garbled than its earliest readers realized"[23].

This understanding probably derives from a comment made by Heath on Book V, definition 5 to the effect that Campanus had a confused understanding of Euclid. According to Heath, "From the revival of learning in Europe onwards, the Euclidean definition of proportion was the subject of much criticism. Campanus had failed to understand it, had in fact misinterpreted it altogether, and he may have misled others such as Ramus (1515-72)"[24].

Murdoch supports this view by suggesting that,

> "Campanus, in a general comment to the definitions of Book V of the *Elements*, denies its application to that infinity of irrational proportions, for, he asserts, their denominations are not knowable. Moreover, he adds, Book V does include irrationals in its domain, and hence Euclid was forced to abandon – unlike the arithmetician – the definition of equal proportions by equal denominations"[25].

Molland initially agreed with Heath and Murdoch, and stated that,

> "This definition [V def. 5], partly as a result of the obscurity of the translation, was not understood in the Middle Ages, and in his version of the *Elements* Campanus flounders hopelessly in search of suitable general criteria of the equality of ratios"[26].

However, when this author reconsidered the matter ten years later, he asserted that,

> "Campanus' explication of the Eudoxian criterion of equality often seems garbled [...] I am not convinced that he completely misunderstood it, for in his comment on the definition of greater ratio we have the following: The ratio of the first of four quantities to the second is never greater than that of the third to the fourth, unless some equimultiples of the first and the third may be found, such that when they are related to some equimultiples of the second and the fourth, the multiple of the first will be found to exceed the multiple of the second, but the multiple of the third will not exceed the multiple of the fourth. And this can never be found unless the ratio of the first to the second be greater than the ratio of the third to the fourth, as we shall demonstrate below"[27].

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[23] I. Grattan-Guiness, ed., *The Fontana History of the Mathematical Sciences* (London: Fontana Press, 1997), 153-4. See also H. L. L. Busard, ed. *Adelard of Bath*, *The first Latin translation of Euclid's "Elements" Commonly Ascribed to Adelard of Bath: books I-VIII* (Toronto [Ont]: Pontifical Institute of Mediaeval Studies, 1983).

[24] See note 5.

[25] Murdoch, 259.

[26] Molland, "Geometrical Background", 115-6.

[27] A. G. Molland, "An examination of Bradwardine's Geometry", *Archive for History of Exact Sciences* 19 (1978): 113-75, on 159.

This reconsideration raises questions concerning the intentionality of Campanus in displaying definition 5 of Book V in terms of denominations, which further raises questions about the need to arithmetize such concept in this context. And it still remains the question on why Campanus inserted arithmetical interpolations from Jordanus of Nemore into Euclid, when acceptable editions containing Book V*,* definition 5 were available. Whether Campanus misinterpreted Euclid, or made this purposefully, this fact does raise important questions concerning competing theories of ratios in the Middle Ages and on the emergence of the arithmetization of such theories in the late Middle Ages.

Equal division of the tone and theories of ratio underlying theoretical music

Besides the scholars mentioned above, many theorists involved in the division of the tone indirectly shaped the conception of ratio throughout the history of the discussions involving the arithmetization of ratio. The equal division of the tone played an important role in the historical process of arithmetization of ratios. Mathematically, the equal division of the tone 8:9 provides incommensurable ratios underlying musical intervals. It means mathematically to find x so that $8:x = x:9$; that results, anachronistically speaking, in irrational numbers, inconceivable in the Pythagorean musical system.

Attempts to divide the tone were made as early as in Antiquity, for instance, by Aristoxenus (4th century B.C.). In contrast with the Pythagoreans, who advocated that musical intervals could be properly measured and expressed only as mathematical ratios, Aristoxenus asserted instead that the ear was the sole criterion for musical phenomena[28]. In preferring geometry to arithmetic to solve problems involving relations between musical pitches, Aristoxenus sustained, also against the Pythagoreans, the possibility of dividing the tone into two equal parts, conceiving musical intervals – and indirectly, ratios – as one-dimensional and continuous magnitudes, making possible in this way their division.

This idea provoked a large number of reactions expressed, for instance, in the *Sectio Canonis*[29]*,* which in Antiquity was attributed to Euclid, and much later in the *De institutione musica* of Boethius, in the early medieval era[30], which gave birth to a strong Pythagorean tradition in theoretical music throughout the Middle Ages. Following the Platonic-Pythagorean tradition, a great part of medieval musical theorists sustained the impossibility of the equal division of the tone, which would mathematically lead to incommensurable ratios underlying musical intervals. Gradually, the need to carry out the temperament gave birth to different attempts to divide the tone.

Goldman suggests that Nicholas Cusanus (1401-1464) was the first to assert in *Idiota de Mente* that the musical half-tone is derived by *geometric division* of the whole-tone and, hence, would be defined by an irrational number[31]. As a consequence, Cusanus would have been the first to formulate a concept that set the foundation for the equal temperament proposed in the work of the High Renaissance music theorists Faber Stapulensis (1455-1537) and Franchino Gafurius (1451-1524), published half a century later[32].

Nevertheless, one can find in the Byzantine tradition, Michael Psellus (1018-1078), who suggested in his *Liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et]*

[28] R. P. Winnington-Ingram, "Aristonexus", in *The New Grove Dictionary of Music and Musicians*, ed. S. Sadie (London: Macmillan, 1995), 592.
[29] A. Barbera, *The Euclidean Division of the Canon* (Lincoln: University of Nebraska Press, 1991), 125.
[30] C. M. Bower & C. V. Palisca, *Fundamentals of Music: Anicius Manlius Severinus Boethius* (New Haven: Yale University Press, 1989), 88.
[31] D. P. Goldman, "Nicholas Cusanus' Contribution to Music Theory", *Rivista Internazionale di Musica Sacra*, 10, No 3/4 (1989): 308-338, on 308.
[32] Ibid.

astronomia[33]*,* a geometrical division of the tone, whose underlying conception implies an understanding of ratio as a continuous magnitude. Also concerning the division of the tone before Cusanus, Marchetus of Padua (1274? --?) proposed, in his *Lucidarium in Arte Musice Planae,* written in 1317/1318, the division of the tone into five equal parts[34], an innovation of extraordinary interest which made Marchettus the first in the Latin tradition to propose such a division, but without any mathematical approach.

At the end of the 15th and beginning of the 16th century, Erasmus Horicius, one of the German humanists gifted in musical matters, wrote his *Musica*[35]*,* where he suggested a geometric division of the whole tone. Erasmus stated that any part of any super-particular ratio could be obtained, in particular the half of 8:9, which corresponds to dividing the whole tone equally. Theoretically based on many geometrical propositions, and unusually modeled on Euclidean style, his *Musica* dealt with ratio as a continuous quantity, announcing perhaps what would emerge as a truly geometric tradition in the treatment of ratios in theoretical music contexts during the 16th century. The attempts to dividing the tone led to a conception of ratio as a continuous quantity in theoretical music, and consequently, to the arithmetization of theories of ratios both in musical and mathematical contexts.

Concluding Remarks

This survey discussed some facts about the complex process associated with the development of theories of ratios and the arithmetization of ratios in the late Middle Ages and the Renaissance, presenting some evidences for the co-existence of arithmetical and geometrical traditions in the treatment of ratios in this period. It was emphasized the use of ratios in musical contexts as an important factor for the permanence of the classical tradition, while at the same time giving rise, through the problem of the division of the tone, to the use of the arithmetical tradition in this context.

This complexity was due to immensurable factors that polarized sometimes the use of ratios in the classical tradition, sometimes in the arithmetical tradition, a process which was extended practically until the 16th century, when conflicts between these two tendencies resulted in the disappearance of the tradition concerning the compounding ratios, and the consolidation of the arithmetical theory of ratios. During the 1500s, the process of arithmetization accelerated, and in the 17th century, the arithmetical theory of ratio became the dominant one.

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[33] M. Psellus, *Pselli, doctiss. uiri, perspicvvs liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia : Graece et latine nunc primùm editus* (Basileae: Oporinus, 1556).
[34] J. W. Herlinger, "Marchetto's Division of the Whole Tone", *Journal of the American Musicological Society* 34 (1981): 193-216, on 193.
[35] Erasmus Horicus, *Musica.* Vatican Library, MS Regina lat. 1245.