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# Einstein's redshift derivations: its history from 1907 to 1921

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## Abstract

Einstein's gravitational redshift derivation in his famous 1916 paper on general relativity seems to be problematic, being mired in what looks like conceptual difficulties or at least contradictions or gaps in his exposition. Was this derivation a blunder? To answer this question, we will consider Einstein's redshift derivations from his first one in 1907 to the 1921 derivation made in his Princeton lectures on relativity. This will enable to see the unfolding of an interdependent network of concepts and heuristic derivations in which previous ideas inform and condition later developments. The resulting derivations and views on coordinates and clocks are in fact not without inconsistencies. However, we can see these difficulties as an aspect of an evolving network understood as 'work in progress.'



## Keywords

Gravitational redshift; Einstein; General relativity


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## Introduction

After reaching his general relativistic field equations in late 1915, Albert Einstein (1879-1955) wrote a streamlined exposition of his new theory, in which he presented the derivations of three testable consequences of his theory.[1] We will consider in detail Einstein's derivations of one of them—the gravitational redshift.

In his detailed analysis of this work, Olivier Darrigol mentions, regarding Einstein's redshift derivation, that it "is problematic, because it relies on the coordinate-dependent notion that clocks slow down in an intense gravitational field."[2] By assuming that the rate of clocks is affected by the gravitational field within the context of his metric theories of gravitation, Einstein was lead to an inconsistency regarding the interpretation of coordinates, which should be 'just' labels without any metrical significance. However, as Darrigol calls the attention to, in his derivation "Einstein accords direct physical significance to the coordinate difference $dx^0$, despite his earlier insistence that coordinates are mere labels."[3, 4]

In the present article, we consider Einstein's redshift derivations from 1907, the date of his first redshift derivation using the equivalence principle,[5] to 1921.[6] This, in particular, sheds light on Einstein's 1916 derivation, which as we will see shares several

[1] Albert Einstein, "The Foundation of the General Theory of Relativity," *CPAE (English translation)* VI: 147-200 (1916). Abbreviations: CPAE—Albert Einstein, *The Collected Papers of Albert Einstein,* ed. John Stachel et al., 14 vols. (Princeton: Princeton University Press, 1987-).

[2] Olivier Darrigol, "Mesh and Measure in Early General Relativity," *Studies in History and Philosophy of Modern Physics* 52 (2015): 163-187, on 164.

[3] Ibid., 172.

[4] According to Einstein, Gaussian coordinates "by themselves have no physical meaning," Einstein, "On the Present State of the Problem of Gravitation," *CPAE (English translation),* IV: 198-222 (1913), on 211. In this way, the "coordinates are thereby reduced to intrinsically meaningless, auxiliary variables that can be chosen arbitrarily," Ibid., 213. Elsewhere, Einstein mentioned that "we refer the four-dimensional space-time continuum in an arbitrary manner to Gauss coordinates. We assign to every point of the continuum (event) four numbers $x_1$, $x_2$, $x_3$, $x_4$ (coordinates), which have not the least direct physical significance," Einstein, "On the Special and General Theory of Relativity," *CPAE (English translation),* VI: 247-420 (1917), on 349.

[5] In its earlier formulation, the equivalence principle entails that an inertial reference frame in a homogeneous gravitational field is physically equivalent to a uniformly accelerated reference frame in a space free of a gravitational field, see, e.g., Einstein, "On the Relativity Principle and the Conclusions Drawn from It," *CPAE (English translation),* II: 252-311 (1907), on 302. Einstein made an alternative formulation of the principle of equivalence in which we consider a four-dimensional space-time region in which special relativity is valid (i.e. in which we can adopt an inertial reference frame K); we then consider a second reference system K' uniformly accelerated with respect to K. According to Einstein, "nothing prevents us from considering [the] system K' as at rest, provided we assume a gravitational field (homogeneous in first approximation) relative to K,'" Einstein, "On Friedlich Kottler's Paper: 'On Einstein's Equivalence Hypothesis and Gravitation,'" *CPAE (English translation),* VI: 237-9 (1916), see 237-8. See also: John Norton, "What Was Einstein's Principle of Equivalence?," *Studies in History and Philosophy of Science Part A* 16 (1985): 203-46, on 205-6.

[6] We decided on 1921 not only because by this time Einstein was already moving his research interests into superseding general relativity, see, e.g., Einstein, "On a Natural Addition to the Foundation of the General Theory of Relativity," *CPAE (English translation),* VII: 224-8 (1921). But also because 1921 is the year when Einstein gave a series of lectures on special and general relativity at Princeton and wrote the corresponding text (which was published in 1922). This is the closest Einstein got to writing his own textbook on relativity. With his review paper from 1916 and his popular book published in 1917, these constitute the most detailed public accounts of general relativity written by Einstein.

presuppositions with Einstein's previous derivations. To enable the analysis of Einstein's derivations we first considered, in section 2, following John Earman and Clark Glymour, a formal redshift derivation.[7] We addressed Einstein's redshift derivations from 1907 to 1921 in section 3. We will see that Einstein's later redshift derivations were clearly informed by earlier derivations made using the equivalence principle, since they shared several presuppositions with them, the most important of which is the idea that the rate of clocks is affected by gravitational fields. This forced Einstein to give a direct physical meaning to the time coordinate, which is taken to express the effect of the field on a clock. What to make of this result? Were Einstein's derivations wrong? In section 4, we tried to make sense of Einstein's redshift derivations not as formal derivations, but as heuristic derivations made within the context of active research in field theory of which general relativity was 'just' a provisional 'stable' theory in the process of being superseded.

**A general relativistic redshift derivation**

In the present paper, we adopted Earman and Glymour's redshift derivation as our 'benchmark' in relation to which analyze Einstein's derivations.[8] The line element for the case of a static gravitational field (due to the Sun) is given by $ds^2 = -g_{\alpha\beta}dx_\alpha dx_\beta + g_{44}(dx_4)^2$ with $\alpha, \beta = 1, 2, 3$, and where $\alpha, \beta = 1, 2, 3$ can depend on $x_\alpha$ but not on $x_4$.[9] Locally, in a freely falling frame, ds is the proper time $d\tau$ measured by a clock at rest in the frame.[10] In this way we have $d\tau^2 = -g_{\alpha\beta}dx_\alpha dx_\beta + g_{44}(dx_4)^2$.

Let us consider two atoms "in the freely falling frames momentarily at rest in the gravitational field;"[11] we can imagine that one is momentarily at rest at the surface of the Sun (A) and the other momentarily at rest on Earth (B). Let $d\tau_{A,A}$ correspond to the proper time of a vibration of the atom located at the surface of the Sun; we can imagine $d\tau_{A,A}$ as corresponding to the time interval between two light signals sent from A as markers of the beginning and end of the vibration.[12] Since the atom is momentarily at rest at A we have $d\tau_{A,A} = \text{sqrt}(g_{44})_A(dx_4)_A$.

We want to determine the proper time interval $d\tau_{A,B}$ between the reception of the two light signals as measured by an atom on Earth corresponding to $d\tau_{A,A}$. This proper time

[7] John Earman, & Clark Glymour, "The Gravitational Red Shift as a Test of General Relativity: History and Analysis," *Studies in the History and Philosophy of Science* 11 (1980): 251-78.

[8] It has been noticed that in the literature there seems to be a good number of erroneous derivations of the redshift; see, e.g., Earman & Glymour; Robert Scott, "Teaching the Gravitational Redshift: Lessons from the History and Philosophy of Physics," *Journal of Physics: Conference Series* 600 (2015), 012055. According to Darrigol's assessment, contrary to Earman and Glymour's view, the early derivations by Weyl, Eddington and Laue are (in different degrees) correct derivations. In the present article we did not make a contextual analysis of Einstein's derivations in relation to derivations made by other physicists at the same time. We only mentioned these when considering that they might have had directly influenced Einstein's work. This kind of contextual analysis can be found, in relation to Einstein's 1916 derivation, in Darrigol.

[9] Earman & Glymour, 182.

[10] Ignazio Ciufolini, & John Wheller, *Gravitation and Inertia* (New Jersey: Princeton University Press, 1995), on 100. In the present article we adopted geometric units in which the velocity of light c = 1.

[11] Ciufolini & Wheller, 100.

[12] Earman & Glymour, 184.

interval is measured by another atom in free fall, momentarily at rest at B when the two light signals arrive. This means that $d\tau_{A,B} = \mathrm{sqrt}(g_{44})_B(dx_4)_B$. We now make the common approximation of considering the atoms 'at rest' in the gravitational field, i.e. instead of being in free fall along a geodesic, they have non-geodesic worldlines. In relation to an atom in free fall momentarily side by side with an atom at rest in the gravitational field, this atom suffers a time dilation; however, it is considered that this approximation renders "negligible special relativistic effects such as time dilation."[13] We then have "for two similar atoms "at rest" in the field,"[14] $d\tau_{A,A}/d\tau_{A,B} = \mathrm{sqrt}(g_{44})_A(dx_4)_A/\mathrm{sqrt}(g_{44})_B(dx_4)_B$. In the case of the static field, the coordinate system was chosen so that the coordinate time interval $(dx_4)_A$ is transmitted without change from the Sun to the Earth; in this way, $(dx_4)_A = (dx_4)_B$.[15] This implies that $d\tau_{A,B} = d\tau_{A,A}\,\mathrm{sqrt}(g_{44})_B/\mathrm{sqrt}(g_{44})_A$. We want to compare $d\tau_{A,B}$ with $d\tau_{B,B}$, which corresponds to the time interval between two light signals sent from B as markers of the beginning and end of the vibration (i.e., corresponds on Earth to the same process occurring on the Sun). At this point, we will take into account explicitly what Einstein called the "central presupposition of the whole theory;"[16] that is: "We assume that, if the clocks initially ticked at the same rate, then when they are again brought side by side, they again tick at the same rate (…) in other words, the ticking rate of a clock does not depend on what has happened to it in the past."[17]

This implies in the case under consideration that $d\tau_{A,A} = d\tau_{B,B}$. This means that $d\tau_{A,B} = d\tau_{B,B}\,\mathrm{sqrt}(g_{44})_B/\mathrm{sqrt}(g_{44})_A$. In terms of frequencies we have $\nu_{A,B} = \nu_{B,B}\,\mathrm{sqrt}(g_{44})_A/\mathrm{sqrt}(g_{44})_B$. This expression shows the relation between identical processes, one occurring on the Sun but as measured on Earth and an identical process occurring on Earth. According to the last expression, "[we] will see a shift towards the red in the spectral lines coming from atoms at [the Sun], as measured relative to the frequency of similar atoms at [the Earth]."[18]

We want to call the attention to the fact that in this derivation the coordinate time $dx_4$ does not appear in the final expression. It was chosen so that the frequency, as determined in terms of the coordinate time, of the light propagating in a static gravitational field does not change along the path of propagation, leading to the equality of $(dx_4)_A$ and $(dx_4)_B$.[19] The role of $(dx_4)_A$ and $(dx_4)_B$ is that of auxiliary variables not having any physical meaning regarding time intervals related to physical processes; these are determined in terms of measurements made with standard clocks; i.e., in terms of proper times. In this case we consider the proper times $d\tau_{A,A}$ and $d\tau_{B,B}$ of atoms at A and B, for identical physical processes, which are equal; and the proper time interval $d\tau_{A,B}$, which is meaningful because, implicitly, we already take into account 'from the start' that we can imagine having standard clocks everywhere and at any time.[20]

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[13] Harvey Brown, & James Read, "Clarifying Possible Misconceptions in the Foundations of General Relativity," *American Journal of Physics* 84 (2016): 327-34, on 329.

[14] Earman & Glymour, 183.

[15] Ibid., 184.

[16] *CPAE*, VII: doc. 31, note 19.

[17] Robert Geroch, *General Relativity: 1972 Lecture Notes* (Montreal: Minkowski Institute Press, 2013), on 8.

[18] Earman & Glymour, 185.

[19] This feature was mentioned by Weyl in 1918 and by Eddington and Laue both in 1920; see, e.g., Darrigol, 173-5.

[20] We can make physical sense of the expressions $d\tau_{A,A} = \mathrm{sqrt}(g_{44})_A(dx_4)_A$ and $d\tau_{A,B} = \mathrm{sqrt}(g_{44})_B(dx_4)_B$ and (mathematically) relate them, because we have the same time scale at A and B; in simple terms because standard clocks have the same rate, which implies, e.g., that $d\tau_{A,A} = d\tau_{B,B}$; or to say the same thing in a different way,

## Einstein's redshift derivations

In the 1916 review paper, Einstein presented a deduction of the redshift which was already a 'prediction' of his earlier approaches in terms of the equivalence principle. In his 1907 paper introducing the equivalence principle, Einstein deduced the redshift of light emitted by the Sun as measured on Earth, using a somewhat cumbersome procedure based on special relativity and the equivalence principle.[21] Einstein arrived at the following conclusion:

> "The process occurring in the clock, and, more generally, any physical process, proceeds faster the greater the gravitational potential at the position of the process taking place.
>
> There exist 'clocks' that are present at locations of different gravitational potentials and whose rates can be controlled with great precision; these are the producers of spectral lines [(i.e. atoms)]. It can be concluded from the aforesaid that the wave length of light coming from the sun's surface, which originates from such a producer, is larger by about one part in two millionth than that of light produced by the same substance on earth."[22]

Here, Einstein presented the redshift as a dynamical-like physical effect of the gravitational field on physical processes 'located' at a particular position in the gravitational field. He gave the example of clocks that are supposed to be affected by the gravitational field—atoms. Einstein addressed again the redshift in a 1911 paper, where he adopted another special relativistic approach also based on the equivalence principle.[23] Like in the previous derivation, the rate of the clock is affected by the gravitational field, which gives rise to the redshift.[24] In his subsequent work on a theory of a static gravitational field, Einstein did not mention explicitly the redshift, but mentioned the effect of the field on clocks, which according to his previous treatment causes the redshift: "A clock runs faster the greater the [gravitational potential] of the location to which we bring it."[25]

We see that in the context of an application of the equivalence principle or the scalar theory of gravitation, the redshift is due to the effect of the gravitational field on clocks (e.g. atoms) 'at rest' in the field. The main aspects of these derivations are then the following:

a) The gravitational field affects the rate of clocks (which leads to the redshift).

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because ds is invariant; see, e.g., Einstein, "Four Lectures on the Theory of Relativity, Held at Princeton University on May 1921," *CPAE (English translation)*, VII: 261-368 (1922), on 323.

[21] Einstein, "On the relativity principle and the conclusions drawn from it," on 302-5; see also Abraham Pais, *'Subtle is the Lord...' The science and the life of Albert Einstein* (Oxford: Clarendon Press, 1982), on 180-1; Earman and Glymour, 177-8.

[22] Einstein, "On the relativity principle and the conclusions drawn from it," on 307.

[23] Einstein, "On the influence of gravitation on the propagation of light," *CPAE (English translation), Vol. 3* (1911), 379-387.

[24] Ibid., 384-5.

[25] Einstein, "The Speed of Light and the Statics of the Gravitational Field," *CPAE (English translation)*, IV: 95-106 (1912), on 104.

b) Clocks are taken to be at 'rest' in the gravitational field (i.e., at rest in an inertial reference frame with a homogeneous gravitational field).

c) Atoms are an example of clocks affected by the gravitational field.

Einstein next approach to the redshift derivation was made already within the context of a metric theory of gravity—the Entwurf theory.[26] Einstein derived the effect of the gravitational field on clocks at rest in the field. Here, Einstein was already working with Gaussian or generalized coordinates. Accordingly, "the coordinates by themselves have no physical meaning."[27] In the Newtonian approximation, the line element ds is given by $ds = \mathrm{sqrt}(-dx^2 - dy^2 - dz^2 + g_{44}dt^2)$, where $g_{44} = c^2(1 - \kappa/4\pi \int\rho_0 d\nu/r)$.[28] Einstein called the attention to the fact that "the coordinate lengths are at the same time natural lengths (dt = 0); thus, measuring rods do not experience any distortion due to the 'Newtonian' gravitational field."[29] It is important to notice that here Einstein considered that measuring rods might be affected by the gravitational field. The situation regarding measuring clocks is similar, and in this case, there is a change in their rates: "The rate of a clock depends on the gravitational potential. For ds/dt is a measure of this rate if one sets dx = dy = dz = 0. One obtains $ds/dt = \mathrm{sqrt}(g_{44}) = \mathrm{const.}\ (1 - \kappa/8\pi \int\rho_0 d\nu/r)$. Thus the greater the masses arrayed in its vicinity, the slower the clock runs."[30]

Again, like in the pre-Entwurf derivations, a clock is at rest in the gravitational field and its rate is affected by the field. In this work, even if Einstein had mentioned that it might be possible arbitrary coordinate transformations,[31] we see that Einstein was actually working not with an arbitrary coordinate system, but with what he later called a reference mollusk (in this case clocks at rest in the gravitational field).[32] In this derivation, Einstein associated to the clock at rest in the gravitational field the coordinate time, to which was given a direct physical meaning as the time reading of the clock under the effect of the gravitational field.

Einstein's next explicit derivation of the redshift has the particularity that it was made within the context of an explicit adoption of Minkowski's concept of proper time.[33] Einstein considered the line element ds along the geodesic of a material point: "[ds] is the 'eigen-time'-differential, i.e., this quantity gives the amount by how much the clock-time of a

[26] Einstein, "Present State."
[27] Ibid., 211.
[28] Ibid., 216-8.
[29] Ibid., 218.
[30] Ibid.
[31] According to Einstein, "since we are in the dark about the class of admissible space-time substitutions, the most natural thing [...] is, at first, to consider arbitrary substitutions of the variables x, y, z, t", Ibid., 209.
[32] In his 1917 book, Einstein mentioned that, instead of using general Gaussian coordinates, we might for reasons of "comprehensibility" specialize to a particular type of coordinate system realized by non-rigid reference-bodies with attached clocks. Einstein named this particular case of Gaussian coordinates a "reference-mollusk;" Einstein, "On Special and General," 354.
[33] Previously, Einstein had already employed variables that we now identify with the proper time; see, e.g., Einstein, "Speed of Light," 104; and Einstein, "Present State," 201 and 218.

clock (which is associated with the moving material point) progresses along the path-element (dx, dy, dz)."[34]

In the Newtonian approximation, Einstein obtained for the line element $ds^2 = -dx^2 - dy^2 - dz^2 + (1+2\Phi)dt^2$, where $\Phi$ plays the role of the gravitational potential in this approximation. According to Einstein:

> "For purely temporal distance [i.e. with dx = dy = dz = 0] we had $ds^2 = (1+2\Phi)dt^2$ or $ds = (1+\Phi)dt$. To the naturally measured duration ds (the clock's proper time) belongs the time duration $ds/(1+\Phi)$ and, therefore, increases with the gravitational potential. One concludes from this that spectral lines of light, generated at the sun, show a redshift relative to corresponding spectral lines generated on earth, the shift amounting to $\Delta\lambda/\lambda = 2 \,.\, 10^{-6}$."[35]

In this passage, Einstein called ds the naturally measured time interval (duration) and identified it as the clock's proper time. Importantly, what increases with the gravitational field is the differential of coordinate time $dt = ds/(1+\Phi)$, which is associated to ('belongs' to) the same clock. It is the (physical) coordinate time that is affected by the gravitational field; this has as a physical consequence the redshift.

There are several related aspects in Einstein's derivations using the Entwurf theory. The three elements identified in the derivations previous to 1913 of the redshift are still present here:

a) The gravitational field affects the rate of measuring clocks (which leads to the redshift).

b) Clocks are taken to be at 'rest' in the gravitational field (i.e., at rest in relation to the masses generating the gravitational field).

c) Atoms are an example of clocks affected by the gravitational field (Einstein does not mention atoms explicitly but refers to spectral lines; i.e., atoms are implicit in Einstein's reasoning).

d) Einstein conflates in the same clock the naturally measured duration ds (i.e., the clock's proper time) and the time coordinate difference dt, which Einstein also calls the time coordinate measure.[36]

e) The coordinate time is the one affected by the gravitational field; it has a direct physical meaning as the time of a clock 'under' the effect of the gravitational field.

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[34] Einstein, "The Formal Foundations of the General Theory of Relativity," *CPAE (English translation),* VI: 30-84 (1914), on 32.

[35] Ibid., 82.

[36] Einstein, & Adriaan Fokker, "Nordström's Theory of Gravitation from the Point of View of the Absolute Differential Calculus," *CPAE (English translation),* IV: 293-9 (1914), on 298.

f) While, in principle, we have generalized time coordinates (time labels), in practice, a time coordinate is given/measured by a clock at rest in the gravitational field (i.e., it has a direct physical meaning, it is not a label).

Let us see some of the problems arising from this state of affairs:

1) If we consider a), c), d) and e) together, it implies that an atom with a proper time difference ds has the physically measurable coordinate time difference dt. We would be forced to conclude that atoms do not actually give/measure a proper time ds since atoms are taken to be affected by the gravitational field in a way that is measurable in terms of their coordinate time.

2) With e) and f), contradicting his view of coordinates as labels, Einstein gave a direct physical meaning to the coordinate time as the time given by the clock under the influence of the gravitational field.

Einstein's next redshift derivation was made in his review paper on the general theory of relativity from 1916. After arriving at the Newtonian approximation in his general theory of relativity, Einstein considered a unit measuring rod for which $ds^2 = -1$; for a particular choice of orientation we have $-1 = g_{11}dx_1^2$. According to Einstein, "the unit measuring-rod appears a little shortened in relation to the system of coordinates by the presence of the gravitational field, if the rod is laid along a radius."[37] For the case of a unit clock "arranged to be at rest in a static gravitational field,"[38] ds = 1. Therefore $1 = g_{44}dx_4^2$; and so $dx_4 = 1 - (g_{44} - 1)/2$. Accordingly: "The clock goes more slowly if set up in the neighborhood of ponderable masses. From this it follows that the spectral lines of light reaching us from the surface of large stars must appear displaced towards the red end of the spectrum."[39]

Again, like in the Entwurf theory, the standard or unit measuring clock is taken to be at rest in the gravitational field and being affected by it so that its physically meaningful coordinate time is changed in relation to its natural time duration ds. Again, we relate to the same clock (rod) ds and dt (dx); and it is the change in dt that gives rise to the redshift. The measuring clock is not considered as in 'free fall," having a proper time ds, but is treated as a clock from what later will be called the reference mollusk giving a coordinate time, to which is given a physical meaning. In fact, this derivation of the clock's rate is the equivalent in general relativity to that of the Entwurf theory.

Einstein's next derivation of the redshift was presented in his 'popular' exposition of the theory written by the end of 1916. In it, Einstein returned to a derivation based on the equivalence principle now applied to the case of a rotating disk. Einstein considered a disk rotating with a constant angular velocity ω (relative to an inertial reference frame K), which constitutes an accelerated reference frame K'. Einstein considered a clock located at a distance γ from the center of the disk; according to special relativity, the frequency of the clock (number of ticks of the clock per unit time) is $\nu = \nu_0(1 - \omega^2\gamma^2/2c^2)$, where $\nu_0$ is the

[37] Einstein, "Foundation of General Theory," 197.
[38] Ibid.
[39] Ibid., 198.

frequency of an identical clock at rest at the origin. Judged from K' the clock "is in a gravitational field of potential Φ,"[40] where $\Phi = -\omega^2\gamma^2/2$. From this result – taken to "hold quite generally"[41] and regarding "an atom which is emitting spectral lines as a clock"[42] – Einstein concluded:

> "An atom absorbs or emits light of a frequency which is dependent on the potential of the gravitational field in which it is situated.
>
> The frequency of an atom situated on the surface of a heavenly body will be somewhat less than the frequency of an atom of the same element which is situated in free space (or on the surface of a smaller celestial body) […] thus a displacement towards the red ought to take place for spectral lines produced at the surface of stars as compared with the spectral lines of the same element produced at the surface of the earth."[43]

It is interesting to notice that this derivation is posterior to the ones made in the context of the Entwurf theory and the derivation made in early 1916 using the general theory of relativity. Einstein went to and fro between derivations made with the equivalence principle and metric field theories as if the procedure adopted with the equivalence principle (and special relativity) was as valid as the one applied with the metric field theories (Entwurf theory and general relativity). This is made more intelligible if we remember that key elements are identical in the different approaches, as we have already seen.

Around the period from 1916 to 1918, there were several related conceptual elements of Einstein's views related to general relativity that were consolidating. These appeared mainly in Einstein's correspondence, before being taken into account in his published papers. In a letter to Michele Besso from October 1916, Einstein recognized that rods and clocks with a different "prehistory" were physically equivalent.[44] By this Einstein meant, as we can see from his later references to this point, that if we consider, e.g., two identical clocks that are subjected to a different physical situation (fields and accelerations)—i.e., a different prehistory—they are still congruent (have the same rate) when brought together. By this time, Einstein regarded this as a basic assumption also present in pre-relativistic physics, not mentioning it in his papers.[45] The advent of Hermann Weyl's (1885-1955) tentative field theory unifying electromagnetism and gravitation forced Einstein to consider this assumption in more detail. In 1918 Einstein presented his views in a series of letters and in a note on a paper by Weyl.[46] In a letter to Weyl from 19 April, Einstein linked the issue of

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[40] Einstein, "On Special and General," 389.
[41] Ibid., 389.
[42] Ibid., 389.
[43] Ibid., 389-90.
[44] *CPAE (English translation)*, VIII: 258.
[45] By this time Einstein had only mentioned in the context of special relativity the so-called boostability assumption; see Einstein, "On Relativity Principle," 260; and Einstein, "The Principle of Relativity and Its Consequences in Modern Physics," *CPAE (English translation)*, III: 117-42 (1910), on 130. see also Harvey R. Brown, *Physical Relativity: Spacetime Structure from a Dynamical Perspective* (Oxford: Oxford University Press, 2005), on 30.
[46] Einstein, "Nachtrag zu H. Weyl, Gravitation und Elektrizität," *CPAE*, VII: 61-2 (1918), 61-2.

the independence of the length of rods and rate of clocks of their prehistory to the existence of atoms:

> "If light rays were the only means of establishing empirically the metric conditions in the vicinity of a space-time point, a factor would indeed remain undefined in the distance ds (as well as in the $g_{\sigma\sigma}$'s). This indefiniteness would not exist, however, if the measurement results gained from (infinitesimal) rigid bodies (measuring rods) and clocks are used in the definition of ds. A timelike ds can then be measured directly through a standard clock whose world line contains ds.
>
> Such a definition for the elementary distance ds would only become illusory if the concepts 'standard measuring rod' and 'standard clock' were based on a principally false assumption; this would be the case if the length of a standard measuring rod (or the rate of a standard clock) depended on its prehistory. If this really were the case in nature, then no chemical elements with spectral lines of a specific frequency could exist, but rather the relative frequencies of two (spatially adjacent) atoms of the same sort would, in general, have to differ."[47]

Two atoms of the same chemical element always have the same spectral line when side by side, independently of their past history—they are stable; as such, the atoms, which are not described as a complex solution of general relativity, provide a standard for length and time duration that can be used to justify the invariance of ds.

In an (unpublished) paper from 1920, Einstein again presented the redshift of spectral lines using his derivation made in terms of a rotating frame K' and the application of the equivalence principle. Regarding two clocks, one located at the center of the disk, the other at its periphery, Einstein concluded "when judged from K' the two clocks are situated at different points in a gravitational field, and the latter is the cause that the two clocks run at different rates."[48] This example is considered as general enough to conclude that "a clock on the surface of a celestial body runs more slowly than the same clock […] when it sits on the surface of a smaller celestial body."[49] One example of a clock is "an atom that can emit or absorb a certain spectral line."[50] In this way, "when compared to the spectral lines generated by an element on earth the spectral lines generated or absorbed by the same element at the surface of the sun then must show a shift toward the red."[51]

In this paper Einstein only referred to the "independence of measuring rods and clocks from their past history"[52] in his treatment of special relativity. It is not clear in the text,

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[47] *CPAE (English translation)*, VIII: 533. See also Marco Giovanelli, "But One Must Not Legalize the Mentioned Sin: Phenomenological vs. Dynamical Treatments of Rods and Clocks in Einstein's Thought," *Studies in History and Philosophy of Modern Physics* 48 (2014): 20-44.

[48] Einstein, "Fundamental Ideas and Methods of the Theory of Relativity, Presented in Their Development," *CPAE (English translation)*, VII: 113-50 (1920), on 141.

[49] Ibid., 142.

[50] Ibid.

[51] Ibid.

[52] Ibid., 127.

as it is in his private correspondence, the relevance of this assumption (experimentally grounded on atomic stability) in the 'definition' of the invariant ds (in terms of measurements made with standard rods and standard clocks). Einstein did not mention, like in a letter from June 1920, that "with the interpretation of ds as a result of measurement that can be obtained in a very specific way by means of measuring rods and clocks, the theory of relativity stands and falls as a physical theory."[53]

By now atoms had a central role in the theory; they 'sustain' the notion of standard clock whose rate is independent of its past history. However, we still see that Einstein considered that atoms run slower at the surface of the Sun. From section 2 we know that this cannot be the case. To be rigorous, we should consider atoms in free fall, momentarily at rest at the surface of the Sun, when the light is emitted. However, since they are not affected by, e.g., an electromagnetic field or sudden accelerations (i.e., they are independent of their past history), we can make the approximation that when at 'rest' in the gravitational field the atom's proper time is equal to its proper time when momentarily at rest falling freely in the field (i.e., we disregard the time dilation). There is no effect of the gravitational field on the rate of the atom.

In a couple of letters from July 1920, Einstein presented a view that at first sight might seem to be altogether different from his previous considerations. This view might have been informed by Weyl's earlier considerations regarding the role of the time variable in a static gravitational field, which he called cosmic time, and his derivation of the redshift.[54] In the first letter, Einstein mentioned "in the general theory of relativity only ds is defined as a measurement result."[55] This much we already know from the first paper on the Entwurf theory.[56] Then Einstein wrote that "dt initially has a purely conventional meaning."[57] Using the phrasing that Einstein adopted in his papers, we could say that dt is an auxiliary variable or an arbitrary coordinate. The important part comes next:

> "In the consideration about line displacements, however, t again receives an absolute meaning in that the 4 coordinates are chosen so as to have the field of an isolated mass-point become static; thus, the number of wavelengths that are traveling between the sun and the observer cannot depend on t."[58]

According to Einstein, the time coordinate becomes a physical time due to a choice of the Gaussian coordinates for which the field is described as static and, in particular, the wavelength of light propagating through space does not depend on t. In the next letter,

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[53] *CPAE (English translation)*, X: 182.

[54] Weyl made a redshift derivation in the book based on his lectures on Einstein's theory held in 1917. Weyl sent to Einstein the drafts of his book in early 1918; see, e.g., Erhard Scholz, "Einstein and H. Weyl: Intertwining Paths and Mutual Influences," in D. Ria et al., eds., *Albert Einstein et Hermann Weyl, 1955-2005: questions épistémologiques ouvertes* (Manduria: Barbieri Selvaggi Editori, 2009), 215-30. In his derivation, Weyl used the notion of cosmic time and called the attention to the fact that "the light waves emitted from an atom, when measured in cosmic time, naturally ha[d] the same frequency everywhere," cited in Darrigol, 173.

[55] *CPAE (English translation)*, X: 210.

[56] Einstein, & Marcel Grossmann, "Outline of a Generalized Theory of Relativity and of a Theory of Gravitation," *CPAE (English translation)*, IV: 151-88 (1913), on 156-7.

[57] Ibid., 210.

[58] Ibid.

Einstein wrote: "I would like to give you the argument in detail again about the influence of the gravitational field on the clocks."[59] This sentence shows that Einstein still considered that the gravitational field affects clocks as in 1907. According to Einstein, "ds is the time measured by a standard clock at rest relative to the coordinate system."[60] Again, like in his equivalence principle derivations, the standard clock is simply at rest; Einstein did not mention the possibility of treating it as momentarily at rest, falling freely in the field. Regarding the time coordinate:

> "Time t is conventional; but in the static case, time t is universally given (with the exception of an additive correction) if it is given at one location. For in the static case, time t has physical meaning to the extent that it is chosen so as to make static or stationary processes appear respectively static or stationary. If I allow, e.g., monochromatic light to travel from the sun to the Earth, then the time ◎t for generating 100 oscillations is equal to the time interval Δt for receiving the 100 oscillations on Earth. With another time choice, the process of light propagation between Sun and Earth would not appear stationary."[61]

We see that in the case of a field generated by a single body we can choose a Gaussian coordinate system so that "all the components of the metric tensor are independent of the time coordinate;"[62] the gravitational field is said to be static. In this case, there is still some freedom regarding the adoption of the time coordinate; we can add to t an "arbitrary function of the space components."[63] After this t is "universally" determined everywhere if we define it at one location, i.e. "in such a field synchronization of clocks is possible over all space."[64] With this stipulation of a universal, cosmic, or world time, the frequency of light propagating in space as determined using the adopted time coordinate is a constant.

The fact that in the static field case we can synchronize clocks of a reference mollusk does not mean that we attribute a direct physical (metrical) meaning to the time coordinate. As noticed by Laudan and Lifshitz, "to the same interval of world time $x^0$ there corresponds, at different points of space, different intervals of proper time τ."[65] In the same vein, Reichenbach remarked "we may therefore say that it is impossible to define in a gravitational field a time coordinate that corresponds everywhere directly to the measure of time."[66]

In the letter, after the exposition of the notion of universal time, Einstein presented a redshift derivation. Einstein considered two standard clocks resting at two locations, location 1 corresponding to the Sun, and location 2 to a region of space-time were special

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[59] *CPAE (English translation)*, X: 223.
[60] Ibid..
[61] *Ibid.*, 224.
[62] Lev Landau, & Evgeny Lifshitz, *The Classical Theory of Fields* (Oxford: Pergamon Press, 1971), on 247.
[63] Ibid., 247.
[64] Ibid., 248; see also Hans Reichenbach, *The Philosophy of Space and Time* (New York: Dover Publications, 1958), on 259.
[65] Laudan & Lifshitz, 247.
[66] Reichenbach, 260.

relativity still applies. We then have $\Delta s_1 = \text{sqrt}(g_{44})_1 \Delta t_1$ and $\Delta s_2 = \text{sqrt}(g_{44})_2 \Delta t_2$. At location 2, $\Delta s_2 = \Delta t_2$ since $\text{sqrt}(g_{44})_2 = 1$. According to Einstein "in the conventional unit of time $\Delta t_2 = 1$, the standard clock hence strikes exactly once ($\Delta s_2 = \Delta t_2 = 1$)." The situation is different at location 1, on the Sun: "For the same conventional unit $\Delta t_1 = 1$, which according to the foregoing case of the static field has direct physical meaning, $\Delta s_1 = \text{sqrt}(g_{44})_1$. Hence, in the conventional unit of time, the clock makes less than one stroke (since $g_{44} < 0$)."[67]

This derivation is very similar to that of Weyl[68] and Max von Laue,[69] which according to Darrigol are basically correct derivations.[70] As we have seen in Earman and Glymour's derivation,[71] the rate of both clocks is the same as given by their identical proper times; also the time coordinate interval associated with the emitted light pulses does not change during the transmission from the Sun to the Earth; this is a contingency, resulting from considering a static field and taking advantage from that in choosing the Gaussian coordinates. What changes is the proper time interval measured on Earth corresponding to the proper time interval measured on the Sun (related to the emission of two light signals). That is, the rates of the atoms (standard clocks) are the same $d\tau_{A,A} = d\tau_{B,B}$ and the time coordinate intervals are also the same $(dx_4)_A = (dx_4)_B$. What is different is $d\tau_{A,A}$ (or $d\tau_{B,B}$) and $d\tau_{A,B}$. Einstein's reasoning is essentially the same. We have $(dx_4)_A = (dx_4)_B = \Delta t_1 = 1$, corresponding to the time coordinate interval associated with the light pulses emitted at the Sun. The proper time of the atom at the Sun emitting the light is $d\tau_{A,A} = \Delta s_1 = \text{sqrt}(g_{44})_1$. Measured on Earth, using an atom identical to the one on the Sun, the proper time interval associated with the light pulses is given by $d\tau_{A,B} = \Delta s_2 = \text{sqrt}(g_{44})_2 \Delta t_2 = 1$ (using Einstein's approximation that the Earth is in a flat region of space-time). From this, we arrive at the result $d\tau_{A,B} = d\tau_{A,A}/\text{sqrt}(g_{44})_1$ (corresponding to Earman and Glymour's $d\tau_{A,B} = d\tau_{A,A}\,\text{sqrt}(g_{44})_B/\text{sqrt}(g_{44})_A$). However, immediately after making a mathematically, if not conceptually, formally correct derivation, Einstein wrote "alternatively, one can also reason like this. If the clock strikes once on the Sun, $\Delta s = 1$, then the elapsed conventional time (which preserves the system's static character) $\text{sqrt}(g_{44})\Delta t_1 = 1$ [and so] $\Delta t_1 = 1/\text{sqrt}(g_{44}) > 1$."[72] This is the derivation made since the Entwurf theory up to the 1916 review paper on general relativity. As we have seen, implicit in this derivation is the idea that the clock's reading is given by t (not by the proper time τ). Einstein presents the two approaches as interchangeable.[73]

[67] CPAE *(English translation)*, X: 224.
[68] Hermann Weyl, Raum, Zeit, Materie: Vorlesungen über allgemeine Relativitätstheorie, 4th ed. (Berlin: Springer, 1921), on 223.
[69] Max von Laue, "Theoretisches über neuere optische Beobachtungen zur Relativitätstheorie," *Physikalische Zeitschrift* 21 (1920): 659-662, on 661-2.
[70] Darrigol, 173-5.
[71] Earman & Glymour.
[72] *CPAE (English translation)*, X: 224.
[73] It is important to notice that Einstein considered an alternative derivation of the redshift that relies on an "underlying physical mechanism" (i.e. the effect of the gravitational field on rods and clocks) present since his early heuristic derivations based on the equivalence principle. In fact, as already mentioned, Einstein wrote in the letter that he was going to give "the argument in detail again about the influence of the gravitational field on the clocks;" Ibid., 223. We can only call the first derivation 'formally correct' by abstracting from the context in which it was made—i.e. by adopting a presentist perspective that relies only on the sketchy mathematical aspects of the derivation.

That the derivation adopted in the letters did not imply a radical change (if any) in Einstein's approach to the issue of the redshift derivation can be seen in Einstein's lectures at Princeton.[74] In these lectures, Einstein clarifies one aspect of the redshift derivation: when speaking of clocks at rest in the gravitational field we refer to clocks momentarily at rest. First of all, we should consider clocks in free fall:

> "In the immediate neighborhood of an observer, falling freely in a gravitational field, there exit no gravitational field. We can therefore always regard an infinitesimal small region of the space-time continuum as Galilean. For such an infinitely small region there will be an inertial system (with the space coordinates $X_1$, $X_2$, $X_3$, and the time coordinate $X_4$) relative to which we are to regard the laws of the special theory of relativity as valid."[75]

We can extend this view to other local coordinate systems which are momentarily at rest relative to our local inertial reference frame, and at rest in relation to the adopted Gaussian coordinate system, in this way making it possible to consider clocks (momentarily) at rest in the gravitational field. According to Einstein:

> "The metrical relations of the Euclidean geometry are valid relatively to a Cartesian system of reference of infinitely small dimensions, and in a suitable state of motion (freely falling, and without rotation). We can make the same statement for local systems of coordinates which, relative to these, have small accelerations, and therefore for such systems of coordinates as are at rest relatively to the one we have selected."[76]

Regarding the issues of the meaning of Gaussian coordinates and the effect of the gravitational field on clocks, Einstein views did not depart much from his 1916 and earlier views. In the Newtonian approximation, Einstein's view regarding measuring rods is as follows:

> "The unit measuring rod has therefore the coordinate length, $1 - \kappa/8\pi \int \sigma dV_0/r$ in respect to the system of coordinates we have selected. This particular system of coordinates we have selected insures that this length shall depend only upon the place, and not upon direction. If we had chosen a different system of coordinates this would not be so. But however we may choose a system of coordinates, the laws of configuration of rigid rods do not agree with those of Euclidean geometry; in other words, we cannot choose any system of coordinates so that the coordinates differences, $\Delta x_1$, $\Delta x_2$, $\Delta x_3$, corresponding to the ends of a unit measuring rod, oriented in any way, shall always satisfy the relation $\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = 1$. In this sense space is not Euclidean but 'curved'."[77]

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[74] Einstein, "Four Lectures."
[75] Einstein, "Four Lectures," 322-3.
[76] Ibid., 350. Einstein mentioned local coordinate systems momentarily at rest in relation to the free-falling system; not that the free-falling system is momentarily at rest in the Gaussian coordinate system. But it seems that here Einstein was trying to provide a 'bridge' to consider standard clocks as momentarily at rest in the field.
[77] Ibid., 351-2.

The view expressed in this sentence is more elaborated than that of the letter; the generalized spatial coordinates do not attain an absolute meaning. It is clear that we could choose different auxiliary coordinates and that for these different coordinates the coordinate length would be different. In this way, the specific value of the coordinate length is not given a too direct metrical meaning. What it shows is that the disposition of rigid bodies is such that it does not correspond to a Euclidean geometry. In this sentence, it is not clear that Einstein ascribes this feature to a physical change in the measuring rod due to the gravitational field. But with respect to clocks, Einstein still maintains this view:

> "The interval between two beats of the unit clock (dT = 1) corresponds to the 'time' $1 + \kappa/8\pi \int \sigma dV_0/r$ in the unit used in our system of coordinates. The rate of a clock is accordingly slower the greater is the mass of the ponderable matter in its neighborhood. We therefore conclude that spectral lines which are produced on the sun's surface will be displaced toward the red, compared to the corresponding lines produced on the earth, by about $2 \cdot 10^{-6}$ of their wave-lengths."[78]

Here, we do not see any nuanced attitude regarding the generalized time coordinate as with the case of the spatial coordinates; this might be due to the specificity of the case under consideration, that of a static gravitational field. However, this is presented as an absolute result not dependent on the choice of coordinate system, and, importantly, the rate of the clock as given in coordinate time is affected by the gravitational field.

In the end of the day, even if we find nuanced views regarding the Gaussian or generalized coordinates—be it that we can have different coordinates implying differences regarding the coordinate length, or that in the static case time regains an 'absolute meaning'— Einstein maintained the view that a clock has its rate affected by the gravitational field; and this led Einstein to derivations of the redshift which were not without inconsistencies.

## Conclusion: Einstein's redshift derivations as heuristic derivations made in the context of 'work in progress'

None of Einstein's redshift derivations qualify as formal derivations; from our perspective, we must consider them as heuristic derivations. What to make of this result? Should Einstein not have made formal 'correct' derivations? In our view, we have to let go of the idea that there were heuristic derivations made in the context of discovery and that there should have been formal derivations made in the context of justification. According to Jürgen Renn and Tilman Sauer:

> "The triumph of November 1915 was hence not the victory of new concepts over old ones but just the temporary stabilization of a complex network made up of still largely traditional concepts, of Einstein's original heuristic

[78] Ibid., 352.

> arguments with only slight adjustments, and of unforeseen results on the level of the mathematical representation of the new theory […] Einstein's continued search for a realization of his original ideas after 1915, beyond the field equation of general relativity, provides in fact strong support in favor of this interpretation."[79]

Einstein's work on gravitation was, to him, a work in progress. After exploring for some years the Entwurf theory, Einstein found in late 1915 the field equations of general relativity. But even if we call general relativity a physical theory with all the explicit and implicit ideas we might have regarding what a physical theory is—in particular, the idea that is a finished *oeuvre*—that is not how Einstein, in practice, interacted with his theory: after 1915 Einstein did work that we can classify as developments made within general relativity (e.g., gravitational waves, cosmology, particles as singularities of the field, etc.),[80] but also worked on its extension/superseding by a theory unifying gravitation and electromagnetism and eventually providing a field description of matter (including the elusive quantum aspects).[81] In fact, in 1925, while working on unified field theory, Einstein wrote regarding his masterpiece that he had become "convinced that $R_{ik} - g_{ik}R/4 = T_{ik}$ is not the right thing."[82] In this context, of 'work in progress,' it might not make much sense to demand that Einstein had made formal derivations. If we look at Einstein's derivations within this context, we can see Einstein's derivations made within general relativity as adaptations of the previous derivations made using the Entwurf theory, which were conditioned by the early equivalence principle derivations. In fact, the 'underlying physical mechanism' leading to the redshift (i.e. the effect of the gravitational field on rods and clocks) is the same since the very first derivation in 1907. More than that, this 'mechanism' conditioned the use and interpretation of elements of Einstein's general relativity. The interplay of different elements in the derivations made in general relativity produced a more entangled situation with added inconsistencies, at the same time that conceptual clarifications of other elements occurred.

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79 Jürgen Renn, & Tilman Sauer, "Heuristics and Mathematical Representation in Einstein's Search for a Gravitational Field Equation," in H. Goenner, J. Renn, & T. Sauer, eds., *The Expanding Worlds of General Relativity* (Boston: Birkhäuser, 1999), 87-125, on 119.

80 See, e.g., Pais, 266-92.

81 Hubert Goenner, "On the History of Unified Field Theories," *Living Reviews in Relativity* 7 (2004): 2 doi: 10.12942/lrr-2004-2

82 *CPAE (English translation)*, XIV: 449.