Pure Alethic Modal Logic[1]

Lógica Modal Alética Pura

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Abstract: This paper deals with pure alethic modal logics, i.e. logics having only two connectives, possibility and necessity. The different articulations of these two modal operators is studied. These logics are presented from the viewpoint of the Polish framework of a structural consequence relation. Matrix semantics, bivalent non truth-functional semantics and sequent systems are also provided.

Keywords: Modal logic. Necessity. Possibility. Bivalent Semantics. Many-valued Semantics. Sequent Systems.

Abstract: *Este trabalho trata de lógicas modais aléticas puras, ou seja, tendo apenas dois conectivos, possibilidade e necessidade. As diferentes articulações desses dois operadores modais são estudados. Estas lógicas estão apresentadas na perspectiva Polonesa da relação de consequência estrutural. Semânticas matriciais, semânticas bivalentes não verofuncionais e sistemas de sequentes também são apresentadas.*

Palavras-chaves: *Lógica modal. Necessidade Possibilidade. Semânticas Bivalentes. Semânticas Polivalentes. Sistemas de Sequentes.*

"Purity and simplicity are the two wings
with which man soars above the earth"
Thomas à Kempis

Birth and baptism of PAM

The aim of this paper is to study pure modal logics, that is to say modal logics with modalities as the only connectives.

There is a huge variety of modal operators, negation can itself be considered as a modal operator. We will rather concentrate here on alethic modalities, i.e. possibililty and necessity. The study of such modalities can be considered as part of a generally study of pure unary zero-order logics. We have already developed part of such theory by studying logics of pure negation (BÉZIAU, 1994).

To develop a general study of unary zero-order logics, one can use the Polish framework of structural consequence relation as presented by Łos and Suszko

1 Dedicated to Lafayette de Moraes for his 80 birthday.

(1958). In the present paper for the study of pure modal logics we are mixing this framework with logical matrices, theory of valuation and sequent calculus, using therefore tools of universal logic.

One interest of studying pure modal logic is that it can be seen as a first step towards the construction of more complex logics. We can then combine pure modal logics with other logics such as logics of negation, logics of implication, etc, and also different modalities to get multimodal logics.

A study of pure alethic modal logic can be seen at first as quite simple and almost trivial. Simplicity is maybe the sister of triviality but strangely enough so is complexity, as it can be understood through the concept of maximalilty: a maximal non-trivial theory is a very strong theory at the edge of triviality. We must not be afraid of simplicity. It is important, as pointed out by Descartes, to have clear and distinct ideas, and to do so, decomposition and analysis is a way, leading to purity.

As we know with birthhood and baptism, to start it is good to give a name. A nickname is especially good, because it is handy and easy to remember. In this spirit we will call pure alethic modal logic PAM.

1. Pure alethic modal logic

1.1. The backbone of PAM

One picture is worth a thousand words says an old Chinese proverb. Let us therefore first contemplate the following picture which represents the basic relation between necessity, possibility and proposition. To fully understand this picture, one has to see that transitivity is implicit as well as negation: two arrows can be composed, and if a reverse arrow is not drawn, it means the relation does not hold. Arrows can be interpreted intuitively as meaning “there is a way”, it is not necessarily the way of material implication. This picture does not hold for all modalities, for example in deontic logic, one may sustain that obligation of voting does not lead to the polls.

$$\begin{tikzcd}
\boxed{\Box p} \arrow[d] \\
\boxed{p} \arrow[d] \\
\boxed{\Diamond p}
\end{tikzcd}$$

1.2. Consequence relation ∏PAM

A very simple and direct way to construct logics matching the backbone of pure alethic modal logic is to use the Polish framework.

We are using here the Polish framework of structural consequence relation. This means that we consider a consequence relation $\vdash$ defined on a set of formulas which is the absolutely free algebra generated from a set of atomic formulas with two unary functions: the connective of necessity and the connective $\Diamond$ of possibility. The consequence relation is a relation between theories (sets of formulas) and formulas. It obeys the three basic Tarskian axioms of reflexivity, monotony and transitivity, Moreover it is structural in the sense that: if $T \vdash F$ then $sub(T) \vdash sub(F)$, where $sub(T)$ and $sub(F)$ are the result of uniform substitutions of an atomic formulas p by any formula F (substitutions can be defined as endomorphisms of the algebra of formulas).

Let us consider the following axioms:

Axy$\Box$	For an atomic formula p,	$\Box p \vdash p$
Axy$\Diamond$	For an atomic formula p,	$p \vdash \Diamond p$
Axn$\Box$	For some formulas F and G,	$F \nvdash G$
Axn$\Diamond$	For some formulas F and G,	$\Diamond F \nvdash G$

We are using here the word "axiom" not in the sense of proof theory but in the sense of the theory of structures, i.e. like axioms for groups. We have two positive axioms and two negative axioms, but the negative axioms are not axioms of a proof system of refutation. These axioms do not define a logic, but a class of logics (logical structures). Logics of this class are pure alethic modal logics, PAMs.

The two negative axioms are useful to avoid circularity and trivial identification - a snake eating its tale - and consequently not to have anything in the head and in the foot of our backbone: we have $\Diamond F \nvdash \Box G$ for some formulas F and G, and also $\Box H$ and $\Diamond H$ are not in general tautologies and antilogies respectively.

Among PAMs there are logics such that $\Box p \vdash \Box\Box p$ and there are logics such that $\Box p \nvdash \Box\Box p$, so there is a wide diversity of PAMs. But in all PAMs we have, due to transitivity: $\Box p \vdash \Diamond p$ and due to structurality: $\Box\Box p \vdash \Box p$, $\Box\Diamond p \vdash \Diamond p$, $\Diamond p \vdash \Diamond\Diamond p$, $\Box p \vdash \Diamond\Box p$

We call $\prod$PAM the logic such that $T \vdash F$ holds iff it holds in all PAMs.

1.3. The Sequent system SPAM

We consider the system of sequents SPAM which has all the standard structural rules (including the cut-rule) and the following two rules (we are writing the rules without contexts and using the arrow not for the conditional, but for sequents, as it was originally used by Hertz and Gentzen):

$$\frac{p \rightarrow}{\Box p \rightarrow} \qquad\qquad \frac{\rightarrow p}{\rightarrow \Diamond p}$$

Here is an example of proof in this system:

$$\dfrac{\dfrac{p \to p}{p \to \Diamond p}}{\Box p \to \Diamond p}$$

Using such system, we define a consequence relation (and the correlated logical structure) as follows: $T \vdash F$ iff there are formulas $G1, \ldots, Gn$ of T such that $G1,\ldots,Gn \to F$ is provable in SPAM.

The above proof shows that $\Box p \vdash \Diamond p$ holds in this logic. Note that we didn't use the cut-rule to prove that $\Box p \to \Diamond p$ is derivable in SPAM. It is easy to show that SPAM enjoys cut-elimination.

The logic generated by SPAM is a pure alethic logic. Using cut-elimination, it is possible to see that $p \to \Box p$ and $\Diamond p \to p$ are not derivable in SPAM. It is also possible to see that $\Box p \to \Box\Box p$ is not derivable in SPAM.

The logic generated by SPAM is in fact ∏PAM.

1.4. Bivalent semantics

We now consider bivalent semantics BIV given by the following conditions:

For any element *bi* of BIV (functions from the set of formulas to {0,1}), we have :

if $bi(F)=0$ then $bi(\Box F)=0$

if $bi(F)=1$ then $bi(\Diamond F)=1$

Such semantics is a da Costa-type semantics, it is non truth-functional and the bivaluations are not defined starting with distributions on atomic formulas (see DA COSTA and BÉZIAU, 1994). With such semantics one can define a consequence relation in the usual way: a formula F is a semantical consequence of a theory T iff when all the values of formulas in T are 1 the value of F is 1.

As da Costa and his team have shown (see DA COSTA and ALVES, 1977), it is also possible to draw for this kind of semantics some truth-tables. We have for example the following tables which correspond to the two above conditions:

p	$\Box p$
0	0
1	0
1	1

p	$\Diamond p$
0	0
0	1
1	1

The two tables can be synthesized in the following table:

$\Box p$	p	$\Diamond p$
0	0	0
0	0	1
0	1	1
1	1	1

We can also draw the two following tables which show that in the logic generated by this semantics $\Box p \nvdash \Box\Box p$ and $\Diamond\Box p \nvdash \Box p$.

p	$\Box p$	$\Box\Box p$
0	0	0
1	0	0
1	1	0
1	1	1

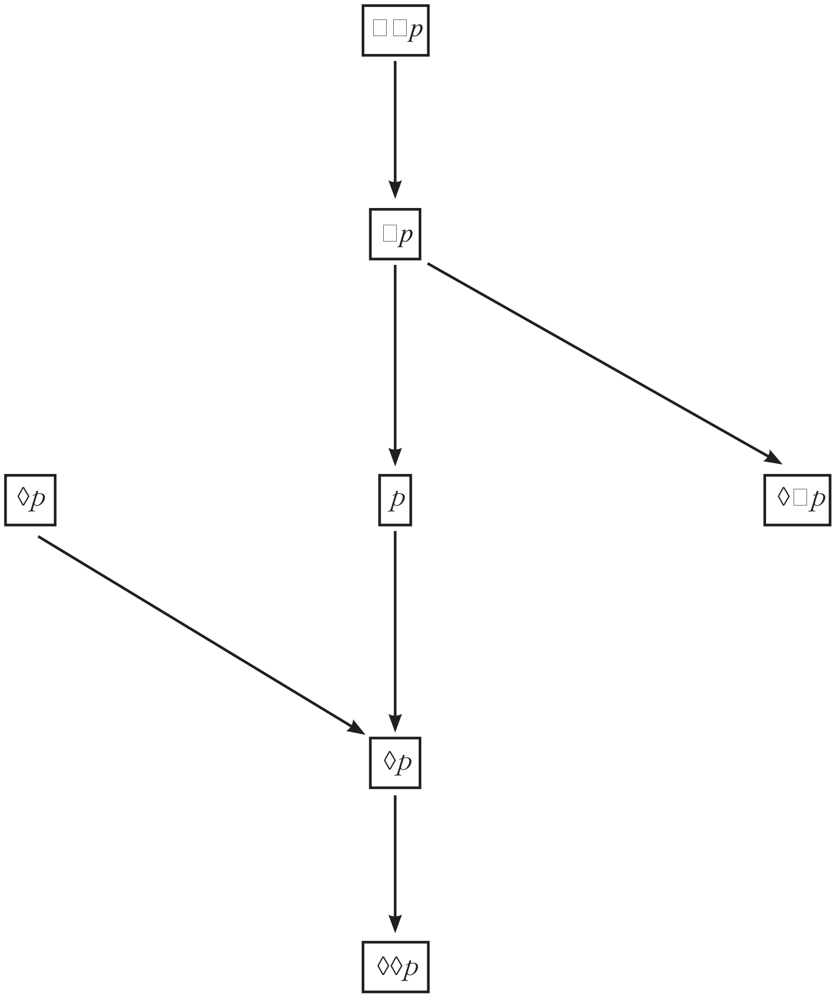
p	$\Box p$	$\Diamond\Box p$
0	0	0
0	0	1
1	0	0
1	0	1
1	1	1

The logic generated by this bivalent semantics is a PAM, in fact this logic is $\prod$PAM. It is easy to show this using the general completeness theorem presented in (BÉZIAU, 2001).

The following table presents a general picture of the relation between modalities in $\prod$PAM:

$\Box\Diamond p$	$\Box p$	p	$\Diamond p$	$\Box\Diamond p$
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
1	0	0	1	0
0	0	0	1	1
1	0	0	1	1
0	0	1	1	0
1	0	1	1	0
0	0	1	1	1
1	0	1	1	1
0	1	1	1	1
1	1	1	1	1

This can be described with the following diagram which furthermore describes the positions of repetitions of modalities:



The skeleton of PAM - pure alethic modal logic

1.5. Logical matrices

Now we will consider logical matrices. A logical matrix for a given logic is an algebra ALG of similar type as the algebra of formulas, together with a subdomain of ALG called the set of designated values.

It is easy to show that bivalent and trivalent semantics cannot do the job for pure alethic modal logic, i.e. they cannot be used to characterize such logics (BÉZIAU, 2011). But four-valued logical matrices can do the job. We consider four-valued matrices with two non-designated values 0- and 0+, and two designated values 1- and 1+. The names of these values, supporting an intuitive interpretation, are the following:

0-	Necessarily false
0+	Possibly false
1-	Possibly true
1+	Necessarily true

We use the notation 0 for non-designated values and 1 for designated values. The following table describes therefore a class of 256 logical matrices:

□***F***	***F***	◊***F***
0	0–	0
0	0+	1
0	1–	1
1	1+	1

Which logical matrix among these 256 matrices does define ∏PAM, if any?

Consider the following matrix:

◊□***F***	□***F***	***F***	◊***F***
0-	0–	0–	0–
0-	0–	0+	1–
0-		1–	1+
1+	1+	1+	1+

In the logic characterized by this matrix we have ◊□*F*├*F and F*├◊ *F* as shown by the following table:

◊□***F***	***F***	***F***	◊***F***
0-	0–	0–	0–
0-	0–	0+	1–
1-	1+	1–	1+
1+	1+	1+	1+

So the logic defined by this table is not ∏PAM.

∏PAM is in fact what holds in all matrices. At first logical matrices are not necessarily interesting, since we can characterize PAM with a bivalent semantics. It is nice to shave additional values with Ockham's razor, but in the next section we will show how these additional values can be useful and nice to clearly explain reduction of modalities, reduction which can seduce an Ockham's aficionado.

2. Pure reductive alethic modal logics

In a PAM there are infinite different modalities. Are they all interesting and meaningful? It is not obvious. One may want to reduce these modalities and there are different ways to do so.

2.1. Reduction of repetition PRRAM

One typical feature one may want to have is $\Box p \vdash \Box\Box p$ and $\Diamond\Diamond p \vdash \Diamond p$, so that we have $\Box p \equiv \Box\Box p$ and $\Diamond p \equiv \Diamond\Diamond p$ where $\equiv$ means logical equivalence.

From the point of view of sequent system, we just have to add the two following rules:

$$\frac{\rightarrow \Box p}{\rightarrow \Box\Box p} \qquad \frac{\Diamond p \rightarrow}{\Diamond\Diamond p \rightarrow}$$

The semantics of bivaluation describes by the following table is a sound and complete semantics for this logic we call PRRAM:

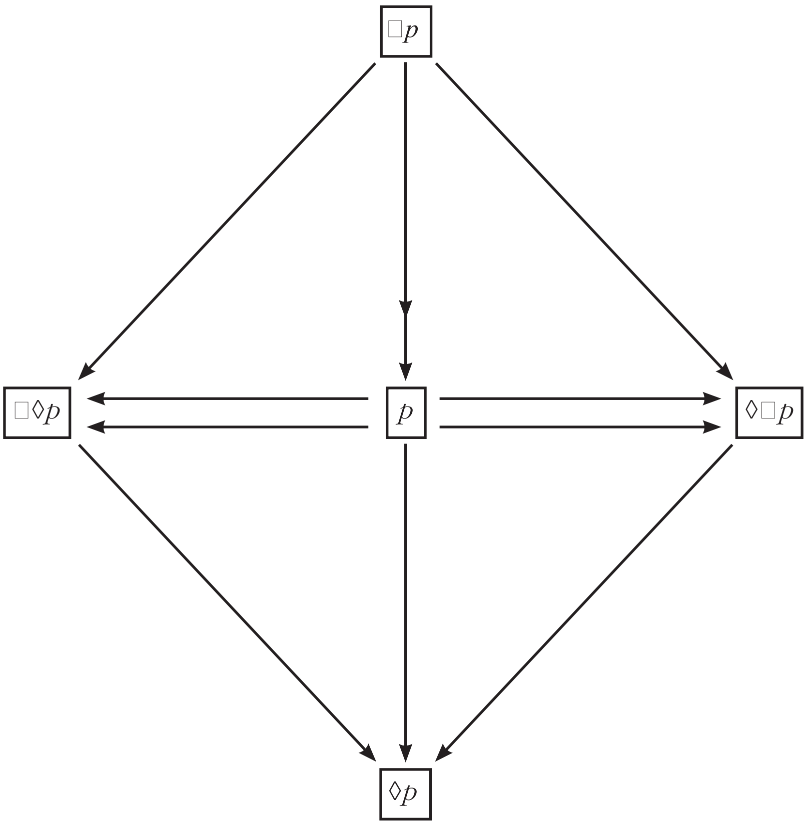
$\Box\Box p$	$\Box p$	p	$\Diamond p$	$\Diamond\Diamond p$
0	0	0	0	0
0	0	0	1	1
0	0	1	1	1
1	1	1	1	1

From the viewpoint of our four-valued matrix semantics, the following table gives necessary and sufficient conditions:

$\Box\Box p$	p	$\Diamond p$
0	0–	0–
0	0+	1
0	1–	1
1+	1+	1

2.2. Central collapse: PCCAM

Even reducing repetition of modalities, we still have 5 modalities considering that the absence of modality is a modality. There are basically two ways to reduce these 5 modalities to 3 modalities. The first method is the central collapse described by the following picture:

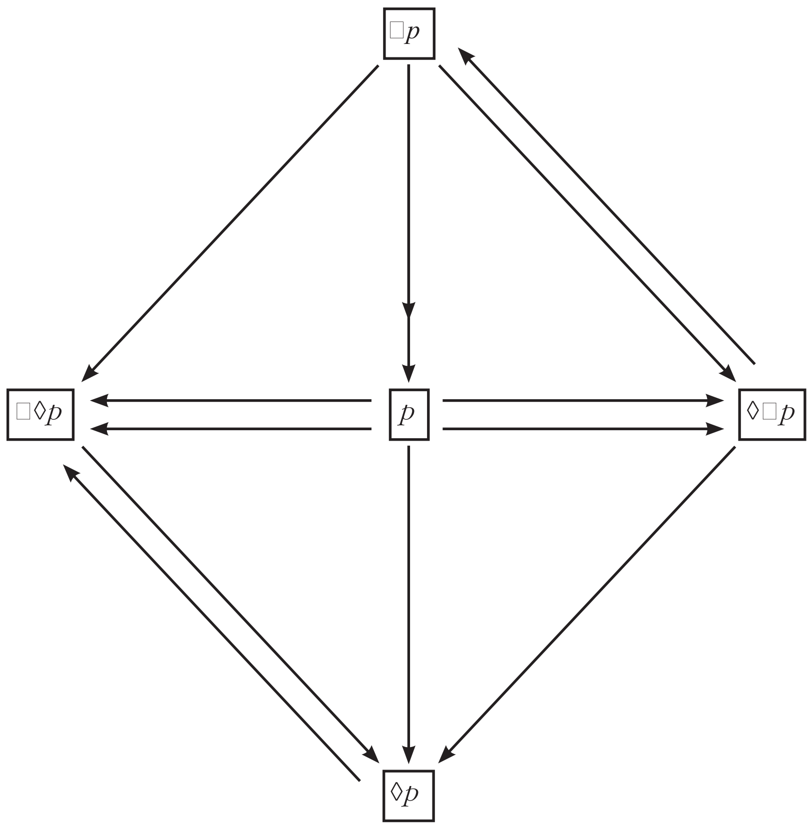


This corresponds to the following table defining a class of four-valued matrix semantics:

□◊p	**□p**	***P***	**◊p**	**◊□p**
0-	0-	0-	0	0
0+	0-	0+	1-	0
1	0+	1-	1+	1-
1	1	1+	1+	1+

2.3. Lateral collapse: PLACAM

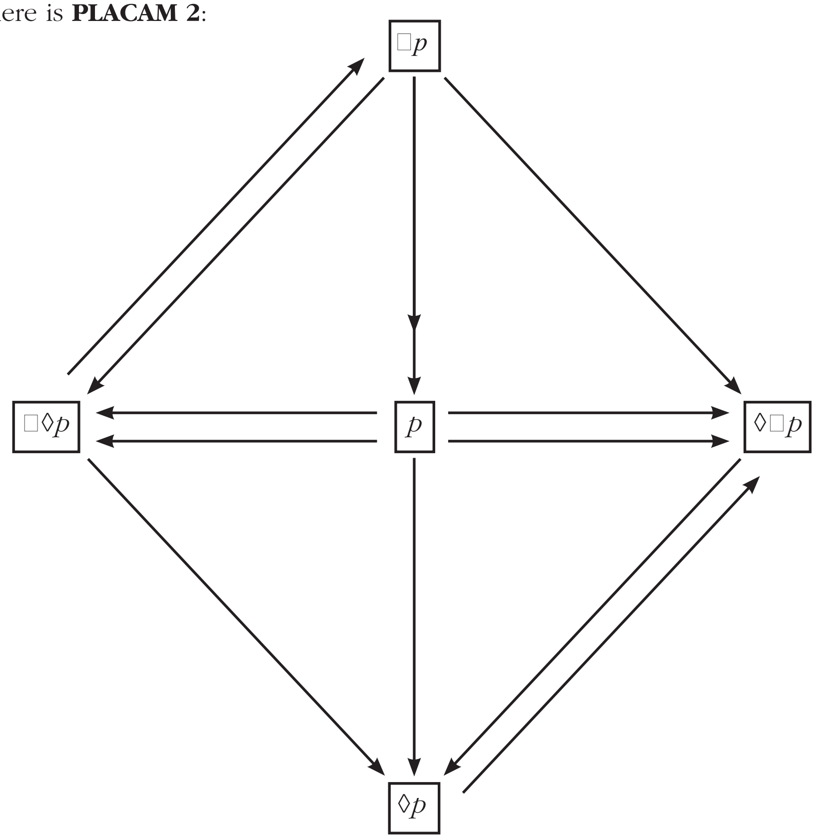
The second way to reduce the 5 basic modalities to 3 is the lateral collapse which has two versions dual of each other, here is **PLACAM 1**:



This corresponds to the following table defining a class of four-valued matrix semantics:

◊*p*	*p*	*P*	◊*p*	◊□*p*
0	0-	0-	0	0-
1+	0-	0+	1+	0-
1+	0-	1-	1+	0-
1+	1	1+	1+	1

And here is **PLACAM 2**:



This corresponds to the following table defining a class of four-valued matrix semantics:

◊p	**p**	***P***	**◊p**	**◊□p**
0	0-	0-	0	0
0	0+	0+	1-	1
0	0+	1-	1-	1
1	1	1+	1+	1

For central collapse and lateral collapse we can also easily built sequent systems and bivaluation semantics.[2]

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References

BÉZIAU, J.-Y. (1994). "Théorie Législative de la Négation Pure", *Logique et Analyse*, 147-148, pp. 209-225.

_____. (2001). "Sequents and Bivaluations", *Logique et Analyse*, 44, pp.373-394.

_____. (2005). "Les Axiomes de Tarski", in La philosophie en Pologne 1918-1939. R. Pouivet and M. Rebuschi (eds). Paris: Vrin, pp.135-149.

_____. (2011). "A New Four-valued Approach to Modal Logic", in *Logique et Analyse*, 54, pp.18-33.

_____. (2012). "History of truth-values", in D.M.Gabbay and J.Woods (eds). *Handbook of the History of Logic , Vol. 11 - Logic: a history of its central concepts*, Elsevier: Amsterdam, 2012, pp.233-305.

COSTA, N.C.A. da & ALVES, E.H. (1977). "A Semantical Analysis of the Calculi Cn". *Notre Dame Journal of Formal Logic,* 16, pp.621-630.

COSTA, N.C.A. da & BEZIAU, J.-Y. (1994). "Théorie de la Valuation". *Logique et Analyse*, 146, pp. 95-117.

LOS, J. & SUSZKO, R. (1958). "Remarks on Sentential Logics". *Indigationes Mathematicae*, 20, pp. 177-183.

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