Scotistic Structures

Estruturas Scotisticas

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Abstract: This essay ties together two strands of Peircean exegesis: Hookway's suggestion that Peirce's mathematical ontology is akin to contemporary structuralism, and the interpretation of his realism as a transmogrified Scotism. I show how Peirce's critical appropriation of scholastic realism plays out in his fragmentary but highly suggestive remarks on the nature of mathematical objects. Though much reconstruction is required, I argue that these remarks point clearly towards a version of mathematical realism (and structuralism) that is tightly integrated into the larger framework of Peirce's mature realism.

Key words: Abstraction. Mathematics. Scholastic realism. Structuralism.

Resumo: *Este ensaio une duas linhas da exegese Peirciana: a sugestão de Hookway de que a ontologia matemática de Peirce é similar ao estruturalismo contemporâneo, e a interpretação do seu realismo como um Scotismo metamorfoseado. Demonstro como a apropriação crítica de Peirce do realismo escolástico se desenvolve em suas observações fragmentárias, porém altamente sugestivas, sobre a natureza dos objetos matemáticos. Embora uma reconstrução significativa seja necessária, argumento que essas observações apontam claramente para uma versão do realismo matemático (e estruturalismo) que está firmemente integrada ao arcabouço maior do realismo maduro de Peirce.*

Palavras-chave: *Abstração. Estruturalismo. Matemática. Realismo escolástico.*

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In a number of texts spanning the years from 1897 to 1906, Peirce articulates a conception of mathematical objects which is both broadly Scotistic and closely akin (as Christopher Hookway has argued, in a forthcoming essay, to which this one is much indebted)[2] to what we nowadays call mathematical structuralism. Those texts include “Recreations in Reasoning” (PEIRCE, 1897b); a discarded draft of the second Harvard lecture on pragmatism (PEIRCE, 1903e); the fifth Lowell lecture (PEIRCE, 1903d) from the same year; a draft of the third (PEIRCE, 1903c); and “Prolegomena to an Apology for Pragmaticism” (PEIRCE, 1906). I will refer to these as *RR*, *PT*, *LF*, *LT* and *PA*, respectively.[3]

When I call Peirce’s conception *broadly* Scotistic I mean, as John Boler says of Peirce’s realism, that it is Scotistic “not for working out from Scotus’ conclusions, but for adapting the framework of the solution” (BOLER, 1963, p. 65). After spelling this out in some detail, I will turn to the affinities with structuralism, and argue that Peirce’s proto-structuralism is of more than historical interest for the philosopher of mathematics.

The scholastic and Aristotelian vocabulary flies pretty thick and fast in these texts. In PT (p. 161), for example, Peirce writes that “an abstraction is something denoted by a noun substantive [...] and therefore, whether it be reality or a figment it belongs to the category of substance.” Soon thereafter he denies that “an abstraction [is] an ordinary primary substance [...] you couldn’t load a pistol with dormitive virtue and shoot it into a breakfast roll” (p. 162) and goes on to characterize dormitive virtue (the soporific power of opium) in terms redolent of immanent Aristotelian forms: “it is wholly and completely in every piece of opium in Smyrna.” He uses similarly evocative language in connection with a directly mathematical example in *PA* (p. 530): “[an] algebraic diagram [. . .] presents to our observation the very, identical object of mathematical research.” This is by analogy with “Molecular Structure, which in all

2 Hookway’s paper is a contribution to a collection of new essays on Peirce’s philosophy of mathematics, forthcoming under my editorship from Open Court. I have had the great (and arguably unfair) advantage of reading those papers in advance of their public appearance. In connection with the present project I want to record my particular indebtedness to Claudine Tiercelin, whose essay builds on and deepens the insights of (Tiercelin 1993); to Susanna Marietti, Daniel Campos and Elizabeth Cooke, who taught me much about the diagrammatic nature of mathematical reasoning on Peirce’s view; and to Sun-joo Shin’s paper on hypostatic abstraction.

3 Citations to *RR* and *PA* will be by paragraph number to their reprintings in Volume 4 of the *Collected Papers* (PEIRCE, 1931-1958); citations to *PT*, *LT* and *LF* will be by page number to their reprintings in Volume 3 (*LT* and *LF*) and 4 (*PT*) of *The New Elements of Mathematics* (PEIRCE, 1976). Other citations to Peirce’s writings will be, in each case, to the best readily available edition. References to the *Collected Papers* will have the form ‘C<volume>.<paragraph>’: e.g., ‘C6.164’ refers to paragraph 164 of Volume 6. References to other standard editions will be by volume and page number, but otherwise the general scheme is the same, using the following codes: ‘E1’ and ‘E2’ for the first (PEIRCE, 1992) and second (PEIRCE, 1998) volumes, respectively, of *The Essential Peirce*; and ‘N’ for *The New Elements of Mathematics*. These codes will be used in the reference list to give abbreviated bibliographic data for these reprintings. Where appropriate I will also give, in the reference list, the manuscript number in Robin’s catalog (ROBIN, 1967) or its supplement (ROBIN, 1971).

[the chemist's] samples has as complete an identity as it is in the nature of Molecular Structure ever to possess." He tops this with a sprig of scholastic subtlety: "do not let me be understood as saying that a Form possesses […] Identity in the strict sense […] that is, what the logicians…call 'numerical identity'."

The specifically Scotistic influence in these texts is most strongly evidenced by Peirce's subtle handling of the terms 'abstraction' and '*ens rationis*' (being of reason). He defines the former in *PT* as "a substance whose being consists in the truth of some proposition concerning a more primary substance" (p. 162). This is clearly a variety of what he usually calls

> hypostatic abstraction, [which …] consists in taking a feature of a percept or percepts…so as to take propositional form in a judgment […] and in conceiving this fact to consist in the relation between the subject of that judgment and another subject which has a mode of being that merely consists in the truth of propositions of which the corresponding concrete term is the predicate. Thus, we transform the proposition, 'Honey is sweet' into 'honey possesses sweetness'. (PEIRCE, 1902, C4.235)

Peirce's favorite example of hypostatic abstraction is the "dormitive virtue" of opium, from Molière's *Le malade imaginaire*. Here the grammatical starting point is not an adjective but the description of a class of phenomena — the falling asleep of those who have recently taken opium — to which the entity denoted by the abstract term bears the uniquely identifying relation of being their cause (cf. SHORT, 1988, p. 54). Peircean abstraction, which enables us to talk about properties and relations as if they were things, is thus similar in function to Scotistic abstraction, whereby the Common Nature "in the mind is given a numerical unity so that it can be predicated as one thing of many things" (BOLER, 1963, 61).

Peirce tells us that "an abstraction […] is an ens rationis," that is, "a creation of thought" (*LF*, 367). A few lines up from there he gives a general definition of ens rationis that is virtually identical to the definition of 'abstraction' in *PT*: "a subject whose being consists in a Secondness, or fact, concerning something else." Peirce was well aware that for the scholastics '*ens rationis*' marked one side of what Boler calls "the basic distinction between existence outside the mind […] and existence within the mind" (Boler 1963, 42). So in using these two terms interchangeably[4] he

4 For evidence of the interchangeability, see (ZEMAN, 1983, p. 299). Short (1988, p. 55, n. 4) maintains that in the discussion of dormitive virtue in *PT* Peirce ought to have distinguished the two notions, and more generally that the distinction is crucial to a clear understanding of how abstractions work in empirical science. Certainly Peirce does not tell the whole story in *PT*; I try to tell a little more of it below (p. 14ff), with particular attention to the distinction, highlighted by Short, "between *entia rationis* and physically efficacious generals." This is not the place to try to sort all of that out, but let me say for the record that while I think Short is basically right, I suspect that his insights can be preserved in a formulation that does not distinguish, as Peirce does not, between abstractions and *entia rationis*.

was inviting a nominalistic dismissal of abstractions as mere creations of thought.5 In *PT* (163) he points out that for a believer in atoms this dismissal results in the dismissal of macroscopic physical objects.[6] But his primary, and deeper, line of defense is a rejection of the very notion of reality that underlies the nominalistic view. In *PT* that rejection is portrayed, suitably to the context, as a corollary of pragmatism: "[on] pragmatistic principles *reality* can mean nothing except the *truth* of statements in which the real thing is asserted."[7] In *LT* he says, without invoking pragmatism by name, that though a collection "is an ens rationis […] that reason or ratio that creates it may be among the realities of the universe" (LT, 353).

Here we see Peirce pushing his conception of mathematical objects right up to the brink of nominalism, and then resolutely refusing to push it over the edge. This brinksmanship is itself a clue, to a reader with some awareness of Peirce's dialogue with scholasticism, that Scotus is making some kind of contribution. Recall the well-known encomium in the review (PEIRCE, 1871) of Fraser's edition of Berkeley (*RFB*

5 In the version of this paper that I read in São Paulo I underestimated the force of the invitation. This was subsequently brought home to me by Fred Michael's criticism of Murphey (1961, p. 132) for failing to recognize that "the view that universals are mere *entia rationis*, beings of reason, was the position of the nominalists, not of realists such as Scotus" (MICHAEL, 1988, p. 339).

6 Peirce would reject this as nominalistic, not only in its attitude towards abstractions, but also in its obliviousness to the indispensable reference to Thirdness (law) in an adequate metaphysics of individuals: see, e.g., (BOLER, 1963, p. 138-143) and (MAYORGA, 2007, p. 130-141). Moreover, as Zeman persuasively argues (ZEMAN, 1983, p. 297-299), what Peirce deems an abstraction is heavily context-dependent, so he would be unlikely to take macroscopic bodies to be abstractions in any absolute sense. The definition of abstraction in PT has a decidedly relativistic ring to it: Peirce talks not of primary substances *simpliciter* but of *more* primary substances, and remarks that the existence of absolutely "primary substance [… is something] we may leave the metaphysicians to wrangle about" (PT, 162).

7 The likeness to one of Peirce's earliest positive references to scholastic realism is striking, and surely no accident: "a realist is simply someone who knows no more recondite reality than that which is represented in a true proposition" (PEIRCE, 1868, E1.53). The excerpted portion of *PT* is worth quoting more fully:

> But the question is whether an abstraction *can* be real. For the moment, I will abstain from giving a positive answer to this question; but will content myself with pointing out that upon pragmatistic principles an abstraction may be, and normally will be, *real*. For according to the pragmatistic maxim this must depend upon whether all the practical consequences of it are true. Now the only practical consequences there are or can be are embodied in the statement that what is said about it is *true*. On pragmatistic principles *reality* can mean nothing except the *truth* of statements in which the real thing is asserted. To say that opium has a dormitive virtue means nothing and can have no practical consequences except what are involved in the statement that there is some circumstance connected with opium that explains its putting people to sleep. If there truly be such a circumstance, that is all that it can possibly mean, — according to the pragmatist maxim, — to say that opium really has a dormitive virtue (*PT*, 161-162).

from now on),[8] that Scotus "was separated from nominalism only by the division of a hair" (*RFB*, 87). Nominalism was not just a whipping-boy for Peirce. The positive role of the nominalist tradition in the development of Peirce's thought, and even in his mature realism, has been insightfully explored by Fisch (1967), Boler (1983), Michael (1988), Mayorga (2007, p. 73-89) and others. I submit that in his philosophical analysis of mathematical ontology, as in his philosophical analysis of generality, Peirce strives for a realism that gives nominalism its due, and that in the latter instance as in the former, the "framework of the solution" conforms in important respects to the pattern of Scotus's barely realistic realism.

There can be no doubt that Peirce is deliberately echoing Scotus's doctrine of the Common Nature when he denies in *PA* that "the very, identical object of mathematical research" has "numerical identity." When he observes in *PT* that one can't shoot an abstraction into a breakfast roll, he is making the same point in more colorful and less technical terms. But in both places the stout denial of numerical identity qualifies an equally stout affirmation that the "object of mathematical research" is the same in its several instantiations. Also in both places he makes much of the analogy between the identity of mathematical objects and that of chemical structures, writing in *PT* (162) that dormitive virtue is "wholly and completely in every piece of opium in Smyrna." So there is a real identity of structure across multiple instances, and it is in just such a paradox-prone environment that Scotus's Common Nature, with its real but less than numerical unity, finds a home.

Peircean mathematical objects, like Scotistic Common Natures, are thus not really objects, as they are for a Fregean or Quinean platonist, who regards a number as no less an object than a table or a chair, though of course unlike such concreta in having no spatiotemporal location. Peirce and Scotus both cut the Quinean tie that binds entity to identity, as they must do to make their respective theories go. The heavy doses of Scotistic phraseology very efficiently signal, then, the distinctive brand of mathematical realism that Claudine Tiercelin (1993) has ascribed to Peirce, non-platonistic and yet steering clear (by the narrowest of margins) of both conceptualism and conventionalism. Here we must take care not to be misled by the way philosophers of mathematics now customarily define 'platonism' and 'nominalism', as belief as disbelief, respectively, in abstract (non-physical, non-mental) objects. With those definitions in mind, we might all too easily reason that it is the rejection of platonism that brings Scotus within a hair's breadth of nominalism, and so that when Peirce uses the nominalistic catch-phrase '*ens rationis*' in his discussions of mathematical ontology he is simply aligning himself with his subtle forbear on that point. But Scotus and Ockham both reject Platonic forms, not because they both incline (to varying degrees) towards nominalism, but because they are both Aristotelians. And in *RFB* Peirce holds Berkeley up as a particularly striking example of "that strange union of nominalism with Platonism, which has […] been such a stumbling-block to the historians of philosophy" (*RFB*, 85). So Scotus's praiseworthy closeness to nominalism cannot consist in his anti-Platonism.

In that review Peirce takes particular pains to get to the heart of nominalism

8 Page references are to the reprinting in the second volume of *The Essential Peirce* (PEIRCE, 1998).

and realism as general tendencies in the history of thought, and to show that the two sides of the fourteenth century debate over universals fit his characterizations of those tendencies. The defining commitment of the medieval nominalist is to the mind-dependence of the universal, for whom "a resemblance [...] consists solely in the property of the mind by which it naturally imposes one mental sign upon the resembling things" (*RFB*, 94). As a result Peirce's resumé of Scotus's realism aims precisely at explaining how "the nature which in the mind is universal and is not in itself singular, exists in things" (*RFB*, 93). At the same time Peirce acknowledges that there are important respects in which the universal is mind-dependent on the realistic view as well: "whiteness…is a real which exists only by virtue of an act of thought knowing it" (*RFB*, 90). Stepping back from what he says about Ockham and Scotus, and setting it alongside Peirce's larger objectives in the Berkeley review, we see that it is very much in line with those objectives to temper the opposition between mind-dependence and reality. If what "will hold in the final opinion" (which Peirce invokes to square the foregoing remark about whiteness with realism) is the lynchpin of the realistic conception of reality, then you can hardly insist that dependence on the mind is flatly incompatible with being real. This, I suggest, is where realism comes within a hair's breadth of nominalism.[9]

Just how, then, can Peirce's conception of a mathematical object be justly deemed "Scotistic"? I see no reason to think that his guiding principle, in arriving at the conception, was to stay close to the Scotistic party line. He was never a great one for orthodoxies, and by the turn of the century he was even less of an orthodox Scotist than when he wrote *RFB*. In the latter the medievals came into play, no doubt, as formative influences on Peirce's thinking, but also as pegs on which to hang the (families of) views over against which he defined his own, with Scotus serving as the peg on which Peirce hung the realism he was striving to articulate. He is much less systematic, in these texts on mathematical ontology, in his use of Scotus, than he was in *RFB*. What we have are outbursts of Scotistically fraught vocabulary that function primarily as pegs for Peirce's mathematical realism. Viewed as such, and taken together with his use of '*ens rationis*', these outbursts betoken a delicate balancing of mind-dependence and objectivity. This is not an *ad hoc* maneuver, but the application to mathematics of a move Peirce makes again and again, with more or less consequential variations, throughout his philosophical career, most notably in his convergence theory of truth and the accompanying theory of reality.

It would take several essays, each at least as long as this one, to spell out the ramifications for mathematical ontology of these larger Peircean themes.[10] I do want to say just a few words, before turning from Scotism to structuralism, about one way —

9 In my reading of *RFB*, and my understanding of the theory of reality that Peirce develops there, I am heavily indebted to the third chapter of (MAYORGA, 2007), and of course to (BOLER, 1963).

10 If Hookway (2004, p. 138-143) is right about the disentangling of Peirce's theories of truth and reality that gets underway around the turn of the century, more or less contemporaneously with the texts I have focused on here, that would complicate my picture of Peirce's mathematical realism quite considerably. Hookway makes some tantalizing remarks about the implications for the philosophy of mathematics on p. 141-142. He also argues (p. 136-138) that Peirce's openness to mind-dependent realities is one reason his convergence theory of truth does not commit him to an "absolute conception of the world."

Scotistic in nature, and perhaps also in origin — in which Peirce performs this balancing act. In a number of places in these ontological reflections we come across beings whose reality consists in what we might call an objective conceivability, in the fact that it *would be* correct for a mind to conceive of something, even if no mind has in fact so conceived of anything. The formal distinction is Scotus's most famous contribution to this genre. Scotus denies that there is a real distinction, within a concrete individual, between the Common Nature and the individuating difference: they are not "really distinct [...] to the extent that one of the two at least may exist apart from the other" (WOLTER, 1990, p. 28). But though these are, to quote now from Grajewski's summary of the doctrine, "really identical formalities," they are nonetheless formally distinct, which is to say "that one [...] before the operation of the intellect, is conceivable without the [other] though inseparable from it even by divine power" (GRAJEWSKI, p. 93). As Boler writes, a few pages after quoting this summary, "a formality is what the mind correctly *conceives* of the real thing [...] But if a thing is intelligible, then the conception which a truly knowing mind would have of it deserves to be called objective" (BOLER, 1963, p. 56; cf. MAYORGA, 2007, p. 128-129). The result is a delicately balanced blend, as we have found in Peirce, of dependence on and independence of the mind:

> If no intellect could exist, there would be no formal distinction. While Scotus insists the distinction is prior to the *act* of thinking (and hence is not created by the mind), he never says that it is prior to the possibility of thought. (WOLTER, 1990, p. 33)

Scotus uses a variety of terms to designate what is distinguished by a formal distinction: "His usual designation for it is *realitas* or *formalitas*, though he occasionally refers to it as an *intentio* or a *ratio realis*" (WOLTER, 1990, p. 32). Peirce *may* be thinking in these terms (and equivocating, admittedly, on 'reason') when he says in *LT* (353) that "the reason or *ratio* that creates [a collection] may be among the realities of the universe."

Now where is the structuralism in all of this? Where does Peirce say that number theorists, for example, study not some particular set of abstract objects, but rather the natural number *structure*, the structure exemplified by both von Neumann's and Zermelo's set theoretic constructions? The most forthright expression of something like a structuralist conception is found in the extended analogy Peirce draws in *PA* between chemistry and mathematics. He argues that the object of chemical inquiry is not the particular sample upon which the chemist experiments, but the molecular structure that is present in all of her samples; in just the same way, when the mathematician experiments on diagrams "the Object of Investigation [...] is the form of a relation [...] the very *form of the relation* between the two corresponding parts of the diagram" (*PA*, 530). He then offers, by way of illustration, the equation

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f_0}$$

and argues that

> [it] is a diagram of the form of the relation between the two focal distances and the principal focal distance; and the conventions of algebra […] in conjunction with the writing of the equation, establish a relation between the very *letters* f_1, f_2, f_0 […] the form of which relation is the *Very Same* as the form of the relation between the three focal distances that these letters denote […] Thus, this algebraic Diagram presents to our observation the very, identical object of mathematical research, that is, the Form of the harmonic mean, which the equation aids one to study.

Hookway rightly pays a lot of attention to this passage in his essay on Peirce's structuralism, but he just as rightly brings in other texts as well; I (and this essay) have benefited in particular from his penetrating discussion of *RR*. Peirce sounds like a structuralist when he says there that the natural numbers form "a cluster of ideas of individual things, but […] not a cluster of real things," a dictum we will spend a good deal of time unpacking later on. The argument for a discernibly structuralist impulse in Peirce has to rest on the cumulative force of scattered hints like these. One other, less scattered, bit of evidence is the definition of mathematics that Peirce advances repeatedly, and so far as I know unswervingly, in the closing decades of his life. A version of it opens the second paragraph of *PT*: "pure mathematics is the study of pure hypotheses regardless of any analogy that they may present to the state of our own universe" (*PT*, 157). This is already a departure from a straightforwardly platonistic view of mathematics as the study of objects which exist alongside (as it were) and on a par with concrete objects. In *LT* Peirce all but says that what matters in these hypotheses, from the mathematician's point of view, is the structure they describe. Having observed that "the mathematician is interested in his hypotheses solely on account of the ways in which necessary inferences can be drawn from them" (*PT*, 343), he writes that as a consequence of that interest "the pure mathematician generalizes his hypotheses so as to make them perfectly applicable to all conceivable states of things in which precisely analogous conclusions could be drawn" (*PT*, 344). Here again the emphasis is on the hypotheses themselves and the structure they define, and not on any particular exemplar of that structure.

Now what, if anything, does Peirce have to offer to a 21st-century mathematical structuralist? Let us begin our search for an answer with the philosophical overture, in the first section of *RR*, to his axiomatization of the natural numbers. A principal concern of this remarkably rich discussion is a semiotic analysis of numerals, the "meaningless vocables invented for the experimental testing of orders of sequence" (*RR*, 154), that is, for counting. To understand that analysis, and its implications for Peirce's ontology of structure, we need to look at the antepenultimate paragraph (4.157 in the *Collected Papers*). Peirce begins that paragraph by listing "ideas of feeling, acts of reaction and habits" as the "three categories of being." He goes on to characterize the "Outer World, or universe of existence" as "the ensemble of all habits about acts of reaction" and "the Inner World, the world of Plato's forms" as the "ensemble of all habits about ideas of feeling." Correlatively we have "two modes of association of ideas: inner association, based on the habits of the inner world, and outer association, based on the habits of the universe." The semiotic payoff is delivered in the final sentence of the paragraph:

> What we call a thing is a cluster or habit of reactions, or, to use a more familiar phrase, is a centre of forces. In consequence, of this double mode of association of

> ideas, when man comes to form a language he makes words of two classes, words which denominate things, which things he identifies by the clustering of their reactions, and such words are proper names, and words which signify, or *mean*, qualities, which are composite photographs of ideas of feelings, and such words are verbs or portions of verbs, such as are adjectives, common nouns etc.

Though Peirce is deliberately abstaining here from his usual categorical and semiotic vocabulary, he is clearly talking about Firstness, Secondness and Thirdness at the beginning of this paragraph and about indices and icons at the end; and he clearly names Seconds (clusters or habits of reaction) as the typical objects of names and other indices, and Firsts (qualities, composite photographs of ideas of feelings)[11] as the typical objects of adjectives, verbs, common nouns and other icons. This falls far short of the full complexity of Peirce's semiotics, but in explicating his analysis of number in *RR* I will (continue to) follow his lead and be somewhat loose in my usage of such terms of semiotic art as 'icon' and 'index'.

When Peirce says that numerals such as 'seven' are meaningless, what he means is that in counting they function as indices, as descriptively empty means of pointing things out: "the cardinal numbers in being called meaningless are only assigned to one of the two main divisions of words" (*RR*, 158).12 He underscores this in his reply to the objection that numerals cannot be altogether meaningless because "when a number is mentioned […] the idea of a succession, or transitive relation, is conveyed to the mind" (*RR*, 155). Fair enough, he concedes, but "this same idea is suggested by the children's gibberish

> 'Eeny, meeny, moni, mi,'[13]

Yet all the world calls these meaningless words, and rightly so." The indexical character of the children's nonsense syllables is highlighted by the pointing that typically accompanies their utterance, and we often point in a similar fashion at objects we are counting as we utter the numerals in sequence. This dovetails nicely with Peirce's general treatment of categories and signs in *RR*; for objects of the sort we can point to while counting them will indeed be Seconds.

Peirce ends the paragraph just quoted with a change of semiotic and categorical key, from index to icon and from Second to First:

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11 The metaphor of composite photographs is a favorite of Peirce's. He uses it in arguing for the continuity of time in (PEIRCE, 1896?, N3.59-60), which is very close in date to *RR*. He also uses it several years later, in the "Syllabus" to the Lowell Lectures, for more or less the same semiotic purpose that it serves in *RR*: "Passing now to the consideration of the Predicate, it is plain enough that [the proposition 'Every rose will be killed'], or any at all like it, only conveys its signification by exciting in the mind some image or, as it were, a composite photograph of images, like the Firstness meant" (PEIRCE, 1903f, E2.281).

12 This is perhaps the most vivid illustation of the rigorous parsimony that Peirce imposes on the presentation of his system in *RR*: having no immediate analytical need for symbols, he lops off one third of his best-known semiotic trichotomy (icon, index, symbol) and pretends that there are only *two* "main divisions of words."

13 Hartshorne and Weiss have substituted 'miney, mo' for Peirce's 'mony mi'.

> The only difference [between numerals and the children's gibberish] is that the children count on to the end of the series of vocables round and round the ring of objects counted; while the process of counting a collection is brought to an end exclusively by the exhaustion of the collection, to which thereafter the last numeral word used is applied as an adjective. This adjective thus expresses nothing more than the relation of the collection to the series of vocables. (*RR*, 155)

As a numeral used in counting, 'seven' is meaningless and functions as an index, but if we are counting a septet then upon completion of the count we apply 'seven' adjectivally, as in 'I have seven pebbles here.' Though 'seven' is not strictly speaking an icon (a sign that represents its object by resembling it),[14] in its adjectival employment it will be at least broadly iconic, within the rather coarse semiotic framework Peirce erects in *RR*, and will signify a quality or attribute (a First). Peirce's initial characterization of the attribute in question hews very closely to the practice of counting: the numerical "adjective […] expresses nothing more than the relation of the collection to the series of vocables." But that relation leads us straightaway to

> a real fact of great importance about the collection itself […] namely, that the collection cannot be in a one-to-one correspondence with any collection to which is applicable an adjective derived from a subsequent vocable, but only to a part of it; nor can any collection to which is applicable an adjective derived from a preceding collection be in a one-to-one correspondence with this collection, but only with a part of it; while, on the other hand, this collection is in one-to-one correspondence with every collection to which the same numeral adjective is applicable. This, however, is not essentially implied as a part of the significance of the adjective. On the contrary, it is only shown by means of a theorem, called "The Fundamental Theorem of Arithmetic," that this is an attribute of the collections themselves and not an accident of the particular way in which they have been counted. Nevertheless, this is a complete justification for the statement that quantity — in this case, multitude, or collectional quantity — is an attribute of the collections themselves. (*RR*, 156)

The attribute that 'seven' signifies thus turns out to be the "multitude, or collectional quantity" common to all seven-membered collections. It *turns out* to be that multitude; for it is not until we have proved the Fundamental Theorem of Arithmetic that we can be sure "that this is an attribute of the collections themselves." In particular, as Peirce explains later on (*RR*, 163), the Fundamental Theorem assures us that "a finite collection counts up to the same number in whatever order the individuals of it are counted."

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14 Though the numeral 'seven' is not iconic in isolation, the sequence of numerals beginning with 'one' and ending with 'seven' does *resemble* a seven-membered collection: the resemblance is established by setting up a one-one correspondence between collection and sequence, just as we do in counting. It would be an interesting exercise, for one more thoroughly steeped in Peirce's semiotics than I am, to find just the right pigeonhole for such sequences in Peirce's classification of signs. A structuralist, for whom numbers are not free-standing entities but places in structures, would be encouraged to find that it is not individual numerals, but initial segments of the whole structure of numerals, that are (more) truly iconic.

It would be natural for Peirce, at this juncture, to identify this attribute of *sevenness* as the number (as opposed to the numeral) seven. He does identify them in *L3* (among many other places):

> Dr. Georg Cantor is justly recognized as the author of two important doctrines, that of *Cardinal Numbers* and that of *Ordinal Numbers*. But I protest against his use of the term *Cardinal Number*. What he calls cardinal number is not number at all. A cardinal number is one of the vocables used primarily in the experiment called counting a collection, and used secondarily as an appellative of that collection. But what Cantor means by a cardinal number is the zeroness, oneness, twoness, threeness, etc. — in short the multitude of the collection. By ordinal numbers Cantor means certain symbols invented by him to denote the place of an object in a series in which each object has another next after it. [...] The doctrine of ordinal numbers, then, is a theory of pure mathematics [... whereas the] doctrine of multitude is not pure mathematics. Pure mathematics can see nothing in multitudes but a linear series of objects, having a first member, each one being followed by a next, and with a few other such formal characters. [...] Multitudes are characters of collections; and the idea of a collection is essentially a logical conception. (*L3*, p. 346-347; p. 350)

If this is Peirce's thought in *RR* as well, one might be tempted to object that this makes his philosophy of arithmetic less than fully structuralist. The foregoing quote from *L3* is open to the interpretation that multitudes (cardinal numbers, as we now follow Cantor in calling them) are one among the many kinds of things that can be slotted into an ordinal succession, which strongly suggests that the multitudes themselves are independent of the succession. But this is uncomfortably close to saying that they are independent of one another and of the structure they compose.[15] Moreover, in *RR* itself attributes like *sevenness* are identified by acts of counting that can hardly be held to depend, either ontologically or epistemologically, upon the entire natural number structure.

This last feature of the story Peirce tells in *RR* is surely a virtue and not a defect. Structuralism, or any other philosophy of mathematics for that matter, will be in a bad way epistemologically if it has to insist that individual numbers cannot be singled out independently of the natural number structure as a whole. But the "slogan of structuralism [...] that mathematical objects are places in structures" (SHAPIRO, 1997, p. 82) does not require so much. It is not so easy — no more than with any philosophical position that attracts a decent number of adherents — to say precisely what it does require. It is worth noting, though, in defense of Peirce's structuralist credentials, that on this interpretation of his view the number seven is itself a structure, the one common to all septets. He is thus well positioned to be unmoved, as any good structuralist should be, by such questions as whether seven is identical to the set assigned its place by Zermelo or that assigned by von Neumann (let alone whether seven is identical to Julius Caesar). Moreover, we find Stewart Shapiro, who is a structuralist if anybody is, outlining an abstractionist epistemology for the natural

15 The interaction of cardinal and ordinal notions in Peirce's conception of number is a large topic, which I am passing over as somewhat tangential to his analysis of mathematical structure. The interested reader should consult Levy's excellent study (LEVY, p. 1986).

numbers (SHAPIRO, 1997, p. 112-120) with more or less the same ontological underpinnings Peirce offers on the interpretation we are now considering. According to Shapiro what we abstract — in a different sense of the term from Peirce's16 — from counting n-membered collections of objects is the "structure exemplified by all systems that consist of exactly n objects" (p. 115). To get from there to the natural number structure, we "form a system that consists of the collection of these finite structures with an appropriate order. Finally, we discuss the structure of this system" (p. 119). He freely admits that this "strategy depends on construing the various finite structures, and not just their members, as objects that can be organized into systems."

So Peirce would be no less of a structuralist if he were to identify sevenness and seven in *RR*. But it is not altogether clear that he does identify them there. He sharply distinguishes between pure arithmetic and the theory of counting: "pure arithmetic has nothing to do with the so-called Fundamental Theorem of Arithmetic. For […] pure arithmetic considers only the numbers themselves and not the application of them to counting" (*RR*, 163). If multitudes belong not to pure but to applied arithmetic, then it would be odd for Peirce to employ them as the basis of his arithmetical ontology. Odd perhaps, but not altogether out of the question, and we have just seen that there is evidence on the other side. In *RR*, at any rate, Peirce does not definitively identify *sevenness* and seven, nor does he definitively distinguish them. In fact, once he gets to the second section of the paper, where he gives his own theory of number, he shows very little concern with metaphysical questions about the identity of seven or of any other number. He makes a very important metaphysical remark, which we will turn to in a moment, about the *system* of natural numbers. But at this point in his paper, where he begins to speak freely of numbers rather than numerals, exactly what the numbers are becomes a decidedly secondary issue, if not a downright "don't care." The structuralist's point is just that mathematicians do not, and hence philosophers of mathematics should not, care about such issues. So whatever Peirce's official stance on *sevenness* and seven, his actual handling of the matter has a distinctly structuralist cast.

The comparison with Shapiro is worth pressing a bit further. Shapiro's account appeals, in the first of its three stages, to our capacity for pattern recognition to explain our knowledge of small finite structures; at the second stage we grasp arbitrarily large finite structures; at the third we regard these as forming a single system, and articulate *its* structure. Peirce has reached the third of Shapiro's stages by the time he gives axioms for the natural numbers in the second section of *RR*, but he does not tell us exactly *how* he reached it. The closest he comes to addressing that question is in the final sentence of §1:

16 As noted below, the abstraction at work in Shapiro's outline is pattern recognition. No doubt some such faculty would need to be invoked in a full-dress version of Peirce's sketch as well, but not to do the logical and ontological duties that Peirce calls upon hypostatic abstraction to perform. Later on in his book (p. 120-124) Shapiro builds on work by Robert Kraut to develop a notion of abstraction that can do those duties. Of course since Shapiro's approach to the natural numbers requires him to treat structures as objects, he will need *some* way of accomplishing what Peirce accomplishes by means of hypostatic abstraction.

> But the system of numerals having been developed during the formative period of language, are taken up by the mathematician, who, generalizing upon them, creates for himself an ideal system after the following precepts. (*RR*, p. 159)

Given all that has gone before, and the pride of place enjoyed by abstraction in Peirce's philosophy of mathematics, we have good reason to think that Peirce is hinting here at a multi-stage abstraction much like Shapiro's. We might miss the multiplicity of stages if we were to read the foregoing quote too literally, and out of context. Peirce would then seem to be saying that the mathematician abstracts directly from the numerals themselves to numbers. This leaves out the practice of counting and the transition from numeral to adjective, from index to icon, from Second to First. We make that transition by performing a hypostatic abstraction. This gets us to *sevenness* and may or may not get us all the way to seven, depending on where we come down on the interpretive question we have just finished turning over. Be that as it may, we are unquestionably ready to treat 'seven' as a proper name, and to talk about numbers as well as numerals, by the time Peirce gives his axioms for the natural numbers in §2 of *RR*. Peirce appears to leapfrog over the second of Shapiro's three stages, and to go directly from small finite structures to the full-blown infinite structure of the natural numbers without lingering over the problems posed by the large finite. We should not jump to the conclusion that he was insensitive to those problems. In *RR* he does not systematically lay out successive stages as we are now attempting to do on his behalf. But we do not have to go too far beyond what he actually says to discern at least two moments of hypostatic abstraction in our progress towards an axiomatized arithmetic: the abstraction of numerical attributes from the practice of counting, and the abstraction from those attributes of what we might call, for lack of a better word, *natural-numberhood*. This latter attribute can be spelled out, just as a structuralist would expect, only by specifying the relations that numbers possessing it bear to one another and to the overarching structure as a whole.

Peirce tells us in *RR* that the "[natural number structure] is a cluster of ideas of individual things, but it is not a cluster of real things" (p. 161). A few paragraphs up from this he has written that "there are three categories of being; ideas of feelin-gs, acts of reaction, and habits" (*RR*, p. 157). This is of course his triadic system of categories, which are both universal features of experience (uncovered by phenomenological reflection) and real operative principles of nature. Translated into his customary categorical jargon, Peirce's characterization of the natural numbers says that the structure is a Third governing Firsts. Thirdness being the category of law and Firstness that of possibility, we may offer the following approximate translation into a jargon nearer to our own: the natural number structure is a law-governed complex of possibilia. If this is (approximately) correct, then we can reasonably count Peirce as a forebear, not just of structuralism, but of *modal* structuralism;[17] and this

17 We must not overlook a crucial difference between Peirce and more recent advocates of a modal analysis of mathematical structure. Unlike, say, Geoffrey Hellman (1989) or Hartry Field (1984), Peirce has no desire to cut his ontological commitments down to concreta alone. (To be sure, he would agree that concreta alone exist, but would flatly deny that they alone are real.) Hellman and Field both want to minimize the ontological impact of their modal involvements, whereas Peirce is an unabashed realist about modality.

turns out to be a characterization, not just of his distinctive brand of mathematical structuralism, but also of his distinctive brand of mathematical realism. A defining commitment of Peirce's "extreme scholastic realism" is a full-blooded modal realism; the "doctrine of substantive possibility" that he sketches in *LT* (p. 350-353) involves a thoroughgoing realism about qualities, where "a quality is anything whose being consists in such logical possibility as there may be that a definite predicate should be true of a single subject" (*LT*, p. 352). Mark the echo of the *PT* definition of abstraction (p. 2). Actual primary substances have dropped out of the picture here, so the doctrine of substantive possibility appears to be ontologically more generous than that of hypostatic abstraction. Jay Zeman maintains that they are at bottom one doctrine (ZEMAN, 1983, p. 299). Be that as it may, we need the more generous standard even for arithmetic, which should surely not be contingent on the existence of arbitrarily large finite collections, let alone of arbitrary completed acts of counting. Moreover, Peirce regularly insists that the mathematician is responsible, not for the truth of her hypotheses, but only for their possibility.

This is already a substantial benefit accrued from Peirce's modal structuralism: it readily explains the creative freedom of the mathematician, a datum which would have cried out for explanation to one as knowledgeable as Peirce was about the revolutionary innovations of the 19th century — and which cries out no less audibly to us. At the same time his view explains why "the last achievement that [mathematics] has in view is an achievement of knowing" (PEIRCE, n.d., N3.527). Though the being of the mathematician's possibilia consists in the logical possibility of a true predication, they also — given a modal realism like Peirce's —have a kind of mind-independence. The mathematician need not care about whether her structures *are* actually realized, but only about what they *would be* like *if they were*, and Peirce's later realism is famously hospitable to such "would be"s. The subtlety of the view is Scotistic not just in degree, but in kind; for the being of possibilia and hence of the mathematician's objects is grounded precisely in a kind of objective conceivability.

Seasoned Peirceans will have caught the covert change of subject in the last two sentences. Substantive possibility is the doctrine of "may be"s and not of "would be"s. Here is how Peirce draws that distinction in "Logic of Mathematics":

> Generality is either of that negative sort which belongs to the merely potential, as such, and this is peculiar to the category of quality; or it is of that positive kind which belongs to conditional necessity, and this is peculiar to the category of law. (PEIRCE, 1896a, C1.427)

So we are now concerned with Thirdness, with laws that necessitate how things *would be* under specified conditions. An unreserved commitment to "would be"s is another defining tenet of Peirce's "extreme scholastic realism" and also of his pragmaticism. To understand his *epistemology* for mathematical structures we must make this transition from First to Third, from may be to would be. We must look more closely at the laws that govern — or rather, to get ahead of myself, that *are* — the mathematician's structures.

In *RR* the laws in question are Peirce's axioms (essentially the same as Dedekind's) for the natural numbers. Now axiomatization is of the utmost importance to structuralists: it is because, as Quine says, "numbers are known only by their laws

[…] that any constructs obeying those laws […] are eligible in turn as explications of number" (QUINE, 1969, p. 44). It is Peirce's profound and original analysis of law as a fundamental constituent of reality that makes his epistemology for mathematical structure so original and, in my view, so profound. When we apply the *PA* account of structure (p. 6) to the natural numbers, what we get is that the natural number structure is realized, not in the axioms alone, but in the axioms *together with* the linguistic conventions that constitute the practice of theorem-proving in number theory. Peirce says there (and in countless other places) that we prove theorems by manipulating diagrams; and in 1903 (the same year as the Pragmatism and Lowell Lectures) he writes that "a diagram is a representamen which is predominantly an icon of relations" (PEIRCE, 1903b, C4.418). If we are to understand the symbiotic bond between Peirce's metaphysics and epistemology for structures, we will have to explicate this remark; here again a slight semiotic detour is called for.

In his "Syllabus" for the Lowell Lectures Peirce calls a sign *iconic* when "it represents its object mainly by its similarity" to that object (PEIRCE, 1903f, E2.273). Not every iconic sign is an icon, as strictly defined in Peirce's classification of signs according to their relations to their objects: "a possibility alone is an Icon purely by virtue of its quality; and its object can only be a Firstness." Peirce coins 'hypoicon' as a term of art for iconic signs in the looser sense; Short helpfully notes that "the prefix suggests a substratum supporting an icon" (SHORT, 2007, p. 216). Peirce goes on in the "Syllabus" to say that hypoicons "which represent the relations…of the parts of one thing by analogous relations in their own parts, are *diagrams*" (PEIRCE, 1903f, E2.274). A mathematical diagram, then, is a hypoicon — with written marks, for example, serving as its substratum — that resembles its object by having the same relational structure. In *PA* Peirce presents an equation as a hypoicon whose object is a relation among the focal distances of a lens, but it functions as such only by virtue of the conventions that govern the use of the equation in algebraic reasoning. I submit that the same analysis holds good for mathematical structures like the natural numbers: the axioms, together with the conventions whereby we prove theorems from them, are a complex hypoicon of the structure. The object of that hypoicon is a First, a "may be," not a simple quality like a color, but rather a way in which objects could be related to one another. We know that "may be", just as Quine says, only through its laws. Peirce adds that the laws themselves, and the process whereby we discover them, have the very structure they express.

The lawlike nature of mathematical theorems, and of the structures they embody, flow directly from the nature of proof as Peirce understands it. The statement of a theorem hypothesis H, according to him, is a diagram D_H of the system S_H of relations described by H, and the diagrammatic transformations licensed by the conventions of proof ensure that the diagrams

$$D_H = D_1, \ldots, D_n = D_C$$

obtained thereby are icons of the systems

$$S_H = S_1, \ldots, S_n = S_C$$

of relations that obtain whenever *H* does. In particular the last diagram D_C is an icon of a system S_C in which the conclusion *C* of the theorem is true. So to take Peirce's favorite geometrical example, D_H might be a diagram (pictorial or pictorial-cum-verbal) of an isosceles triangle and D_C a diagram of that same triangle in which it is explicit that the base angles are equal. What we discover through this sequence of diagrammatic transformations is that any situation in which *H* is true is *necessarily* one in which *C* is true. That is, we discover that the conditional "if *H* then *C*" is necessarily true.[18]

The laws of mathematical structures are thus, like other laws of nature, conditional necessities: they tell us what necessarily *would be* the case under stipulated conditions. So another advantage of Peirce's structuralism is that it integrates mathematics, but not too tightly, into the larger system of the sciences. He does not hesitate to call the mathematician's study of her diagrams "experimental" and "observational," and the fruits of that study are generalizations having exactly the logical form of scientific laws. But the integration is not *too* tight; for the mathematician is answerable, not to the physical facts, but only as it were to the logic of her own imagined reality. Finally, Peirce's view yields an attractively straightforward account of *applied* mathematics: the scientist investigating a physical system whose structure can be represented in terms of a given mathematical structure can learn from the mathematician what inferences can be drawn about the system by virtue of that structure (PEIRCE, 1898, C3.559).

At the same time, one might object, there appears to be an enormous disanalogy between mathematical and physical experimentation: experimental evidence can confirm *or refute* physical hypotheses, but how could a mathematical experiment refute its hypotheses? Part of the answer, of course, is that the mathematician is not concerned, as the scientist is, with the truth of her hypotheses. But there are other hypotheses involved in mathematical practice that do admit of a kind of experimental refutation: those that figure in hypostatic abstraction.

In his seminal writings on abstraction, Thomas Short has called our attention to the fact that in empirical investigations hypostatic abstraction occurs at an early stage and is — just as Peirce says — at once trivial and based upon a methodologically

18 I take it that Peirce has something like this in mind when he endorses his father's definition of mathematics as "the science which draws necessary conclusions" (PEIRCE, n.d., p. 2). Note also its consonance with his own definition of mathematics, discussed on p. 6 above. In (PEIRCE, 1898, C3.558) he argues that the two definitions are at bottom the same. The view of mathematical truths as necessitated conditionals is a staple of modal structuralism: see, for example, (PARSONS, 1990, p. 289-292) and the works cited there and elsewhere in that paper. Peirce's definition of 'theorem' in (PEIRCE, 1903?, N4.289) seems to say that theorems have the form $H \rightarrow C$, which would also result from a literal reading of his father's slogan that mathematicians "draw necessary conclusions." Such a view runs a well-known risk of vacuity (PARSONS, 1990, p. 279); moreover, it is at odds with Peirce's insistence that the mathematician is unconcerned with the truth of her hypotheses. It makes much more sense to read him as necessitating the whole conditional, and not just the consequent.

essential hypothesis. The triviality is obvious in Molière's joke about "dormitive virtue": here the abstraction "is defined only in relation to its supposed effect" (SHORT, 1988, p. 54). But it is based on the hypothesis that there is something, and a unique something, that enables opium to put people out. Because "[no] direct characterization of [the power] is given" (SHORT, 1988, p. 54) in this hypothesis, I will call it the *indirect hypothesis*. Consisting as it does of two claims, existence and uniqueness, an indirect hypothesis is doubly vulnerable to refutation — more on that later.

At the next stage of inquiry the scientist tries to characterize the explanatory entity in more direct terms; in the dormitive virtue example the likeliest candidates would be chemical properties of opium. Let us call a hypothesis that identifies some directly characterized entity with that posited in the indirect hypothesis a *direct hypothesis*. This will ultimately be combined (and here I go beyond Short's *ipsissima verba*)[19] with a *nomological hypothesis* stating a lawlike connection between the entity named in the direct hypothesis and the phenomena to be explained. In keeping with Peirce's general account of natural law, the nomological hypothesis will state a conditional necessity. Experimental results then settle — in a manner whose details we can thankfully pass over here — the fate of the nomological hypothesis, which in turn settles that of the corresponding indirect hypothesis. The fate of the latter hangs, not on that of any one direct hypothesis, but on our eventual success in confirming exactly one direct hypothesis.

Though Short might disagree,[20] I claim that hypotheses of all three kinds are to be found in Peirce's treatment of the natural numbers in *RR*. The indirect hypothesis is that there is a unique structure embracing all the counting-based structures of finite collections; the direct hypothesis is that the structure is the one given by Peirce's axioms. The nomological hypotheses are the theorems we derive from the axioms, or at least those theorems whose consequents[21] describe the "phenomena" that set the process of mathematical theorizing into motion. Peirce gives an apt example in RR: the thesis he labels, in grudging deference to the usage of his time,[22] the

19 It was Short's criticism of Peirce (see note 4) for failing to distinguish *entia rationis* from "really effective laws of nature," reinforced by Boler's emphasis on the centrality of prediction to Peirce's realism, that suggested the idea of nomological hypotheses to me.

20 Short lists the two-fold refutability of direct hypotheses as a point of difference from their mathematical counterparts (SHORT, 1988, p. 53). It would seem to follow (though Short does not, so far as I know, say so) that there are no direct or nomological hypotheses in mathematics.

21 Here I take for granted the modal-structuralist reading of theorems as necessitated conditionals.

22 This usage clashes, not just with Peirce's terminological preferences, but with our own standard nomenclature. When we speak of the Fundamental Theorem of Arithmetic, we typically mean the existence of a unique prime factorization. Peirce thinks the title ought to be reserved for what he calls the Fermatian inference (that is, mathematical induction). The underlying trouble is that the honorific 'fundamental' can be awarded on the strength of at least three different varieties of mathematical importance. Relative to a given axiomatization of a structure, a proposition can be regarded as fundamental simply by virtue of being an axiom. Fundamentality of this first sort is not intrinsic to the proposition, but varies from

"Fundamental Theorem of Arithmetic," which says that "a finite collection counts up to the same number in whatever order the individuals of it are counted" (*RR*, 163). The invariance of cardinality under reordering of the count is fundamental to our conception of the natural numbers in the sense that any theory of those numbers according to which cardinality was not thus invariant would thereby be disqualified as untrue to the practice of counting collections that led us to formulate the theory in the first place. If, on the other hand, the Fundamental Theorem is a necessary consequence of some proposed system of axioms for the natural numbers — that is, if we can prove a necessitated conditional (A→F), where A is a conjunction of axioms from the system, and F is the Theorem — then our success in proving that partially confirms the axioms.[23] I submit that much as the phenomenon of falling asleep figures in the consequents of nomological hypotheses whose confirmation contributes to that of some direct hypothesis about the dormitive virtue of opium, the "phenomenon" of order invariance figures in the consequents of nomological hypotheses whose confirmation contributes to that of some direct hypothesis about the natural number structure.

I say "much as," not "just as"; for there are of course any number of disanalogies between the two kinds of confirmation. For example, in mathematics we have a highly uniform and straightforward experimental procedure — the diagrammatic manipulations whereby we establish our results — while in the physical sciences, by contrast, experiments are harder to set up and as variegated as the materials with which they have respectively to do. Because the objects of mathematical reasoning, and the reasoning itself, are so straightforward, mathematicians are much less prone to error than, say, chemists.[24] But these are differences of degree, and not of kind.

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axiomatization to axiomatization. Mathematical induction is fundamental in this sense, for example, relative to Peano's axioms for the natural numbers, but not to the Dedekindian axioms Peirce gives in *RR* (p. 160), where induction is proved as a theorem (*RR*, p. 165). At the same time, mathematical induction is arguably fundamental to our conception of the natural numbers in a second, more intrinsic sense. To put the matter crudely, one might reasonably maintain that induction is essential to the conception in a way that what Peirce reluctantly calls the Fundamental Theorem is not. A proposition can be fundamental in this sense even if, perhaps for reasons of mere technical convenience, it is not included among the axioms, as is the case with induction in *RR*. Finally, a proposition may be fundamental in being at least partially consitutive of our initial awareness or specification of a structure: in failing to satisfy that proposition, a structure would thereby fail to be the one we had in mind. It is in this third sense that what Peirce calls the Fundamental Theorem is rightly so called. He would rather bestow the title on mathematical induction because it is fundamental in the second of these three senses, as the "Fundamental Theorem" is not (*RR*, p. 165). I owe the inspiration for these remarks about fundamentality to André De Tienne's essay on that topic, in this issue of *Cognitio*.

23 I am being sloppy here about the distinction between pure and applied arithmetic, which according to Peirce disqualifies the Fundamental Theorem as a theorem of pure arithmetic (*RR*, 163). If we give that distinction all the weight Peirce would have us give it, then $(A \rightarrow F)$ belongs to the applied, rather than the pure, theory of the natural numbers. Well and good. It is only to be expected that applications of a structure will figure prominently among the phenomena that lead us to formulate the theory of that structure.

24 For a typical treatment of the straightforwardness of mathematical methods and objects,

Mathematicians *do*, for example, make mistakes, though these are less frequent, and more readily corrected, than errors in the physical sciences (PEIRCE, 1896b, C3.426). And overall what is really striking, and what strikes the dominant note in Peirce's account of mathematical method, is the strength of the analogies with the physical sciences. Most momentously, Peirce really means it when he says that mathematics *has* an experimental procedure.

The experimental refutability of the mathematician's nomological hypotheses is thus but one among many strong and satisfying analogies to physical science that fall out of Peirce's brand of structuralism. Less comforting, perhaps, is the refutability of indirect hypotheses in mathematics. Such hypotheses, mathematical or physical, are vulnerable to failures not just of existence, but also of uniqueness, and in Gödel's first incompleteness theorem we have what looks like an iron-clad case against the uniqueness of any natural number structure whose identity is supposedly fixed, as it is on Peirce's view, by our practice of proving theorems. It is hard to know how Peirce himself might have responded to this challenge; better in any case to ask rather how *we* should respond if we have appropriated his insights while avoiding his errors. If we are impressed, as I am, with Peirce's anti-logicism, and therefore skeptical of logical remedies for mathematical maladies, we will not be satisfied by resorts to second-order logic, or to such weaker cures for incompleteness as w-logic.[25] Peirce's overall approach to mathematical ontology, as interpreted here, is more in the spirit of the conceptual realism so forcefully expressed in some of Gödel's philosophical writings on sets (GÖDEL, 1944, p. 456-457; 1947, p. 483-485). Whether Peirce can help us flesh out these Gödelian hints in a compelling way only time will tell.

Peirce's stock among scientifically-minded philosophers is still low enough that many will reflexively look askance at the suggestion that he might have something to offer towards the solution of such recalcitrant puzzles about mathematics. Even if they find the conception I have sketched here attractive — say, for its judicious placement of mathematics among the sciences — they may nonetheless take its dependence upon Peirce's larger philosophical system as reason enough to reject it. But if one finds the conception sufficiently attractive, why not revalue Peirce's stock instead?

and the implications thereof, see (PEIRCE, 1902, C4.232).

25 In the earlier version of this paper delivered in São Paulo I listed realism about second-order logic as a possible Peircean response to the challenge of incompleteness. This prompted Professor Edelcio Gonçalves de Souza to ask, in his comments on the paper, whether Peirce would have been content with w-logic as a remedy for incompletness. This very apposite question led me to the generally negative assessment of such remedies that I express in the text. My hunch is that Peirce would have taken a realistic view of second-order logic, because his view of sets has such a strongly realistic (which is not to say platonistic) cast. That does not completely settle the question, however, because we now have a lot more information than Peirce did about the mathematical and philosophical difficulties of sets. In any case, even with the limited information at his disposal, Peirce was arguably more acutely critical about the concept of set than we are: see (DIPERT, 1997, p. 53-58).

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