

Mathematical Individuality in Charles Sanders Peirce

Individualidade Matemática em Charles Sanders Peirce

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Abstract: Peirce regards mathematics as an informative science capable of really increasing our knowledge. This means that mathematics is not limited to conceptual analysis but possesses a real object of investigation. The core of Peirce's view of mathematics is that mathematical reasoning is not developed through general concepts alone but deals with an unavoidable element of individuality. The conclusion of a deductive inference can contain information that is not at all present in its premises and can only come into being through concrete work on the part of the mathematician. Peirce describes this work as observation and experimentation on individual diagrams. While the idea of an individual element in mathematics is already present in Kant (and can also be traced back to Aristotle), the different location Peirce assigns it attests to a marked difference in their conceptions, the basis of which lies in the difference between Kant and Peirce in categorial analysis. Peirce's semiotic approach to mathematics involves a shift from the plane of the object denoted to that of the sign itself. This holds true for geometric as well as algebraic inferences, which Peirce can equate in this respect. In both cases, the individual element of mathematics is thus to be found within the diagram itself with no reference to the object denoted. While the diagram is in any case a token, this cannot explain its essential individuality, to which end the indexical juxtaposition of its parts should be examined.

Keywords: Mathematics. Mathematical reasoning. Individuality. Diagram. Index.

Resumo: *Peirce considera a matemática uma ciência informativa realmente capaz de aumentar nosso conhecimento. Isso significa que a matemática não é limitada à análise conceitual, mas possui um objeto real de investigação. O coração da visão peirciana da matemática é que o raciocínio matemático não é desenvolvido somente por meio de conceitos gerais, mas lida com um inevitável elemento de individualidade. A conclusão de uma inferência indutiva pode conter informação que não está absolutamente presente nas suas premissas, podendo vir a ser somente por meio de trabalho concreto por parte do matemático. Peirce descreve esse trabalho como observação e experimentação sobre diagramas individuais. Enquanto a ideia de um elemento individual na matemática já está presente em Kant (e pode também ser traçada até Aristóteles), a sua base está na diferença entre Kant e Peirce acerca da análise categorial. A abordagem semiótica peirciana da matemática implica uma mudança, do plano do objeto denotado para o do próprio signo. Isso vale tanto para as inferências geométricas como para as inferências algébricas, que Peirce pode igualar nesse aspecto. Em ambos os*

casos, o elemento individual da matemática deve ser encontrado dentro do próprio diagrama, sem referência ao objeto denotado. Enquanto o diagrama é em todo caso um token, isso não pode explicar sua individualidade essencial, para cujo fim a justaposição indicial de suas partes deve ser examinada.

Palavras-chave: Matemática. Raciocínio matemático. Individualidade. Diagrama. Índice.

In manuscript 16¹ Peirce makes the following assertion:

Just as the positive sciences are founded upon the consistency of action of Nature, so mathematics is founded upon the consistency of action of reason. The action of Nature is a wonder to us; but that of Reason is not usually so [...]. All that we demand of science is that it should show nature to be reasonable. Further than that we do not usually go. We seem to comprehend Reason. We flatter ourselves we grasp its very *noumenon*. But it is really as occult as Nature. It is only because its effects are for the most part familiar to us from infancy that they are not surprising. For when we come upon some property of numbers which is new to us, although it can spring from nothing but Reason, we are greatly surprised and begin to talk of the *Mystery of Numbers*, as of something which it is desirable to explore or which is incomprehensible. What we here demand is the mode of evolution of the action of Reason.

Mathematics, Peirce thus says, can be surprising. It can be full of real discoveries that really enlarge our knowledge, showing us, as is said here, the mode of evolution of the action of reason. In proceeding from the premises to the conclusion of a deductive – i.e. mathematical – inference, we can find the initial information leading to entirely new elements and be surprised by what we are discovering. And this surprise is not psychological in nature but rather objective. The new information that can be obtained in the conclusion of the deductive inference is not at all present in its premises. If it is to come into being, concrete activity is required on the part of the mathematician, which Peirce describes as experimentation on diagrams, manipulation and observation of the kind of signs that are employed in mathematics. The new information does not exist at all until this activity is performed.

Now, what does it mean to say that mathematics is able to provide new information and real knowledge? What does this assertion imply? It implies that mathematics has a real object of investigation about which it makes new discoveries. In other words, mathematics is not confined to conceptual analysis but goes beyond the boundaries of the understanding alone to address an object that resists the mathematician, an object that, to use Peirce's terminology, possesses an element of *Secondness*. Mathematics does not work on general concepts and definitions but rather on individual objects. While philosophers, Peirce says, use a kind of "demonstration that employs only general concepts and concludes nothing but what would be an item of a definition if all its terms were themselves distinctly defined", in mathematics "it will not do to confine oneself to

¹ The numbering follows Robin's catalogue of Peirce's manuscripts.

general terms. It is necessary to set down, or to imagine, some individual and definite schema, or diagram – in geometry, a figure composed of lines with letters attached; in algebra an array of letters of which some are repeated.”²

In Peirce’s conception of mathematics, the mathematician works on concrete individual diagrams – drawn on paper or else only imagined – that can be manipulated and observed. These are not conceptual entities but rather endowed with both a formal and a material element. A geometric proof is not worked out on the definition or the general form of triangle but rather on an individual triangle drawn on the blackboard, a composition – using terminology that is not Peirce’s – of form and matter. What kind of matter are we referring to here? The kind that Aristotle calls *intelligible matter*.³ In Aristotle’s *Metaphysics*, as is known, matter plays an individualizing role while form constitutes the conceptual and general element of reality. Two objects sharing the same form can be distinguished from one another only by the circumstance that the form pertains to two different portions of matter. We can have rational knowledge of the formal but not the material elements of reality. Matter can only be grasped intuitively or, to use Aristotle’s words, by means of a *noetic* act. Aristotle maintains that there is no definition of individual entities. Containing a material element, these entities can only be known through intuition or perception and must therefore actually be present to us. Now, while it is intelligible rather than empirical matter that is involved in mathematical objects, the role played is the same. A geometric proof is worked out on an individual triangle that contains a material element and is necessarily – as Peirce also maintains – present to our intuition.

This idea of intuition is of course reminiscent of Kant. In his metaphysical exposition of space, Kant shows that it cannot be a concept and proves that it is a pure *a priori* intuition. One of the arguments used by Kant to support his statement is that space is unique. When we speak of many spaces, Kant says, we are actually speaking of parts of one single space.⁴ Intuitions are always single entities. Now, mathematical knowledge is founded, according to Kant, on the construction of concepts.⁵ Constructing a concept means exhibiting *a priori* the corresponding intuition. It is impossible, he says, to deduce a mathematical proposition from general concepts alone. In order to prove, for example, that in any triangle the sum of any two sides is greater than the third, it is necessary to exhibit the intuition of a single triangle – by actually drawing it – together with the intuitions of all the additional lines required by the demonstration. It is precisely the intuitive nature of the representations employed in mathematics that makes this science really informative. This is Kant’s fundamental discovery: mathematical judgments are synthetic because they are not based on the understanding alone. Every synthesis hinges upon a non-conceptual, intuitional element. This role is played in empirical knowledge by empirical intuition and in mathematics by the pure *a priori* intuitions of space and time. According to Kant, mathematical judgments are synthetic because they synthesize a spatiotemporal manifold that is entirely heterogeneous with respect to the

² CP 4.233.

³ See ARISTOTLE, *Metaphysics* Z, 10, 1036a 1-13.

⁴ See KANT, *Critique of Pure Reason*, B 53, 4-20.

⁵ See KANT, *Critique of Pure Reason*, B 469, 9-11.

understanding. Unlike formal logic, which is analytic, mathematics is synthetic in that it deals with spatiotemporal content. It is, so to speak, on this ontological level that the distinction is made. A logical inference can only clarify what is already present in its premises, whereas a mathematical inference, referring to the intuitions of space and time, can operate a synthesis and add new knowledge to the initial elements.

Returning to Peirce, we can find an analogous distinction between deductive inferences that really increase our knowledge and deductive inferences that are not so powerful. This distinction has, however, entirely different bases in Peirce. In a nutshell, we could say that, unlike Kant's, Peirce's distinction is rooted not in ontology but in methodology. It is not a matter of having or not having spatiotemporal content but rather of using or not using a certain kind of reasoning, which Peirce calls *theorematic reasoning* as opposed to *corollarial reasoning*, the less informative kind (Peirce chose these terms with reference to the theorems and corollaries of Euclidean geometry). According to Peirce, both kinds of reasoning require a spatiotemporal element, which cannot therefore be the basis of the distinction. The spatiotemporal element that we find in Peirce has nothing to do, however, with the Kantian manifold. Space and time are employed, in Peirce's reconstruction, in the diagrammatic representation of mathematical inferences. The core of Peirce's description of mathematical reasoning is that in order to make a deductive inference it is necessary first of all to represent the system of relations expressed by the premises in a diagram; the new relations that emerge in the diagram and constitute the conclusion of the inference must then be observed, sometimes after experimenting on the diagram and altering it. The purpose of the diagram is to make this observation possible. It must possess a material intuitional element in that it is required to represent the relations between the objects involved through corresponding perceptual, observable – i.e. spatiotemporal – relations between its elements. Both theorematic and corollarial reasoning require this perceptual character. The difference between the two lies in the fact that while the diagram remains unchanged in corollarial reasoning, in theorematic reasoning it must be experimented on and modified in order to bring out the new relations (the clearest examples of theorematic proofs are geometric proofs in which the so-called *auxiliary construction* is required).

The distinction between, say, analytic and synthetic *a priori* propositions is thus determined by Peirce at a methodological level, whereas there is no relevant difference in terms of content (the ontological level) in the two cases. While Peirce and Kant both recognize an intuitional, material element in mathematical inferences, the different location they assign it bears witness to a marked difference in their approaches. It is from their difference in categorial analysis that the difference in their approach to mathematics stems. Peirce's categorial analysis, which is the real core of his semiotics, starts from a deep study of Kant's *Critique of Pure Reason* but soon departs from Kantian transcendentalism. Peirce's investigation is not a *critique*, an *a priori* search for the conditions making our experience possible, but rather positive *a posteriori* knowledge, the observation of already formed experience, to which he gives the name *phaneroscopy*. The *phaneron* is defined by Peirce as «the collective whole of all that could ever be present to the mind in any way or in any sense».⁶ Since whatever can be present to the mind always possesses the nature of a sign, phaneroscopy becomes semiotics. This is a

⁶ NEM IV, p. 320.

fundamental difference between Peirce and Kant. The *a posteriori* observational method of phaneroscopy has important implications. Since the categories must be phaneroscopically observed, all of them – and not only the third, i.e. mediation, symbol, generality – must already belong to the sign interpretation. It is this that allows Peirce to jettison the Kantian *thing in itself* – what gave rise to the empirical manifold, which was completely extraneous to the synthesis of understanding and required a special faculty of its own – which he regards as metaphysically dogmatic. Peirce intends to avoid any dogmatic element that is not detectable through direct inspection of the *phaneron*. The Kantian distinction between two different faculties – sensibility and understanding – thus fails. Translated into phaneroscopic terms, the Kantian manifold now becomes the category of *Firstness*.⁷ In a mathematical diagram, this category consists simply in the setting down of the relations between its parts, which is already expressed in semiotic terms because the relations are embodied in a material diagrammatic *token* that can be interpreted symbolically according to its formal structure. The diagram is in itself already particular and general. We find in it the mere *suchness* of *Firstness*, i.e. the simple presentation of concrete elements in mutual relations, and the general mediation of *Thirdness*, i.e. the possibility of interpreting it as referring to a general form of relation. In Kant's terminology, as Peirce himself notes,⁸ the diagram is a *schema*. The schema is, in a certain sense, the only element of Kantian transcendental analysis that survives in Peirce's system. Its role is, however, completely different from Kant's mediation between two different faculties and their heterogeneous contributions to knowledge. It now constitutes the very starting point of phaneroscopic investigation. There is nothing before it.

The diagram-schema is thus individual and general at the same time. In it we find the possibility not only of experimenting and observing, thanks to its individual character, but also of obtaining general conclusions, thanks to the general interpretation it allows. Peirce's explanation of mathematical procedure develops at the level of the diagram-schema without requiring anything else. Mathematics is for Peirce an entirely semiotic activity. As seen above, Peirce explains the informative nature of mathematics at a methodological level with no reference to the content of the signs employed. I regard this as a suggestion to be developed. If we are to understand Peirce's conception of mathematical reasoning, we must move definitively from the plane of the object denoted – which in Peirce is nothing but a limit to which sign interpretation tends – to the plane of the sign itself. It is only at this level that deductive inferences are made.

Given this background and the fact that Peirce saw individuality as the true core of mathematical activity, it is now necessary to understand what kind of individuality he means. If, as I suggest, the object denoted must be set aside, it cannot actually be of individual objects that Peirce is really thinking when he makes assertions such as the following, which is taken from the essay *Description of a Notation for the Logic of Relatives* (1870):

Demonstration of the sort called mathematical is founded on suppositions of particular cases. The geometrician draws a figure; the algebraist assumes a letter to signify a single quantity fulfilling the required conditions.⁹

⁷ See CP 1.302.

⁸ See NEM IV, p. 318.

⁹ CP 3.92.

Peirce is not always clear on this point. Here, for example, he appears to equate geometric and algebraic demonstrations with respect to the kind of individuality they draw upon, calling into play the numerical objects denoted by the letters of algebra. I would insist, however, that we cannot account for Peirce's description of mathematical activity through reference to the object denoted. If we look at the object denoted, the parallel drawn between geometric and algebraic demonstrations is simply incorrect. When we introduce individuals in algebra, calling them *a*, *b* or *c*, we are just saying that these letters can be used as variables. They represent individual but indeterminate quantities. It is a very different matter when individuals are introduced in a geometric proof. Here an individual is a single, determinate object: a single segment of a definite length, a single circle of a definite radius, and so on. In geometry space is represented through space, and the only way to represent a segment or a circle through space is to trace a single segment or a single circle that will necessarily be of particular length or particular radius. Individuality thus differs greatly in geometry and algebra with respect to the object denoted. So, what kind of individuality does Peirce have in mind when he equates the two, claiming that any deductive inference whatsoever involves an individual entity, which introduces a material, intuitional element in mathematics and ensures the possibility of surprising discoveries?

Although Peirce sometimes wavers on this point, I believe that mathematical individuality is to be found within the sign itself. The diagram contains an individual element upon which the deductive inference is made. There is, however, a confusion to be avoided here. As Peirce constantly repeats, the diagram is a *token*, a singular object employed as a sign. This cannot, however, explain its peculiarity. Once drawn or uttered so as to be apprehended, any sign whatsoever is necessarily a *token*. If we write down the word *man*, this word written on paper becomes a *token*. The point is that the mathematical diagram is essentially a *token* and could not be otherwise. What then is its peculiarity?

In *On the Algebra of Logic: A Contribution to the Philosophy of Notation* (1885), Peirce classifies the juxtaposition of algebraic letters to the right and the left of the operational symbol as an index and includes this in the list of the kinds of signs employed by the algebra of logic.¹⁰ This juxtaposition – the concrete juxtaposition of the material letters involved in the formula – is an index because it is the very element of *Secondness* within the algebraic diagram, the element of existence. The totality of these juxtapositions constitutes an icon of the relational situation in the mathematical state of affairs represented. On this view, the individuality of the mathematical diagram is very different from the individuality of the word *man* when considered as a *token*. Nothing new can in fact be extracted from observation of the latter, whereas “a great distinguishing property of the icon is that by the direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction.”¹¹ As Peirce notes few lines earlier, an algebraic expression is not a compound conventional sign. It is not something we can consider a whole, as we do the word *man*. On the contrary, we must go into it and experiment on the juxtapositions between its parts so as to discover new

¹⁰ See CP 3.385.

¹¹ CP 2.279.

truths that are not present in the premises of the inference. The peculiarity of the mathematical sign lies precisely in the fact that the iconicity of the entire formula is based on some material elements of *Secondness* subsisting within the formula itself. The algebraic formula is essentially a *token* because we are required to work concretely on the juxtapositions of the letters.

This holds true in geometry as well as algebra. It is in this sense that geometry and algebra can be equated. It might seem at first sight that geometric demonstrations are developed on individual objects. Pythagoras's Theorem is proved by working on a particular right triangle made up in turn of particular sides and angles. Closer examination shows, however, that the three sides and the two non-right angles are in point of fact functionally treated as variables. Their peculiarities are never mentioned in the proof. The situation is analogous to that of algebra. We are, admittedly, obliged by the nature of geometry to draw one particular triangle, but what actually matters in the demonstration is not the individuality of the object but solely the individuality of the sign. Moreover, the demonstration is not concerned with the individuality of the diagram regarded as a whole but with the individuality of the single juxtapositions of its parts, which are of an indexical nature and serve all together to form the iconic sign.

The real difference between algebra and geometry is the fact that the operational sign between the juxtaposed parts of the diagram is not required in the latter. The relations between the geometric elements do not need to be symbolically expressed, being immediately shown by corresponding spatial relations in the diagram. This is a very important difference and one deserving more attention than it can be given here.

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