A Note on Abstract Consequence Structures

Uma Nota sobre Estruturas Abstratas de Conseqüência

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Abstract: Tarski's pioneer work on abstract logic conceived consequence structures as a pair (X, Cn) where X is a non empty set (infinite and denumerable) and Cn is a function on the power set of X, satisfying some postulates. Based on these axioms, Tarski proved a series of important results. A detailed analysis of such proofs shows that several of these results do not depend on the relation of inclusion between sets but only on structural properties of this relation, which may be seen as an ordered structure. Even the notion of finiteness, which is employed in the postulates may be replaced by an ordered substructure satisfying some constraints. Therefore, Tarski's structure could be represented in a still more abstract setting where reference is made only to the ordering relation on the domain of the structure. In our work we construct this abstract consequence structure and show that it keeps some results of Tarski's original construction.

Keywords: Abstract logic. Consequence operators. Order structures.

Resumo: *O trabalho pioneiro de Tarski sobre lógica abstrata concebia estruturas de conseqüência como um par (X, Cn) tal que X é um conjunto não vazio e Cn é uma função definida no conjunto das partes de X, satisfazendo alguns postulados. Baseado nesses postulados, Tarski demonstra uma série de resultados importantes. Uma análise detalhada de tais demonstrações mostra que vários desses resultados não dependem da relação de inclusão entre conjuntos, mas apenas das propriedades estruturais dessa relação, que pode ser vista como uma estrutura de ordem. Mesmo a noção de finitude, que é empregada nos postulados, pode ser substituída por uma subestrutura ordenada satisfazendo alguns vínculos. Portanto, a estrutura de Tarski pode ser representada em um contexto ainda mais abstrato onde se faz referência apenas à relação de ordem sobre o domínio da estrutura. Neste trabalho, construímos essa estrutura de conseqüência abstrata e mostramos como ela mantém alguns resultados da estrutura de Tarski original.*

Palavras-chave: *Lógica abstrata. Operadores de conseqüência. Estruturas de ordem.*

1. Consequence Structures

A *consequence structure* is a pair (X, Cn) such that X is a non empty set (in fact, Tarski considered X as an infinite enumerable set) and Cn is a map on the parts of X satisfying the following postulates for all $A \subseteq X$ (see TARSKI, 1983a, 1983b; DE SOUZA; VELASCO, 2001, 2002, and VELASCO, 2000):

(i) $A \subseteq Cn(A)$;

(ii) $Cn(Cn(A)) = Cn(A)$;

(iii) $Cn(A) = \cup \{Cn(A'): \text{for all } A' \subseteq A, \text{finite}\}$.

The first theorem is the well known law of monotonicity that can be proved as: Consider $A \subseteq B$ and let $x \in Cn(A)$. So, by (iii), there exists $A' \subseteq A$, finite such that $x \in Cn(A')$. Thus, there exists $A' \subseteq B$, finite such that $x \in Cn(A')$ and, using (iii) again, we have that $x \in Cn(B)$.

A detailed analysis of such a proof shows that monotonicity does not depend on the concept of finite that occurs in axiom (iii), but only on the properties of the relation of inclusion between subsets of X. Even the concept of finite can be modified in a more abstract setting.

Therefore, if we consider an arbitrary family F of subsets of X and if we change the postulate (iii) by:

(iii') $Cn(A) = \cup \{Cn(A'): \text{for all } A' \subseteq A, A' \in F\}$, we would get monotonicity with an analogous proof.

The aim of this work is to study consequence structures in a more abstract level using partial orders.

2. Order Structures

In this section we review the main concepts of order structures that will be used in the next sections.

A *partial order* is a pair $(X,\leq)$ where X is a non empty set and $\leq$ is a binary relation on X such that, for every $x,y,z \in X$, we have the following properties:

a) Reflexivity: $x \leq x$;

b) Anti-symmetry: if $x \leq y$ and $y \leq x$, then $x = y$;

c) Transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$.

Moreover, if it holds that, for every $x, y \in X$, we have $x \leq y$ or $y \leq x$, we say that the order is *linear* or *total*.

Let $A \subseteq X$ and $x \in X$. We say that x is a *upper bound* of A if for every $a \in A$, $a \leq x$. Similarly, we say that x is a *lower bound* of A if for every $a \in A$, $x \leq a$. We say that x is a *supremum* of A if for every upper bound y of A, we have that $x \leq y$. Again, we say that x is a *infimum* of A if for every lower bound y of A, we have that $y \leq x$. The supremum or infimum of A (if they exist) are unique and will be denoted by sup A e inf A, respectively. A partial order has a *greatest element*, if there exists $y \in X$ such that $x \leq y$, for every $x \in X$. Again, a partial order has a *least element*, if the exists $y \in X$ such that $y \leq x$, for every $x \in X$. The greatest and least element of a partial order, if they exist, are unique and will be denoted by 1 e 0, respectively.

A *lattice* is a partial order $(X, \leq)$ such that for every $x, y \in X$, there exist sup $\{x, y\}$ and inf $\{x, y\}$, denoted, respectively, by $x \vee y$ and $x \wedge y$. It is clear that, for lattices, if A finite subset of X, then, by induction, there exist sup A e inf A. On the other hand, a lattice $(X, \leq)$ is called *complete* if for all $A \subseteq X$, there exist sup A and inf A. Again, it is clear that, in this case, $1 \leq \sup X$ and $0 \leq \inf X$ are the greatest and the least elements of the lattice.

The following results of order theory will be used freely in what follows. For $x, y \in X$ and $A, B \subseteq X$ we have:

1. $x \vee y = y$ if and only if $x \leq y$;

2. $x \wedge y = x$ if and only if $x \leq y$;

3. $x \leq x \vee y$ and $y \leq x \vee y$;

4. $x \wedge y \leq x$ and $x \wedge y \leq y$;

5. If $x \leq y$ and $x \leq z$, then $x \leq y \wedge z$. In general, we have that, if $x \leq a$, for all $a \in A$, then $x \leq \inf A$;

6. Se $x \leq z$ and $y \leq z$, then $x \vee y \leq z$. In general, we have that, if $a \leq x$, for all $a \in A$, then $\sup A \leq x$;

7. If $A \subseteq B$, then $\sup A \leq \sup B$.

Consider a lattice $(X, \leq)$ e let $A \subseteq X$. We say that A is a *sub-lattice* (of X) if for all $x, y \in A$, we have that $x \vee y$ and $x \wedge y$ also belong to A (that is, A is closed by the operations sup and inf of whatever two of its elements).

3. Abstract Consequence Structures

An *abstract consequence structure* is a 4-tuple $(X, F, \leq, c)$ where $(X, \leq)$ is a complete lattice, $F \subseteq X$ is a sub-lattice that contains 0 and c is a function on X satisfying the following conditions, for all $x \in X$:

(i) $x \leq c(x)$;

(ii) $c(c(x)) \leq c(x)$;

(iii) $c(x) \leq \sup \{c(y): y \in F \text{ and } y \leq x\}$.

Theorem 1. For $x \in X$, $c(x) \leq c(c(x))$.
Proof. By (i), we have $c(x) \leq c(c(x))$. Then, by (ii) e anti-symmetry, it follows the result.

We define a function Φ on X with values in the parts of X, such that for all $x \in X$, we have that $\Phi(x) = \{y \in F: y \leq x\}$.

Theorem 2. For $x \in X$, $c(x) = \sup c(\Phi(x))$.
Proof. Regarding that if $A \subseteq X$, then $c(A) = \{c(a): a \in A\}$; the result is a direct consequence of (iii) and the definition of Φ.

Theorem 3. For $x \in X$, $F(x)$ is a sub-lattice of F that contains 0.

Proof. As 0 is the least element of X (and is an element of F), then $0 \in \Phi(x)$. Moreover, if $a, b \in \Phi(x)$, then we have that $a, b \in F$ and, thus, $a \vee b, a \wedge b \in F$ (because F is a sub-lattice of X). Therefore, as $a \leq x$ and $b \leq x$, then we have that $a \vee b, a \wedge b \leq x$ and, $a \vee b, a \wedge b \in \Phi(x)$.

Theorem 4. For $x, y \in X$, we have the following properties:

a) If $x \leq y$, then $\Phi(x) \subseteq \Phi(y)$;

b) $\Phi(x) \cup \Phi(y) \subseteq \Phi(x \vee y)$;

c) $\Phi(x \wedge y) = \Phi(x) \cap \Phi(y)$.

Proof. a) Suppose that $x \leq y$ and let $z \in \Phi(x)$. Then, $z \in F$ and $z \leq x$. By transitivity, $z \leq y$. Therefore, $z \in \Phi(y)$. b) e c) are consequence of a) and the fact that $x, y \leq x \vee y$ and $x \wedge y \leq x, y$.

Theorem 5. For $x, y \in X$, we have that if $x \leq y$, then $c(x) \leq c(y)$. That is, the function c is monotone and increasing on X with respect to the order relation $\leq$.

Proof. Suppose that $x \leq y$. By theorem 4a, $\Phi(x) \subseteq \Phi(y)$. Then, $c(\Phi(x)) \subseteq c(\Phi(y))$, and we have that $\sup c(\Phi(x)) \subseteq \sup c(\Phi(y))$. Applying the theorem 2, we have the result.

Theorem 6. For $x, y \in X$, are equivalents: $x \leq c(y)$ and $c(x) \leq c(y)$.

Proof. As, by (i), it holds that $x \leq c(x)$, considering that $c(x) \leq c(y)$, we have, by transitivity, that $x \leq c(y)$. On the other hand, supposing $x \leq c(y)$, applying increasing monotonicity, we obtain $c(x) \leq c(c(y))$. By theorem 1, we have $c(x) \leq c(y)$.

Theorem 7. For $A \subseteq X$, are equivalents:

a) If $x \leq y$, then $c(x) \leq c(y)$;

b) $\sup c(A) \leq c(\sup A)$;

c) $c(\inf A) \leq \inf c(A)$.

Proof. Let $u = \sup A$ and $v = \inf A$. Then, $a \leq u$ for all $a \in A$. Applying a), $c(a) \leq c(u)$ for all $a \in A$. Thus, $\sup \{c(a): a \in A\} = c(u)$, that is $\sup c(A) \leq c(\sup A)$.

Similarly, as $v = a$ for all $a \in A$, applying a), we have $c(v) \leq c(a)$ for all a A. Thus, $c(v) \leq \inf \{c(a): a \in A\}$, that is, $c(\inf A) \leq \inf c(A)$.

We proved that a) entails b) and c).

Consider, now, $x,y \in X$ and let $A = \{x, y\}$. By b), we have that $c(x) \vee c(y) \leq c(x \vee y)$. As $x \leq y$, then $x \vee y = y$. Thus, $c(x) \vee c(y) \leq c(y)$, that is, $c(x) \leq c(y)$.

Similarly, by c), we have that $c(x \wedge y) = c(x) \wedge c(y)$. As $x \leq y$, then $x \wedge y = x$. Thus, $c(x) \leq c(x) \wedge c(y)$, that is, $c(x) \leq c(y)$.

We proved, then, that b) entails a) and also that c) entails a).

Therefore, we have the result.

Theorem 8. a) For $x, y \in X$, we have that: $c(x \vee y) = c(x \vee c(y)) = c(c(x) \vee c(y))$; b) If $A \subseteq X$, then $c(\sup A) = c(\sup c(A))$.

Proof. a) As, by (i), $y \leq c(y)$, we have that $x \vee y \leq x \vee c(y)$. By theorem 5, $c(x \vee y) \leq c(x \vee c(y))$. As $x \leq c(x)$, we have that $x \vee c(y) \leq c(x) \vee c(y)$. Again, by theorem 5, c(x Ú c(y)) $\leq c(c(x) \vee c(y))$.

On the other hand, $x \leq x \vee y$ and $y \leq x \vee y$. by theorem 5, we have that $c(x) \leq c(x \vee y)$ and $c(y) \leq c(x \vee y)$. Thus, $c(x) \vee c(y) \leq c(x \vee y)$. Applying again the theorem 5, we have $c(c(x) \vee c(y)) \leq c(c(x \vee y))$. By theorem 1, $c(c(x) \vee c(y)) \leq c(x \vee y)$. Therefore, we obtain, by anti-symmetry e transitivity, $c(x \vee y) = c(x \vee c(y)) = c(c(x) \vee c(y))$.

b) As $a \leq c(a)$, for all $a \in A$, then $\sup \{a: a \in A\} \leq \sup \{c(a): a \in A\}$. By theorem 5, we have, then, $c(\sup \{a: a \in A\}) \leq c(\sup \{c(a): a \in A\})$, that is, $c(\sup A) \leq c(\sup c(A))$.

On the other hand, by theorems 5 and 7, we have that $\sup c(A) \leq c(\sup A)$. Applying again the theorem 5, we obtain $c(\sup c(A)) \leq c(c(\sup A))$. Now, by theorem 1, $c(\sup c(A)) \leq c(\sup A)$. Finally, by anti-symmetry, we obtain $c(\sup c(A)) = c(\sup A)$.

Consider an abstract structure of consequence $(X, F, \leq, c)$. Let $x \in X$. We say that x is a *theory* if and only if $c(x) \leq x$.

Theorem 9. For $x \in X$, x is a theory if and only if $c(x) = x$. That is, x is a fixed point of the function c.

Proof. Immediate from (i) and the definition of theory.

Theorem 10. If x is a theory, then are equivalents, for $y \in X$: $y \leq x$ and $c(y) \leq x$.

Proof. Suppose that $y \leq x$. By theorem 5, $c(y) \leq c(x)$. As x is a theory, by theorem 9, $c(y) \leq x$. Suppose, now, that $c(y) \leq x$. As by (i), $y \leq c(y)$, by transitivity, we have that $y \leq x$.

Theorem 11. For $x \in X$, we have that:

a) c(x) is a theory and if y is a theory such that $x \leq y$, then $c(x) \leq y$. That is, $c(x) = \inf \{y \in X: y \text{ is a theory and } x \leq y \}$;

b) If x is a theory, then $c(0) \leq x$ That is, $c(0) = \inf \{x \in X: x \text{ is a theory}\}$.

Proof. a) By theorem 1, $c(x) = c(c(x))$. Then, by theorem 9, c(x) is a theory. b) As $0 \leq x$, by theorem 5, $c(0) \leq c(x)$. As x is a theory, by theorem 9, $c(0) \leq x$.

Theorem 12. We have: 1 is a theory and $1 = \sup \{x \in X: x \text{ is a theory}\}$.

Proof. Immediate, because $x \leq 1$, for all $x \in X$.

Theorem 13. a) If $x, y \in X$ are theories, then $x \wedge y$ is a theory. b) If $A \subseteq X$ is a family of theories, then inf A is a theory.

Proof. a) As $x \wedge y \leq x, y$, by theorem 5, $c(x \wedge y) \leq c(x), c(y)$. Thus, $c(x \wedge y) \leq c(x) \wedge c(y)$ and, as x and y are theories, by theorem 9, $c(x \wedge y) \leq x \wedge y$.

b) Let $u \leq \inf A$. Then, $u \leq a$, for all $a \in A$. By theorem 5, $c(u) \leq c(a)$, for all $a \in A$, that is, $c(u) \leq \inf \{c(a): a \in A\}$. As, for all $a \in A$, we have that a is a theory, then $c(u) \leq \inf \{a: a \in A \}$, that is, $c(\inf A) \leq \inf A$.

4. Valuation Structures

A *valuation structure* is a pair (X, K) such that X is a non-empty set and K is a family of subsets of X (see Da COSTA; BÉZIAU, 1994; Da COSTA; LOPARIC, 1984; De SOUZA, 2001, and Da SILVA, 2000).

Let $A \subseteq X$. We define $Mod(A) = \{V \in K: A \subseteq V\}$ the set of *models* of A. Similarly, for $x \in X$, we define $Mod(x) = Mod(\{x\}) = \{V \in K: \{x\} \subseteq V\} = \{V \in K: x \in V\}$.

It is easy to see that $Mod(\varnothing) = K$ and $Mod(X) = \varnothing$, if $X \notin K$.

Theorem 14. For $A, B \subseteq X$, we have the following results:

a) If $A \subseteq B$, then $Mod(B) \subseteq Mod(A)$;

b) $Mod(A) = \cap\{Mod(a): a \in A\}$.

Proof. a) Suppose that $A \subseteq B$ and consider $V \in Mod(B)$. Then, $V \in K$ and $B \subseteq V$. Therefore, $V \in K$ and $A \subseteq V$, that is, $V \in Mod(A)$.

b) $V \in \cap\{Mod(a): a \in A\}$ iff $V \in Mod(a)$ for all $a \in A$ iff $V \in K$ and $a \in V$ for all $a \in A$ iff $v \in K$ and $A \subseteq V$ iff $v \in Mod(A)$.

We define, for $A \subseteq X$, $Cn(A) = \{a \in X: Mod(A) \subseteq Mod(a)\}$ the set of *consequences* of A.

Theorem 15. $Cn(A) = B$ iff $Mod(A) \subseteq Mod(B)$.

Proof. Let $A, B \subseteq X$. Using theorem 14b we have: $Cn(A) = B$ iff $Mod(A) \subseteq Mod(b)$ for all $b \in B$ iff $Mod(A) \subseteq \cap\{Mod(b): b \in B\}$ iff $Mod(A) \subseteq Mod(B)$.

Theorem 16. For $A, B \subseteq X$, we have:

a) $A \subseteq Cn(A)$;

b) If $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$;

c) $Cn(A) = Cn(Cn(A))$.

Proof. a) Consider $a \in A$ and we have to show that $a \in Cn(A)$, that is, $Mod(A) \subseteq Mod(a)$. Let $V \in Mod(A)$, then $A \subseteq V$. Thus, $a \in V$, that is, $V \in Mod(a)$.

b) Consider $A \subseteq B$ and suppose that $a \in Cn(A)$. We have to show that $a \in Cn(B)$, that is, $Mod(B) \subseteq Mod(a)$. Let $V \in Mod(B)$. Thus, $B \subseteq V$ and we have that $A \subseteq V$, that is, $V \in Mod(A)$. As $a \in Cn(A)$, we have $Mod(A) \subseteq Mod(a)$, and then, $V \in Mod(a)$.

c) From a), we have that $Cn(A) \subseteq Cn(Cn(A))$. Now, suppose that $a \in Cn(Cn(A))$ and we have to show that $a \in Cn(A)$, that is, $Mod(A) \subseteq Mod(a)$. Consider $V \in Mod(A)$ and let $D = Cn(A)$. Thus, $Mod(A) \subseteq Mod(d)$ for all $d \in D$. Therefore, $V \in Mod(d)$ for all $d \in D$, that is $d \in V$ for all $d \in D$, i.e., $D \subseteq V$. Then, $V \in Mod(D)$. Now, as $a \in Cn(Cn(A)) = Cn(D)$, then, $Mod(D) \subseteq Mod(a)$. Therefore, $V \in Mod(a)$.

5. Abstract Valuation Structures

We finished this note defining an abstract setting for valuation structures in an analogous way that we done for consequence structures.

An *abstract valuation structure* is a 3-tuple $(X, \leq, K)$ such that $(X, \leq)$ is a complete lattice and K is just a subset of X.

If $A, B \subseteq X$ we say that B *recover* A, in symbols $A < B$, iff for all $a \in A$ there exists $b \in B$ such that $a \leq b$. It is easy to see that $A < A$; if $A < B$ and $B < A$, then $A = B$; and if $A < B$ and $B < C$, then $A < C$.

Consider $x \in X$. We define $\mathrm{Mod}(x) = \{v \in K: x \leq v\}$ the set of *models* of x. It is immediate that $\mathrm{Mod}(0) = K$ and $\mathrm{Mod}(1) = \varnothing$, if $1 \notin K$. Moreover, if $x \leq y$, then $\mathrm{Mod}(x) \subseteq \mathrm{Mod}(y)$ and $\mathrm{Mod}(x)\ \mu\ \mathrm{Mod}(y)$.

Now, we define a function $c: X \rightarrow X$, such that:

$$c(x) = \inf \{y \in X: \mathrm{Mod}(x) < \mathrm{Mod}(y)\}.$$

It is easy to see that c has the properties of an abstract consequence operators:

a) $x \leq c(x)$;
b) if $x \leq y$, then $c(x) \leq c(y)$;
c) $c(x) = c(c(x))$.

6. Final Considerations

Of course the theory developed here can be extended dealing with consistent and complete sets, axiomatizability and so on as the original Tarski's papers. This is just a summary that will be developed in future papers.

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