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DOSSIÊ PEIRCE E A LÓGICA / *DOSSIER PEIRCE AND LOGICS*

Three-valued logic and paraconsistency in Peirce[1]

Lógica trivalorada e paraconsistência em Peirce

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Abstract: Peirce is now recognized as one of the pioneers of mathematical and algebraic logic, but his original work on non-classical logic still receives little attention outside the narrow circle of Peirce scholars. This is particularly evident in his development of the three-valued propositional calculus, recorded in Peirce's Logic Notebook more than a decade before the rise of many-valued logics. The triadic logic, as named by Peirce, was formalized by Turquette in the late 1960s. Turquette presented an axiomatic interpretation of the three-valued tables in a series of articles that have since become a cornerstone in such studies. Recently, I have proposed a new approach, emphasizing a non-explosive fragment of triadic logic. This paper aims to expand the research in the following points: (i) a critical analysis of Turquette's works, including the discussion of the Rosser-Turquette method of axiomatization; and (ii) the reconstruction of a fragment of the triadic logic in a system based on Sobociński's material implication, formalized in sequent calculus. I conclude that Peirce's three-valued matrix induces a paraconsistent, relevant, and substructural logic with investigative potential for contemporary research in non-classical logics.

Keywords: Charles S. Peirce. Logic. Non-classical logics. Paraconsistency. Sequents. Three-valued logics.

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Resumo: *Peirce é hoje reconhecido como um dos pioneiros da lógica matemática e algébrica, mas seu trabalho original em lógicas não-clássicas ainda recebe escassa atenção fora do círculo estreito de especialistas peircianos. Esse é o caso do cálculo proposicional trivalorado que Peirce registrou em seu "Logic Notebook", mais de uma década antes do surgimento das lógicas multivaloradas. A lógica triádica, como Peirce a chamava, foi formalizada por Turquette no final dos anos 1960. Turquette apresentou uma interpretação axiomática das tabelas trivaloradas em uma série de artigos que se tornaram referência nesse estudo. Recentemente, propusemos uma nova abordagem, enfatizando um fragmento não-explosivo da lógica triádica. Este artigo objetiva ampliar a pesquisa nos seguintes pontos: (i) uma análise crítica dos trabalhos de Turquette, incluindo a discussão do método Rosser-Turquette de axiomatização; e (ii) reconstrução do fragmento da lógica triádica em um sistema baseado na implicação material de Sobociński, formalizado em cálculo de sequentes. Concluímos que a matriz trivalorada de Peirce induz uma lógica paraconsistente, relevante e subestrutural, com potencial investigativo para as pesquisas contemporâneas em lógicas não-clássicas.*

Palavras-chave: *Charles S. Peirce. Lógica. Lógicas não-clássicas. Lógicas trivaloradas. Paraconsistência. Sequentes.*

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1 This paper presents the results of Postdoctoral research in Philosophy at the University of São Paulo (FFLCH-USP), titled "Studies in Charles S. Peirce's three-valued logic" (2021-2022).

1 Introduction

Charles S. Peirce is considered, alongside Jevons and Schröder, one of the key figures in the development of modern deductive formal systems in the 19th century, working in algebraic approaches inspired by Boole's pioneering work. Peirce's contributions to formal logic span approximately from 1867 to 1910. They include refinements of Boolean calculus and, later on, the creation of diagrammatic logical systems for first-order logics with identity – the Existential Graphs. Additionally, Peirce holds a prominent position in the model-theoretic tradition, alongside Skolem and Löwenheim (Brady, 2000).

One of his major contributions to mathematical logic was the extension of Boolean algebras to include quantification theory, first published in "Description of a notation for the logic of relatives" (Peirce, 1870). In subsequent years, Peirce's work in collaboration with his students, such as O. H. Mitchel, resulted in a system equivalent to modern first-order logic.

In "On the algebra of logic" (1880), Peirce presented a complete propositional logic system with primitive operators of negation and material implication. Moreover, his propositional logic includes introduction and elimination rules for connectives, anticipating modern natural deduction and Gentzen's sequent calculus. Ma and Pietarinen (2020) recently presented Peirce's propositional logic as a sequent calculus, endorsing this interpretation.

However, Peirce's experiments with non-classical logics, which he never published during his lifetime, are much less known among logicians. In some pages of Peirce's "Logic Notebook", dated from January to February 1909, we find experiments in modal semantics and three-valued logics that preceded advancements in the 20th century. Particularly, in certain handwritten pages Peirce proposed a three-valued propositional logic, a decade before the publications of Łukasiewicz (1920) and Post (1921). The calculus is presented in a list of truth tables for negation, conjunction, and disjunction operators. Peirce referred to this system as *triadic logic,*[2] but did not provide details of its semantics or his philosophical motivations (although some of these manuscripts may have been lost).

These notes were discovered and published in 1966 by Max Fish and Atwell Turquette (1966). In the following years, Turquette published a series of articles in which he developed a formalization of Peirce's three-valued calculus, applying a method he devised with Rosser in the 1950s (Rosser; Turquette, 1952). This material remains, to this day, and to the best of our knowledge in the English-speaking academic literature, virtually the entirety formal analysis of Peirce's three-valued truth tables available to us.[3]

Recently, I proposed a new approach to Peirce's triadic logical matrices (Salatiel, 2023), inspired by Parks' article (1971). In this new approach, Peirce's triadic logic is analyzed as three independent formal systems, varying according to the negations and purposes of the author.[4] Two of these systems were rediscovered years later by Łukasiewicz, Post, Kleene, and Bochvar. The third, however, which I call $\mathbf{P_3}$, is the most original and corresponds to the fragment of relevant logics suggested by Sobociński. Its main characteristic is that a formula receives a third truth value (which Peirce called "limit") only when its components also receive this same value. I also demonstrate that this $\mathbf{P_3}$ calculus can be transformed into paraconsistent or paracomplete logics, and I present a complete and sound calculus using analytic tableaux (Salatiel, 2022).

This paper advances the results of this research in the following aspects: (i) presents a critical analysis of Turquette's articles, pointing out what I consider to be the most problematic aspects that

2 In this article, I will use the term "triadic logic" as a synonym for Peirce's three-valued propositional logic.

3 I do not include here the discussion on Peirce's philosophical motivations, which is considered a separate debate. Regarding this matter, cf. Fish e Turquette (1966), Lane (1999; 2001) and Odland (2020; 2021).

4 Belikov (2021) reached similar conclusions in a recent article published, where he presents a formalization in a natural deduction system without the material implication operator.

justify a new approach to the topic; (ii) provides a detailed discussion of the elements absent in Peirce's calculus, namely the notions of logical consequence and material implication, justifying my choices based on Peirce's work; (iii) examines a fragment of Peirce's triadic logic supplemented by Sobociński's implication, suggesting that the conception of paraconsistency was implicit in Peirce's study of many-valued logics; (iv) formalizes Peirce's calculus based on primitive operators of negation and material implication into a sequent calculus, which is more consistent with Peirce's proposal and according to contemporary research on three-valued paraconsistent systems (Avron, 2003).

This article is divided into four sections: in the second section, I analyze Turquette's articles, and in the third section, I examine the $\mathbf{P}_3$ calculus in correspondence with paraconsistent systems. The original conclusion of this research is that Peirce's triadic logic leads to a relevant and substructural paraconsistent system. I set aside the investigation of Peirce's philosophical motivations, which deserves further complementary study.

2 The Rosser-Turquette method applied to triadic logic

Experiments with Peirce's three-valued logic are recorded in three pages of his "Logic Notebook", dated February 23, 1909, and first published by Max Fish and Atwell Turquette in 1966 (Fish; Turquette, 1966).[5] This makes the philosopher one of the pioneers of non-classical logics (or, as he called them, non-Aristotelian logics)[6], alongside figures such as the Polish Jan Łukasiewicz, the Russian Nicolai Vasiliev, and the Scot Hugh MacColl, who worked in similar directions from the late 19th to the early 20th century.

Two specific handwritten pages, seq. 638 and seq. 645, contain logical tables for four different unary negation operators, $\{\bar{x}, \dot{\bar{x}}, \grave{x}, \acute{x}\}$, and six other tables for binary operators, listed in pairs, $\{\{\Phi, \Theta\}, \{\Psi, Z\}, \{\Omega, Y\}\}$, for a three-valued propositional calculus. Peirce denotes these values by the symbols V, F, and L, respectively, from Latin verum and falsum, and limit.[7] For standardization purposes, in this paper I will use the numbers 1, 3, and 2 for these respective values, instead of Peirce's notation. Below are the tables for the logical connectives of triadic logic:

5 Peirce's "Logic Notebook" compiles unpublished manuscripts written by the author throughout his life. The material is now digitized and available online at: https://nrs.harvard.edu/urn-3:FHCL.Hough:3686182. The focus of my analysis will be on these three main pages of Peirce's draft, but it is necessary to mention at least two other sources related to triadic logic. The first consists of three additional pages from the same manuscripts preceding the mentioned three, which discuss modal logic and Existential Graphs. These six pages are analyzed together by Odland (2020). The second source comes from unpublished texts published under the title "N-valued logic" by Carolyn Eisele in 1976 (NEM III: 739-766). These writings refer to discussions on continuity dating back to 1903, which, according to Eisele, could explain Peirce's motivations in developing his three-valued logical system.

6 The reference is a letter from Peirce to Francis C. Russell cited by Paul Carus, editor of the journal The Monist (Carus, 1910). In this letter, Peirce mentions some results from research on non-Aristotelian logics, which he considered insufficient to be published. Could he be referring to triadic logic?

7 Regarding this terminology, Turquette (1988) comments that Peirce's aim was to emphasize the difference between true and not-false in triadic logic, unlike the equality between these values in classical logic (this difference, related to the notion of designated values, will be explained later). However, Peirce already used the symbols v and f (in lowercase and bold), for verum and falsum, in his bivalent propositional logic (W: 5, 112, 1884).

x	$\bar{x}$	$\dot{\bar{x}}$	$\grave{x}$	$\acute{x}$
1	3	2	3	2
2	2	2	1	3
3	1	2	2	1

Table 1: Negations.

Φ	1	2	3
1	1	1	1
2	1	2	3
3	1	3	3

Θ	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

Ψ	1	2	3
1	1	1	3
2	1	2	3
3	3	3	3

Z	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

Ω	1	2	3
1	1	2	3
2	2	2	2
3	3	2	3

Y	1	2	3
1	1	2	3
2	2	2	2
3	1	2	3

Table 2: Binary operators.

The main question concerning these matrices is whether they indeed constitute a sound and complete calculus of three-valued propositional logic. Peirce, although he did not explicitly address metalogical issues, seemed confident in his work, judging by the last sentence of his manuscript: “Triadic logic is universally true. But dyadic [bivalent] logic is not absolutely false” (seq. 645).[8]

Fish and Turquette (1966) argued, based on these manuscripts, that Peirce should be recognized as the creator of the matrix method not only for bivalent logics but also for many-valued logics.[9] They also argued that he should be acknowledged as the first to develop a three-valued logical system, anticipating the publications of Łukasiewicz (1920) and Post (1921) by nearly a decade. In the following fifteen years, Turquette alone published six articles providing a comprehensive analysis of Peirce’s three-valued calculus (Turquette, 1967; 1969; 1972; 1976; 1978; 1981).[10] This body of work, collectively, constitutes virtually the only literature on the formal aspects of Peirce’s triadic logic.[11]

I will divide Turquette’s articles into two groups to facilitate my exposition. In the first group (Turquette, 1967; 1969; 1972), Turquette emphasizes the discussion regarding hypotheses about Peirce’s motivation for introducing different unary and binary operators in three-valued logic. In the second group of articles (Turquette, 1976; 1978; 1981), he presents axiom systems in a Hilbert-style for triadic logic, employing the method he developed in the 1950s with J. Barkley Rosser (Rosser; Turquette, 1945; 1952). The work is sophisticated and complex, but it has some weaknesses which I will address here.

8 Nevertheless, he did not consider his work complete enough to submit it for publication.

9 There is an ongoing debate about whether Peirce was the inventor of truth tables. Regarding this, consult Rodrigues (2021).

10 Two complementary articles are excluded from this main examination: Turquette (1974; 1988).

11 Some exceptions that should be mentioned are Parks (1971) and, more recently, Belikov (2021). In this list, I exclude works that deal exclusively with Peirce’s philosophical motivations.

In the first three articles, the aim is to explain the motivation behind the exposition of triadic logic with different operators, particularly to explain the function of two connectives, Φ (*phi*) and Ψ (*psi*), which Turquette considered "mysterious" (Turquette, 1967). In the first work, with Max Fisch, they identified correspondences between the negations $\bar{x}$, $\dot{\bar{x}}$, $\grave{x}$ and $\acute{x}$ with, respectively, Łukasiewicz's negation, Słupecki's T-operator, and Post's two cyclic negations; and correspondences between the binary operators Θ, Z, Ω and Y with those later used as conjunctions and disjunctions by Łukasiewicz, Kleene, and Bochvar. However, they curiously did not identify Φ and Ψ as respective disjunction and conjunction connectives, which was only pointed out by Parks (1971).[12]

Turquette then postulates that the choice of these operators is due to metalogical motivations: Peirce, according to him, sought to develop a functionally complete three-valued system (Turquette, 1967; 1969). To achieve this, Turquette identifies dual relations among the connectives. Furthermore, to substantiate this hypothesis, he introduces two partial negations to Peirce's negation tables (partial negations, unlike complete negations, do not transform all truth values). Therefore, the tables for partial negations are as follows (note that $\neg_2 p$ corresponds to Peirce's negation $\bar{x}$):[13]

p	$\neg_1 p$	$\neg_2 p$	$\neg_3 p$
1	1	3	2
2	3	2	1
3	2	1	3

Table 3: Partial three-valued negations.

With this, Turquette demonstrates that the operators relate dually.[14] For example, the operators Θ and Φ are duals relative to the partial negations $\neg_1 p$ e $\neg_3 p$ (which can be verified in a truth table):

(i) $p \, \Theta \, q \coloneqq \neg_1 (\neg_1 p \, \Phi \, \neg_3 q)$.

Thus, he proves that functionally complete calculi can be constructed using the set of primitive operators {O, ¬}, where "O" is a meta-variable representing any binary operator, and "¬", denotes one of Peirce's two complete negations ($\grave{x}$ and $\acute{x}$).

Examples include the operator sets {Φ, ´} (*phi* and left negation) and {Ψ, `} (*psi* and right negation). One issue with this approach, despite its success in demonstrating symmetries in triadic logic, is the assumption that Peirce should have explicitly considered the group of partial negations (Turquette, 1967). Moreover, Peirce did not explicitly address metalogical concerns such as completeness and consistency of logical systems, which could have influenced his choices of logical functions in the calculus.[15]

In the second group of papers, Turquette provides an axiomatic calculus for Peirce's three-valued propositional logic. To accomplish this, he had to define a material implication operator and a notion of consequence for tree-valued logic, both concepts absent from Peirces manuscripts. Importantly, all these considerations take place within the framework of applying a general method of formalizing calculus for

12 Parks observed that these connectives occur as disjunction and conjunction in Sobociński's three-valued logic (1952), in Cooper's discourse logic (1968), and are analogous to Belnap's concept of conditional assertion (1970). I will address this topic in the next section.

13 Turquette uses Polish notation in his papers: Np, Kpq, Apq and Cpq, instead of ¬p, p ∧ q, p ∨ q, p → q, which I adopt here. In Rosser and Turquette (1945, n. 23, p. 77), he justifies his choice by noting that a significant portion of the works on many-valued logic at that time used this notation, which is not the case nowadays.

14 Turquette (1972) generalizes these dual relations into trimorphic relations, inspired by Lewis Carroll. However, I consider that the main point – functional completeness – is highlighted in the dual relations, making it unnecessary to elaborate on their extension into trimorphisms in this paper.

15 Turquette was aware of this difficulty and justified his interpretation: "There is no historical evidence available at the present time which would indicate that Peirce knew that his three-valued logics were functionally complete. The fact that it can be proved that they are, however, suggests that Peirce may also have been motivated by ideas associated with the notion of completeness" (Turquette, 1967, p. 67; see also 1969, p. 207).

finitely many-valued logics, developed by Rosser and Turquette several years earlier (Rosser; Turquette, 1945; 1952).[16] Let me now outline the general principles of this method.

Consider a propositional logic as a pair L = < $\mathcal{L}$, ⊢>, where $\mathcal{L}$ is a propositional language and ⊢ denotes a Tarskian consequence relation among sets of $\mathcal{L}$ formulas. Then, a logical matrix is defined as the triple $\mathcal{M}_{n,k} = < \mathcal{V}_n, \mathcal{D}_k, F >$, where, for $n \geq 2$ and $1 \leq k \leq n$ ($n \in \mathbb{N}$), we have (Malinowski, 1993):

(i) $\mathcal{V}_n = \{1, 2, ..., n\}$ is a non-empty set of truth values, where 1 denotes "true" and n denotes "false";
(ii) $\mathcal{D}_k = \{1, 2, \ldots, k\}$ is a non-empty subset of $\mathcal{V}$ whose elements are called *designated values*;
(iii) $F = \{f_1, ..., f_n\}$ is a non-empty finite set of basic functions of $\mathcal{L}$ such that for every connective with arity $n \geq 0$ there is a truth function $f: \mathcal{V}^n \longrightarrow \mathcal{V}$ associated with it.

Let the primitive (or derived) functions $F = \{\neg, \vee, \wedge$ and $\rightarrow\}$ of the matrix $\mathcal{M}_{n,k}$, denote, respectively, the connectives of negation, disjunction, conjunction, and material implication. Rosser e Turquette then introduce a special unary function j_k ($1 \leq k \leq m$). The aim is to address the interpretation problem of negation in many-valued logics. In classical logic, the formula ¬p is true if and only if p is false. However, this is not the case in many-valued logics, where the set of non-designated values may include both false (3) and an intermediate element (value) between true and false (2). In the case of a function j_k p, p is true when it takes the value k (Rosser; Turquette, 1952, p. 16-17).

Finally, the propositional functions of the calculus must satisfy certain *standard conditions*. A connective satisfies standard conditions when, considering only its classical truth values, it behaves as in bivalent logics (Rosser; Turquette, 1945, p.79; 1952, p. 25). Thus, for example, considering only the truth values 1 and 3 in the truth table for "→", this table is exactly configured as its counterpart in classical propositional logic.[17]

Therefore, for any $x, y \in \mathcal{V}_n$ and $i \in \{1, 2, ..., n\}$, we have the following semantic definitions that satisfy the standard conditions:

(i) $\neg p \in \mathcal{D}_k$ iff[18] $p \notin \mathcal{D}_k$,
(ii) $p \wedge q \in \mathcal{D}_k$ iff $p \in \mathcal{D}_k$ and $q \in \mathcal{D}_k$,
(iii) $p \vee q \in \mathcal{D}_k$ iff $p \in \mathcal{D}_k$ or $q \in \mathcal{D}_k$,
(iv) $p \rightarrow q \notin \mathcal{D}_k$ iff $\in p \in \mathcal{D}_k$ and $q \notin \mathcal{D}_k$.
(v) $j_i\, p \in \mathcal{D}_k$ iff $p = i$.

According to the Rosser-Turquette method, based on these standard conditions, any many-valued propositional logic can be axiomatized (formalized) by taking the rules of *modus ponens* (MP) and substitution (SUB), along with the following schema-axioms:

A1. $(\alpha \rightarrow (\beta \rightarrow \alpha))$;
A2. $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$;
A3. $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$;
A4. $(j_i\, \alpha \rightarrow (j_i\, \alpha \rightarrow \beta)) \rightarrow (j_i\, \alpha \rightarrow \beta)$;
A5. $(j_i\, \alpha \rightarrow \beta) \rightarrow ((j_{n-1}\, \alpha \rightarrow \beta) \rightarrow (\ldots \rightarrow (j_i\, \alpha \rightarrow \beta) \rightarrow \beta) \ldots))$;
A6. $(j_i\, \alpha \rightarrow \alpha)$;
A7. $(j_{i\,r} (\alpha_r) \rightarrow (j_{i\,r-1} (\alpha_{r-1}) \rightarrow (\ldots \rightarrow (j_{i\,1} (\alpha_1) \rightarrow j_v (F (\alpha_1, ..., \alpha_r)))$, for $F \in F$ and v= v (i1, ..., ir).

16 Various systems of many-valued logics emerged by the 1930s, developed by logicians such as Łukasiewicz, Post, Gödel, Jaskowski, Kleene e Bochvar. A completeness proof for these types of logics, in a generalized manner, was first addressed by Rosser and Turquette (1945), followed by Chang (1958), who introduced the notion of MV-algebras.

17 Rescher called this propriety *normal* (Rescher, 1968, p. 78-79).

18 Abbreviation of "if and only if".

This exposition is relevant because Turquette's interpretation of triadic logic is guided by the application of this formalization method for three-valued systems. Thus, for example, the method is crucial in Turquette's choice among three possibilities of conditionals suggested by him: the three-valued implications of Kleene, Łukasiewicz, and Gödel (see the tables below):

$\to_{K3}$	1	2	3
1	1	2	3
2	1	2	2
3	1	1	1

$\to_{Ł3}$	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

$\to_{G3}$	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

Table 4: Kleene (K_3), Łukasiewicz ($Ł_3$) and Gödel's (G_3) implications.

He then opts for the latter, stating that "[...] there is considerable evidence that of these possible choices suggested for triadic implication, Peirce could have definitely selected [$\mathbf{G_3}$]" (Turquette, 1976, p. 170). However, he does not provide reasons or evidence for this choice.[19] In fact, Turquette believed that Gödel's implication would be more suitable for generating a functionally complete and formalized system (Turquette, 1974). Therefore, the decision appears to be grounded less in a hypothetical preference of Peirce and more on formal adequacy to the Rosser-Turquette method.

The formalization of Peirce's three-valued calculus, proposed by Turquette, is based on the set of operators $\{\neg, \to, \bot\}$[20], where "¬" is one of Peirce's complete negations (or , respectively, left negation or right negation), "→" is Gödel's implication, and "⊥" denotes contradiction. Let "¬" be the left negation ($\grave{x}$ or $\acute{x}$), and the following functions J_k (k = 1, 2, 3) defined as:

(i) $J_3 p \coloneqq p \to \bot$;

(ii) $J_1 p \coloneqq \neg p \to \bot$.

(iii) $J_2 p \coloneqq \neg\neg p \to \bot$.[21]

We have the following definitions of conjunction and disjunction operators (Turquette, 1981, p. 374):

(i) $p \vee q \coloneqq (p \to q) \to q))$;

(ii) $\neg_2 p \coloneqq (J_3 p \vee \neg J_1 p)$;

(ii) $p \wedge q \coloneqq \neg_2 (\neg_2 p \vee \neg_2 q)$

(vi) $p \text{ Z } q \coloneqq \neg\neg((J_2 p \wedge J_1 q) \to \neg (p \wedge q))$[22];

(v) $p \, \Theta \, q \coloneqq \neg_2 (\neg_2 p \text{ Z } \neg_2 q)$.

From the operators Z e Θ, by duality, the others are obtained, thus achieving functional completeness. However, note how intricate and difficult this formalization of the calculus is!

The issue of the notion of consequence presents another challenge, as Peirce does not provide semantic definitions in his manuscripts on triadic logic. There are essentially two types of validity in many-valued logics: those where the truth value 2 belongs to the set of designated values, and those where it does not (Avron, 2003).Turquette develop two formal systems (and their duals) for both left and right negations, with sets of schema-axioms valid for calculi where $\mathcal{D}_k = \{1\}$ and where $\mathcal{D}_k = \{1, 2\}$, which he respectively names systems A and B (Turquette, 1976, 1978).[23] Below, I provide as

19 I will delve further into the issue of conditionals in triadic logic in the next section.

20 Turquete use C, N (Polish notation) and f.

21 These definitions correspond to the following tables: $J_3 p = \{3, 3, 1\}$, $J_1 p = \{1, 3, 3\}$ and $J_2 p = \{3, 1, 3\}$.

22 In this particular definition, I identify an error, highlighted in bold: $\neg\neg (\mathbf{J_3} (J_2 p \wedge J_1 q) \to \neg (p \wedge q))$.

23 He also extends this formalization to include quantifiers (Turquette, 1981), but such analysis goes beyond the purposes of this paper.

an example the axioms of the calculus $\mathcal{M}$ = < {1, 2, 3}, {1,2}, ¬, →, ⊥>, where "¬" stands for left negation (system B):

A1. $((p \to q) \to r) \to ((p \to q) \to (p \to r))$;

A2. $(p \to (q \to p))$;

A3. $(J_2\,(p \to q) \to J_2\,p)$;

A4. $(J_2\,(p \to q) \to J_1\,p)$;

A5. $(J_1\,p \to (J_2\,q \to J_2\,(p \to q)$;

A6. $(\neg\neg p \to (\neg p \to (p \to \bot))$;

A7. $(D_3\,p \to p)$,

where $D_3 \coloneqq J_3 J_1\,p \to J_2\,p$.

Once again, Turquette asserts that there "[...] are good reasons to believe that Peirce would have preferred logics of the latter type [$\mathcal{D}_k$ = {1, 2}]" (Turquette, 1981, p. 373). However, once more, he does not explain why.

In the conclusion of his final paper on triadic logic, Turquette suggests intriguing parallels between Peirce's concept of "limit" with Heisenberg's uncertainty and Tarski's undecidability. He states that triadic logic systems were created with the purpose of "[...] reject absurdities, but to accept that which is either valid (Peirce's term is *verum*) or essentially undecidable" (Turquette, 1981, p. 381). According to him, this would explain why Peirce preferred the set $\mathcal{D}_k$ = {1, 2}. Unfortunately, Turquette did not revisit the topic to further develop this hypothesis.

Today, his articles constitute a valuable yet critically underexamined source on many-valued logics and non-classical logics in Peirce. However, I argue that his analysis of triadic logic is problematic in some respects, justifying my proposed approach (see next section).

Firstly, it should be noted that despite the rigor and sophistication of the observed symmetries, these calculations have limited practical use. Furthermore, to prove functional completeness, Turquette had to introduce two additional negations to the system, not defined by Peirce, and a concept of material implication that lacks justification in Peirce's work. Overall, the choices appear to cater more to what best fits the Rosser-Turquette method.

Regarding this formalization method, another issue raised in recent works by Radzki (2017; 2019; 2022) should be mentioned. The concern raised by Radzki is that if a given many-valued logic L_n must satisfies the standard conditions of Rosser-Turquette, the standard functions $(\to, j_1, ..., j_n)$ must be definable in terms of L_n 's basic connectives, say the set {¬, →}, for the system to be considered *consistent*. On the other hand, for this same system to be semantically *complete*, {¬, →} must also be definable in terms of $(\to, j_1, ..., j_n)$. However, Radzki (2017) proves that the Rosser-Turquette method does not satisfies this simultaneous definability, as material implication is not definable through normal connectives, meaning the system does not meet Tarski's equipollence criterion (Tarski, 1994, p. 122). Thus, the axiom system of Rosser-Turquette is functionally incomplete, and consequently, so is Peirce's triadic logic as analyzed by Turquette.

Finally, the dualistic relations among the various operators listed by Peirce and the complex axiomatic formalization developed by Turquette make it difficult to observe the originality of Peirce's three-valued propositional system – compared to those developed later —, whose semantic properties suggest new relations with contemporary non-classical logics. This is what I will show next.

3 Triadic logic and paraconsistency

In (Salatiel, 2023), I proposed that Peirce's three-valued propositional logic could be viewed as three distinct logical systems, each independent of the others and constructed based on conjunction, disjunction, and negation operators. This new approach, inspired by Parks' paper (1971), hypothesizes that Peirce was merely experimenting with different types of three-valued operators, without necessarily aiming at, as Turquette suggests, metalogical considerations. Thus, the listing of six pairs of binary connectives, {{Φ, Θ}, {Ψ, Z}, {Ω, Y}}, as recorded in the margin of the manuscript page (seq. 640), should be read as {Φ, Ψ}, {Θ, Z}, {Ω, Y}, meaning the first two pairs of operators are read vertically and the last horizontally. This reading would give us the following tables:

Φ	1	2	3
1	1	1	1
2	1	2	3
3	1	3	3

Ψ	1	2	3
1	1	1	3
2	1	2	3
3	3	3	3

Θ	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

Z	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

Ω	1	2	3
1	1	2	3
2	2	2	2
3	3	2	3

Y	1	2	3
1	1	2	3
2	2	2	2
3	1	2	3

Table 5: Reorganized three-valued operators.

There are two advantages to this proposed strategy compared to Turquette's. The first is that we can identify a more original logical matrix for three-valued logics, induced by the set of operators {Φ, Ψ}. The second system, formed by the sets of operators {Θ, Z}, constitutes what we now know as Łukasiewicz's (1920) and Kleene's "strong" calculi (Kleene, 1938).[24] Meanwhile, the third system, formed by the operators {Ω, Y}, comprises Kleene's "weak" logics (Kleene, 1971) and Bochvar's "internal" logics (Bochvar; Bergmann, 1981).[25] The first system, however, which I call $\mathbf{P}_3$, formed by the set of operators {Φ, Ψ}, is little known and studied, so much so that Turquette considered these connectives "mysterious", as seen in the previous section. These two operators correspond, respectively, to disjunction and conjunction in Sobociński's logic (1952), as identified by Parks (1971). More recently, these tables were studied by Carnielli, Marcos, and De Amo (2000) within the scope of Formal Inconsistency Logics (cf. Carnielli; Marcos, 2002).

This brings us to a second advantage of my approach. The use of different negations allows us to analyze the triadic logic within the context of other non-classical logics, such as paraconsistent and

24 Kleene refers to the use of these connectives as "strong" to differentiate it from another, created by him in 1952 and called used in the "weak" sense (Kleene, 1971, p. 334). The terms "strong" and "weak" here refer to their functions being more or less close to classical bivalent logic.

25 In this calculus, published by the Russian scientist Bochvar in 1937 and translated into English by Merrie Bergmann in 1981, the "internal" sense differs from the "external" sense because the latter's value 2 ("meaningless" in Bochvar) behaves like value 3 (false).

paracomplete logics (as shown in Salatiel, 2023). For instance, by adopting the right negation ()we can obtain a consistency operator in $\mathbf{P_3}$ logic. Assuming both 1 and 2 as designated values, we establish a non-explosive system.

In this section, I reconstruct the $\mathbf{P_3}$ system based on Sobociński's trivalent material implication and formalize it in sequent calculus, a type of proof that aligns more closely with Peirce's purposes in analyzing inferential processes. I argue that this calculus is a non-explosive logic supported by Peirce's investigations into non-Aristotelian logics. The first step is to complete the functions Φ and Ψ with negation and material implication operators.

Peirce developed several classical propositional calculi in his project to extend Boolean algebra into a logic of relations. In these works, he anticipated various techniques and concepts of modern logic, including truth tables and quantification theory. One of Peirce's innovations compared to Boole was substituting the equality symbol as primitive with the sign of illation (⤙) as the fundamental logical relation (W 2: 359-429, 1870), which he utilized for material implication, inclusion, and logical consequence.

In "On the algebra of logic" (1880, W 4: 163-209), he used this symbol to define the modern concept of logical consequence: "Thus, the form P_i⤙ C_i implies *either*, 1, that it is impossible that a premise of the class P_i should be true, *or*, 2, that every state of things in which P_i is true is a state of things in which the corresponding C_i is true" (W 4: 166).

As early as 1885 (*On the algebra of logic*, W 5: 162-190), Peirce presented a complete system of implicative algebra, anticipating the works of Rasiowa (1974), in which the primitive symbol ⤙ is defined exactly as a material implication: "Here, therefore, the proposition *a* ⤙ *b* is true if *a* is false or if *b* is true, but false if *a* is true while *b* is false" (W 5: 170). The formalization follows a Hilbert-style axiomatic approach, with the selection of five axioms (which Peirce calls *icons*).

Given the importance of conditionals in his propositional calculi, the absence of a corresponding table for material implication in his manuscripts on triadic logic reinforces the hypothesis of an experimental, unfinished work. What then would Peirce's choice be for a trivalent conditional? Based on the defined dual operators, the most natural option would be $\rightarrow \coloneqq (\neg\alpha \vee \beta)$ or $\neg(\alpha \wedge \neg\beta)$, which gives us the following table:

→	1	2	3
1	1	3	3
2	1	2	3
3	1	1	1

Table 6: **Material implication for $\mathbf{P_3}$.**

Hence, the logical matrix for Peirce's three-value logic fragment $\mathbf{P_3}$ can be defined as: $\mathcal{M}\,\mathbf{P_3} = <\{1, 2, 3\}, \{1\}, \neg, \rightarrow>$. I opt for Peirce's overbar negation ($\bar{x}$), while acknowledging the possibility of equally interesting uses of other negations. With these two primitive operators as a basis, disjunction and conjunction can be respectively obtained by the following definitions (thus achieving the complete functionality of the calculus):

(i) $p \vee q \coloneqq (\neg p \rightarrow q)$;
(ii) $p \wedge q \coloneqq \neg(p \rightarrow \neg q)$.

Another important definition is the set of designated values. The first option would be $\mathcal{D} = \{1\}$. This is justified by the following passage in the manuscript:

> Triadic logic is that logic which, though not rejecting entirely the Principle of Excluded Middle, nevertheless recognizes that every proposition, S is P,26 is either true or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not P, but is at the limit between P and not P. (seq. 645).

There is much to be said about this passage, which presents Peirce's philosophical motivations for the third (limit) value. However, what concerns us in the present context is the information that the proposition receiving the "third value" is neither determinately true nor determinately false; it is, in essence, a "gap"[27] between classical truth values. These truth values, being absolute, make the choice of value 1 as the sole designated value the most reasonable option.

Nevertheless, this choice has a downside. No theorem could be proven in the $\mathbf{P_3}$ system, because whenever atomic formulas receive the value 2, the complex formula would also receive the value 2. This seems to contradict Peirce's intention to treat many-valued logic as an extension of classical bivalent logic (which indeed it is, given its standard connectives): "Thus the triadic logic does not conflict with dyadic logic; only it recognizes what the latter does not [...]" (seq. 645).

There are two ways to avoid this problem: using another definition of material implication (as Turquette does), which would require justification; or assuming a set of designated values $\mathcal{D} = \{1, 2\}$, a common strategy for many-valued systems. In the latter case, is there any support in Peirce's writings?

In the second paragraph of the same manuscript page, after mentioning the Principle of the Excluded Middle (which triadic logic would not reject *entirely*), Peirce refers, this time, to the Principle of Non-contradiction:

> Of course it remains true, as far as the principle of contradiction is concerned that the state of things represented by the proposition cannot be V and F, *verum atque falsum* [true and false] and must be V + F if by F is meant L + F (L signifying the limit, i.e. that S is not capable of the determination P or of the determination F [non-P]).

In other words, Peirce states that, *considering* the validity of the Principle of Non-contradiction, a proposition with value L cannot be both true and false. In this case, L is a "gap" between absolute classical truth values and $\mathcal{D} = \{1\}$, which he expressed by saying that "if by F we mean L + F" (the symbol + replaces the notation "∨" for disjunction). However, what happens when the Principle of Non-contradiction is not considered?

Peirce, in recurring passages,[28] questioned the universal validity of Principle of Non-contradiction. Therefore, it wouldn't be difficult to speculate, based on his work, about a triadic logic where we would have a "glut" interpretation for truth values. In this case, the third value would be both true and false, and the designated set of values would be $\{1, 2\}$.[29]

Therefore, the logical matrix would be: $\mathcal{M}\,\mathbf{P_3} = \langle\{1, 2, 3\}, \{1, 2\}, \neg, \rightarrow\rangle$.

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26 Why the Aristotelian representation of the proposition? Could Peirce be contemplating a three-valued logic for first-order logic? It appears to me, however, that this manuscript engages in a dialogue with other works of Peirce where he discusses the principles of Excluded Middle and Non-contradiction in the context of predication, such as Issues of Pragmatism (EP 2: 346-359, 1905).

27 A three-valued is called "gap" if a sentence that receives the truth value 2 is neither true nor false, and "glut" when this sentence is both true and false.

28 In addition to the work cited in note 26, we have, for example, NEM IV: 258 (1904), NEM II: 528 (1904), NEM III: 747 and 762 (n.d), CP 8. 216 (c. 1910) and CP 6.182 (1911).

29 The correct understanding of three-valued logical consequence that Peirce would adopt largely hinges on a philosophical interpretation of what he meant by "limit", the term he used for the third value. If by "limit" Peirce intended vague or undefined information, then we can assert that $L \notin \mathcal{D}$. However, if he meant inconsistent information (because it is both true and false), then $L \in \mathcal{D}$ (cf. Avron, 1991). This ongoing debate about Peirce's philosophical motivations, though inconclusive, has yielded some advances and findings (see, for instance, Lane, 1999; 2001; Odland, 2020; 2021). Nevertheless, our current assessment suggests that these findings are insufficient to resolve the issue.

In this case, interesting correspondences arise with relevance logics and paraconsistent logics. The system $\mathbf{P_3}$, in this instance, corresponds exactly to Sobociński's three-valued calculus for $\mathcal{D} = \{1, 2\}$ and the primitive operators $\{\neg, \rightarrow\}$, which he formalized in the 1950s (Sobociński, 1952 and 1961)[30].

The logician Zane Parks was the first to suggest this interpretation of Peirce's triadic logic. He highlighted that the connectives Φ and Ψ were not "mysterious" but induced from the three-valued propositional calculus proposed by Sobociński as disjunction and conjunction operators (Parks, 1971).[31] Parks also demonstrated that Sobociński's logic is a fragment (implication and negation) of Dunn-McColl's R-*Mingle* logic (**RM**) (Parks, 1972).[32]

More recently, Arieli and Avron (2015) demonstrated that this system is non-explosive, normal, and maximally paraconsistent.[33] A logical system is paraconsistent if a contradiction does not trivialize the system, that is, $p, \neg p \nvdash q$. A logic is maximally paraconsistent in the sense that it is so close to a classical system that extending it would cause it to lose its paraconsistent property.

The logic induced by the matrix $\mathcal{M}$ $\mathbf{P_3}$ is normal, in that all classical tautologies are provable, i.e., $\vdash \Delta$, where Δ is a classical tautology. It aligns with Peirce's aims to be closest to bivalent logic and, as a paraconsistent system, also conforms to Costa's Cn hierarchy for maximality property, which aims to preserve, in principle, "[...] all schemes and rules of classical logic that are compatible" (Costa, 1993, p. xi; cf. Arieli; Avron, 2015; Carnielli; Coniglio, 2016). In the $\mathbf{P_3}$ calculus, it is straightforward to verify that the Principle of Explosion is invalid, but the Principle of Non-contradiction is a theorem.

We can utilize Sobociński's axiomatizations or Avron's sequent calculi for formalizing $\mathbf{P_3}$ (Avron, 2015).[34] However, unlike Turquette's Hilbert-style formalization proposed for triadic logic, I believe that sequent calculi are preferable. This preference arises both because they are closer to propositional logics developed by Peirce from 1880 onwards and because they are more in line with his views on formal systems.

Peircean implicational logics developed from 1880 onwards, for instance, anticipate various aspects of natural deduction and sequent theories formulated by Gentzen in the 1930s, such as the emphasis on illation (deduction) and the formulation of introduction and elimination rules for connectives (Brady, 2000, p. 6; cf. Roberts, 1973). Recently, Ma and Pietarinen (2020) argued that Peirce's calculi are closer to sequent calculus than to natural deduction, primarily due to the primacy of the logical consequence relation.

Furthermore, Peirce was more interested analyzing the inferential steps of a deduction than in deriving proofs through an axiomatic method. As Rodrigues (2017, p. 465) states, according to Peirce, the task of the logician "is not to reach the conclusions, but to study how the conclusions are related to the premises, and which are the necessary steps of this link".

By sequent I understand an expression in the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas from $\mathcal{L}$ and ⇒ is a new symbol for the consequence relation. Γ is called the antecedent (conjunctive-like) and Δ is called the succedent (disjunctive-like) of a sequent. Let $\mathbf{P_3}$ be a sequent calculus; then we say that the *proof* of a sequent **s** from a set of sequents **S** is a finite sequence that ends in **s**, and whose other formulas in the tree are either an axiom (occupying the top position in this structure) or a formula obtained from a set of rules.

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30 Other axiomatizations were proposed by Rose (1956) and Meyer and Parks (1972).

31 A question of historical interest arises: why did Turquette not recognize the correspondence between Peirce's and Sobociński's operators? The logicians mentioned here – Turquette, Sobociński, Parks, Rose, and Meyer – worked around the same time in the United States on the formalization of many-valued logics. Sobociński extensively reviewed Rosser and Turquette's 1952 book (Sobociński, 1955), and Turquette reviewed Sobociński's article (Turquette, 1966), both texts published in the *Journal of Symbolic Logic*. Therefore, Turquette was well acquainted with the work of his Russian colleague from the University of Notre Dame.

32 Parks (1971) was also the first to suggest an approximation of Peirce's triadic logic with Belnap's relevant logics.

33 In Salatiel (2023), I showed that Peirce's three-valued paraconsistent logic also belongs to the family of *Logics of Formal Inconsistency* (LFI).

34 I proposed an analytical tableau calculus in Salatiel (2022).

Supposing now the matrix $\mathcal{M}$ for $\mathbf{P}_3$, we say that:

(i) A *valuation* v is a model-$\mathcal{M}$ of a sequent $\Gamma \Rightarrow \Delta$ if $v(\varphi) \notin \mathcal{D}$, for some $\varphi \in \Gamma$, or $v(\psi) \in \Gamma$, for some $\psi \in \Delta$.

(i) A set of formulas Δ is said to be a *logical consequence* in $\mathcal{M}$ $\mathbf{P}_3$, $\Gamma \vdash \Delta$, iff every model of Γ is also a model of some formula in Δ;

(i) A sequent $\Gamma \Rightarrow \Delta$ is *valid* in $\mathcal{M}$ iff $\Gamma \vdash_{\mathcal{M}} \Delta$.

Therefore, a sequent calculus for three-valued propositional logic $\mathbf{P}_3$ is obtained from the rules of classical logic by removing the negation rules, which are replaced by the axiom $\Rightarrow \varphi, \neg \varphi$ below, and adding the negation rules for classic connectives. The schema-axioms are as follows:

Ax. 1: $\varphi \Rightarrow \varphi$;
Ax. 2: $\Rightarrow \varphi, \neg \varphi$.

In addition to the structural rules of substitution (**SR**) and contraction (**CR**), and the classical rules for logical connectives, we have the following rules for double negation and negations of connectives:

$(\neg \neg \Rightarrow)$
$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg \neg \varphi \Rightarrow \Delta}$$

$(\Rightarrow \neg \neg)$
$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg \neg \varphi}$$

$(\neg \wedge \Rightarrow)$
$$\frac{\Gamma, \neg \varphi \Rightarrow \Delta \qquad \Gamma, \neg \psi \Rightarrow \Delta}{\Gamma, \neg (\varphi \wedge \psi) \Rightarrow \Delta}$$

$(\Rightarrow \neg \wedge)$
$$\frac{\Gamma \Rightarrow \Delta, \neg \varphi, \neg \psi}{\Gamma \Rightarrow \Delta, \neg (\varphi \wedge \psi)}$$

$(\neg \vee \Rightarrow)$
$$\frac{\Gamma, \neg \varphi, \neg \psi \Rightarrow \Delta}{\Gamma, \neg (\varphi \vee \psi) \Rightarrow \Delta}$$

$(\Rightarrow \neg \vee)$
$$\frac{\Gamma \Rightarrow \Delta, \neg \varphi \qquad \Gamma \Rightarrow \Delta, \neg \psi}{\Gamma \Rightarrow \Delta, \neg (\varphi \vee \psi)}$$

$(\neg \rightarrow \Rightarrow)$
$$\frac{\Gamma, \varphi, \neg \psi \Rightarrow \Delta}{\Gamma, \neg (\varphi \rightarrow \psi) \Rightarrow \Delta}$$

$(\Rightarrow \neg \rightarrow)$
$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \neg \psi}{\Gamma \Rightarrow \Delta, \neg (\varphi \rightarrow \psi)}$$

Table 7: Rules for P3 system.

An interesting characteristic (but expected from a relevant logic) is that the structural rule of weakening does not hold in $\mathbf{P}_3$. Thus, it is easy to verify, for example, that the Principle of Explosion does not hold either (see *Proof 1*), since this theorem is demonstrated in classical logic through the application of negation and weakening rules. Consequently, this distinguishes $\mathbf{P}_3$ logic as a type of substructural logic. Other theorems, such as the Positive Paradox, also prove to be invalid (*Proof 2*).

Proof 1:

$$\frac{p, \neg p \Rightarrow q}{p \wedge \neg p \Rightarrow q}\ \neg \wedge \Rightarrow$$

Proof 2:

$$\cfrac{\cfrac{p \Rightarrow q \qquad p \Rightarrow p}{p \Rightarrow q \rightarrow p}\ \Rightarrow \rightarrow}{\Rightarrow p \rightarrow (q \rightarrow p)}\ \Rightarrow \rightarrow$$

Other laws and theorems of classical logic, such as Non-contradiction, Excluded Middle, Identity, and *modus ponens* are valid in $\mathbf{P}_3$, as well as the expected *mingle* axiom: $(\varphi \rightarrow (\psi \rightarrow \varphi)$. It is straightforward to verify the validity of the Principle of Non-contradiction (*Proof 3*), as well as the equivalence of material implication (*Proof 4*).

Proof 3:

$$\cfrac{\cfrac{\Rightarrow p, \neg p}{\Rightarrow p, \neg\neg p}\ \Rightarrow \neg\neg}{\Rightarrow \neg(p \wedge \neg p)}\ \Rightarrow \neg \wedge$$

Proof 4:

$$\cfrac{\cfrac{\cfrac{\Rightarrow p, \neg p}{\Rightarrow p, \neg p \vee q}\ \Rightarrow \vee \qquad \cfrac{q \Rightarrow q}{q \Rightarrow \neg p \vee q}\ \Rightarrow \vee}{p \rightarrow q \Rightarrow \neg p \vee q \ \neg p \vee q}\ \Rightarrow \rightarrow}{p \rightarrow q \Rightarrow \neg p \vee q}\ \mathbf{CR}$$

The proofs that the calculus is sound, complete, and admits cut elimination are given by Avron (1991; 2015).

4 Conclusion

Peirce's triadic logic is one of the most interesting topics in his studies on deductive systems, but also one of the most neglected. In this paper, I first conducted an analysis of Turquette's articles, which are practically the only references on the formal aspects of triadic logic. I then showed that these papers present a formalization that is not very consistent with Peirce's work and is difficult to use in practice. Turquette was one of the main theorists of many-valued logical systems in the 1950s and made a significant contribution to Peircean studies after the discovery of the triadic logic manuscripts. However, the absence of critical literature on this subject is a flaw that this work aims to highlight and, perhaps, correct.

In the second part of this paper, I present an alternative approach to Turquette's, inspired by Parks' work (1971), which solved the "mystery" of the *phi* and *psi* operators. This new approach suggests that the problem of Peirce's triadic operators could be resolved by viewing them as three different three-valued propositional systems, each operating with different types of negation for different purposes. The fact that Peirce's matrices are incomplete, lacking the material implication operator, for example, might be a reason why Peirce never published them. I then show that two of these logical systems are well-known in the works of Łukasiewicz, Kleene, and Bochvar, but one of them, which I called $\mathbf{P_3}$, is the most original because it has been little studied. In my research, I proposed notions of material implication and logical consequence, based on Peirce's work, to complete the calculus of $\mathbf{P_3}$, which proves to be sound and complete.

The results indicate that Peirce's three-valued system leads to paraconsistent,[35] relevance, and substructural logical systems, placing his experiments at the core of important discoveries in contemporary non-classical logics. The greatest advantage of my approach, compared to Turquette's, is precisely to show the originality of the $\mathbf{P_3}$ calculus, which Parks demonstrated to be the same as that rediscovered by Sobociński decades later. This research also has the advantage of shedding new light on Turquette's work, who may have been the first logician to point out the relations between triadic logic and the many-valued systems of Łukasiewicz, Kleene, and Sobociński.

Future research can further explore the relationships between Peirce and paraconsistent formal systems, based on an examination of his philosophical motivations — two perspectives that still seem to remain ununified today. I hope this research also highlights the importance of Peirce as one of the pioneers in the field of non-classical logics.

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Abbreviations of Peirce's works:

Collected papers of Charles Sanders Peirce: volume (v), paragraph (p) (CP v. p).
The essential Peirce: volume (v), page (p) (EP v: p).
The new elements of mathematics: volume (v), page (p) (NEM v: p).
Writings of Charles S. Peirce: volume (v), page (p) (W v: p).
Philosophy of Mathematics: page (p) (PM p)

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