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An algebraic model for the modal logic *KD*

Um modelo algébrico para a lógica modal KD

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Abstract: Deontic logic is a branch of symbolic logic interested in notions such as obligatory, permissible, optional, ought, and others similar. There are some equivalent ways to present the Standard Deontic Logic or **KD**. In this paper, we will mention some of them and highlight one that is of interest. With this presentation we can propose a simple algebraic model for the Standard Deontic Logic.

Keywords: Algebraic model. Axiom *D*. Deontic logic.

Resumo: *A lógica deôntica é um ramo da lógica simbólica interessada em noções como obrigatório, permitido, opcional, dever e outras semelhantes. Existem algumas maneiras equivalentes de apresentar a Lógica Deôntica Padrão ou **KD**. Neste artigo, vamos citar alguns deles e destacar um que é de interesse. Com esta apresentação podemos propor um modelo algébrico simples para a Lógica Deôntica Padrão.*

Palavras-chave: *Axioma D. Lógica deôntica. Modelo algébrico.*

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1 Introduction

Since Antiquity the alethic sense of "necessarily" and "possibly" has attracted the interest of logicians. These terms are used to qualify the truth of a proposition.

The pioneers of Modal Logic, at the beginning of the XX century, investigated the formal behaviour of expressions such as "it is necessary that" and "it is possible that" using formal propositional language with two modal operators for necessary and possible. Currently we have used the symbols $\Box$ and $\Diamond$ for these two notions, respectively.

Currently, the term "modal logic" is broader and characterizes a family of logical systems, each with several different modalities.

This family is always increasing; however, it includes tense or temporal logics, epistemic logics, deductively valid aspects of logics, doxastic logics, among others, and deontic logics.

We are particularly interested in the case of deontic logic, that analyses expressions such "it is obligatory that", "it is permitted that", and "it is forbidden that".

It is common, in introductory texts on modal logics, to present a modal family constructed from a weak logic indicated by **K**, in honour of Saul Kripke, who, in the 1950s introduced the Kripke models for this family of logics.

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This family has as its language the set L = {¬, ∧, ∨, →, □} such that the four first operators are the classical ones and the last one is the modal operator for 'necessary'.

The operator for possibility must be defined from □ by $\Diamond\varphi =_{df} \neg\Box\neg\varphi$.

System **K** is obtained by adding the following two principles to classical propositional logic.

Necessitation Rule: If φ is a theorem of **K**, then □φ is also a theorem of **K**.

Axiom *K*: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

The axiom *K* is also known as the Distributivity Axiom.

From the Necessitation Rule, any theorem of logic is necessary. Axiom **K** says that if an implication $\varphi \to \psi$ is necessary, then for every necessary φ there is a necessary ψ.

We are particularly interested in a case of deontic logic, Standard Deontic Logic (**SDL**), which introduces the primitive symbol O for "it is obligatory that", a deontic necessity, in the place of □.

From the operator O, we can define the operator P for "it is permitted that", deontic permission, by $P\varphi \Leftrightarrow \neg O \neg \varphi$, and F for "it is forbidden that" by $F\varphi \Leftrightarrow O\neg\varphi$.

The usual modal axiom *T*: $O\varphi \to \varphi$ is not appropriate for deontic logic. Even if some action is obligatory, it may not always be the case.

However, the logic **SDL** admits the axiom (*D*): $O\varphi \to P\varphi$, which says that if φ is obligatory, then φ is permissible; moreover, axiom *D* is a weakening of axiom *T*.

Therefore, we start with the logical system **SDL** presented in different but equivalent axiom systems.

Considering one of these presentations we introduce D-algebras, which are planned as algebraic models for **SDL**.

Finally, we show that D-algebras are completely adequate models for **SDL**.

2 Standard deontic logic

Standard Deontic Logic is known as the system **KD** generated by the inclusion of axiom *D*.

Axiom *D*: $\Box\varphi \to \Diamond\varphi$.

In Deontic Logics we use the following formalization: O for "it is obligatory that", P for "it is permitted that" and F for "it is forbidden that".

Thus, axiom *D* says that if is obligatory, then φ is permitted.

If we take the operator of obligation O as basic, then we can define the operators of permitted and forbidden by:

$P\varphi =_{df} O\neg\varphi$

$F\varphi =_{df} O\neg\varphi$.

Now, we will present three different but equivalent presentations of **KD**. The last one is less usual but it is formidable for the algebraic model.

Let us consider the propositional language L = {¬, ∧, ∨, →, O}, with Var(**KD**) = $\{p_1, p_2, p_3, ...\}$ the set of propositional variables, and the operators P and F defined as above. "'For(**KD**)"' denotes the set of formulas of **KD**.

As we are presenting axiomatic deductive systems, we consider the usual concepts of deduction and proof.

Definition 1.1: If $\Gamma \subseteq$ For(**KD**) and Ax denotes a set of axioms for **KD**, then C(Γ) = { ∈ For(**KD**) : Γ ∪ Ax ⊢ ψ} is the set of consequences of Γ.

Definition 1.2: We say that is derivable in **KD**, or ψ is a theorem of **KD**, when ψ ∈ C(∅).

Thus, we have $C(\varnothing)$ as the set of theorems of **KD**. That is, $\in C(\varnothing) \Leftrightarrow\ \vdash \psi$.

Definition 1.3: A theory of **KD** is a set $\Delta \subseteq For(\mathbf{KD})$, such that $C(\Delta) = \Delta$.

The logical system **KD** can be characterized in several ways, for example, by some of the following deductive systems, $\mathbf{KD_1}$, as in (CARNIELLI; PIZZI, 2001), and $\mathbf{KD_2}$ or $\mathbf{KD_3}$, as in (CHELLAS, 1980), such that:

- $\mathbf{KD_1}$:

(**CPC**)	φ, if φ is a tautology
(**K**)	$O(\varphi \to \psi) \to (O\,\varphi \to O\,\psi)$
(**D**)	$O\,\varphi \to P\,\varphi$
(**MP**)	$\to \psi,\ \varphi \,/\, \psi$
(**Nec**)	$\vdash \varphi \,/\, \vdash O\,\varphi$.

- $\mathbf{KD_2}$:

(**CPC**)	φ, if φ is a tautology
(**OD***)	$\neg\,(O\,\varphi \wedge O\,\neg\,\varphi)$
(**MP**)	$\varphi \to \psi,\ \varphi \,/\, \psi$
(**ROK**)	$\vdash (\varphi_1 \wedge \ldots \square\ \varphi_n) \to \psi \,/\, \vdash (O\,\varphi_1 \wedge \ldots \wedge O\,\varphi_n) \to O\,\psi$, for $n \geq 0$.

- $\mathbf{KD_3}$:

(**CPC**)	φ, if φ is a tautology
(**OC**)	$(O\,\varphi \wedge O\,\psi) \to O\,(\varphi \wedge \psi)$
(**ON**)	$O\,\top$
(**OD**)	$O\,\bot$
(**MP**)	$\to \psi,\ \varphi \,/\, \psi$
(**ROM**)	$\vdash \varphi \to \psi \,/\, \vdash O\,\varphi \to O\,\psi$.

Proposition 1.4: The systems $\mathbf{KD_2}$ and $\mathbf{KD_3}$ are deductively equivalent.
Proof: We need to achieve **OD*** and **ROK** in $\mathbf{KD_3}$.

From **OC** we have $(O\varphi \wedge O\neg\varphi) \to O\,(\varphi \wedge \neg\varphi) \Leftrightarrow (O\varphi \wedge O\neg\varphi) \to O\bot$. Considering **OD**, it follows that $\neg\,(O\,\varphi \wedge O\,\neg\varphi)$.

As $(\varphi \wedge \psi) \to \varphi$ and $(\varphi \wedge \psi) \to \psi$, then by **ROM** we get $O(\varphi \wedge \psi) \to (O\,\varphi \wedge O\,\psi)$. Compounding with **OC** it holds that $O(\varphi \wedge \psi) \leftrightarrow (O\,\varphi \wedge O\,\psi)$, that can be extended for [1] $O(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow (O\,\varphi_1 \wedge \ldots \wedge O\,\varphi_n)$.

Now, if we take $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_n)\ \ \psi$, by **ROM** we obtain $\vdash O(\varphi_1 \wedge \ldots \wedge \varphi_n) \to O\,\psi$. With the equivalence [1] we get $\vdash (O\,\varphi_1 \wedge \ldots \wedge O\,\varphi_n) \to O\,\psi$.

In the other direction, we need to establish **OC**, **ON**, **OD**, and **ROM** in $\mathbf{KD_2}$.

From **ROK**, considering $n = 0$, we have $\vdash \varphi \,/\, \vdash O\,\varphi$. Since $\vdash \top$, then $\vdash O\,\top$.

Again, from **ROK**, considering $n = 1$, we have **ROM**.

From $\vdash (\varphi \wedge \psi) \to (\varphi \wedge \psi)$ and **ROK** we get **OC**.

From **OD*** we have $\neg\,O\bot \vee \neg\,O\,\neg\bot \Leftrightarrow \neg\,O\bot \vee \neg\,O\top$. Considering **ON** then $\neg\,O\bot$.

■

Proposition 1.5: The systems $\mathbf{KD_1}$ and $\mathbf{KD_2}$ are deductively equivalent.
Proof: The axioms and rules of $\mathbf{KD_1}$ are obtained in $\mathbf{KD_2}$.

From **ROK** with $n = 0$ we get **Nec**.

From **OD*** we obtain **D** in the following way $\neg\,(O\,\varphi \wedge O\neg\,\varphi) \Leftrightarrow \neg\,O\,\varphi \vee \neg O\,\neg\,\varphi \Leftrightarrow \neg\,O\,\varphi \vee P\,\varphi \Leftrightarrow O\,\varphi \to P\,\varphi$.

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Finally, considering the tautology $((\varphi \to \psi) \wedge \varphi) \to \psi$, using **ROK** we get $(O(\varphi \to \psi) \wedge O\,\varphi) \to O\,\psi$. Now using **CPC** we have $O(\varphi \to \psi) \to (O\,\varphi \to O\,\psi)$.

The axioms and rules of $\mathbf{KD_2}$ are obtained in $\mathbf{KD_1}$.

As above, **OD*** and **D** are equivalent.

We need to prove **ROK** in $\mathbf{KD_1}$.

If $\vdash (\varphi_1 \wedge ... \wedge \varphi_n) \to \psi$ by **Nec** it follows that $\vdash O((\varphi_1 \wedge ... \wedge \varphi_n)\ \ \psi)$ and using **K** we get $\vdash O(\varphi_1 \wedge ... \wedge \varphi_n) \to O\,\psi$. With the equivalence [1] we have $\vdash (O\,\varphi_1 \wedge ... \wedge O\,\varphi_n) \to O\,\psi$. ■

3 D-algebra

D-algebra describes the fundamental properties of $\mathbf{KD_3}$ in a Boolean algebra with an operator.

Definition 2.1: D-algebra is a 7-tuple $\mathbb{D} = (D, 0, 1, , \vee, \sim, \pi)$ such that $(D, 0, 1, \wedge, \vee, \sim)$ is a Boolean algebra and π: $D \to D$ is an operator for which:

(i) $\pi\ 1 = 1$

(ii) $\pi\ 0 = 0$

(iii) $\pi(a \wedge b) = (\pi\ a \wedge \pi\ b)$.

Definition 2.2: A D-algebra is non-degenerate if its universe D has at least two elements.

Proposition 2.3: If $\mathbb{D} = (D, 0, 1, , \vee, \sim, \pi)$ is a D-algebra and $a, b \in D$, then:

(i) $a \leq b \Rightarrow \pi\ a \leq \pi\ b$

(ii) $(\pi\ a \vee \pi\ b) \leq \pi(a \vee b)$.

Proof: (i) $a \leq b \Rightarrow a = a \wedge b \Rightarrow \pi\ a = \pi\ (a \wedge b) = \pi\ a \wedge \pi\ b \Rightarrow \pi\ a \leq \pi\ b$.

(ii) As $a \leq a \vee b$, then $\pi\ a \leq \pi(a \vee b)$. Hence, $(\pi\ a\ \ \pi\ b)\ \ \pi\ (a \vee b)$. ■

Theorem 2.4: For each D-algebra $\mathbb{D}$ there exists a monomorphism h from $\mathbb{D}$ into a D-algebra defined in a Boolean algebra of sets $\mathbb{B}$.

Proof: From the Stone's isomorphism, it is known that for each Boolean algebra $(B, 0, 1, , \vee, \sim)$ there is a monomorphism h from it into a Boolean algebra of subsets $\mathbb{B}$.

We denote this Boolean algebra determined by $Im(h)$ by $\mathbb{B} = (P(B), \varnothing, B, \cap, \cup, ^C)$ and h: $\mathbb{D} \cong \mathbb{B}$.

Next, we introduce a D-algebra of sets in $\mathbb{B}$ and extend the isomorphism h to an isomorphism between $\mathbb{D}$ and $\mathbb{B} = (P(B), , B, \cap, \cup, ^C, \pi)$.

For each set $a \in D$ we define $\pi\ h(a) = h(\pi\ a)$.

We need to show that π satisfies the definition of a D-algebra:

(i) $\pi(B) = \pi(h(1)) = h(\pi\ 1) = h(1) = B$.

(ii) $\pi(\varnothing) = \pi(h(0)) = h(\pi\ 0) = h(0) = \varnothing$.

(iii) $\pi\ (h(a) \cap h(b)) = \pi\ h(a \wedge b) = h\ (\pi\ (a \wedge b)) = h(\pi\ a \wedge \pi\ b) = h(\pi\ a) \cap h(\pi\ b) = \pi\ h(a) \cap \pi\ h(b)$. ■

In the next section, we show that the D-algebras are appropriate models for **KD**.

4 Algebraic adequacy of KD

Now, we will take a generic D-algebra **D** as an algebraic model for **KD**.

Definition 3.1: A restrict valuation is a function $\underline{v}$: $Var(\mathbf{KD}) \to \mathbb{D}$, that maps each variable of **KD** in an element of $\mathbb{D}$.

Definition 3.2: A valuation is a function v: For(**KD**) $\mathbf{D}$, that extends natural and uniquely $\underline{v}$ as follows:

(i) $v(p) = \underline{v}(p)$

(ii) $v(\neg \varphi) = \sim v(\varphi)$

(iii) $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$

(iv) $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$

(v) $v(\mathrm{O}\, \varphi) = \pi\, v(\varphi)$.

As usual, the operator symbols on the left side represent logical operators and those on the right side represent algebraic operators.

Definition 3.3: A valuation v: For(**KD**) $\to \mathbf{D}$ is a model for a set $\Gamma \subseteq$ For(**KD**) if $v(\gamma) = 1$, for each formula $\gamma \in \Gamma$.

In particular, a valuation v: For(**KD**) $\to \mathbf{D}$ is a model for a formula $\varphi \in$ For(**KD**) when $v(\varphi) = 1$.

Definition 3.4: A formula $\in$ For(**KD**) is valid in a D-algebra $\mathbf{D}$ if each valuation v: For(**KD**) $\to \mathbf{D}$ is a model for φ.

Definition 3.5: A formula is D-valid, what is denoted by $\vDash \varphi$, when it is valid in every D-algebra.

Let (For(**KD**), $\wedge, \vee, \to, \neg, \mathrm{O}, \bot, \top$) be the algebra of formulas of **KD**, such that , $\vee$ and $\to$ are binary operators, $\neg$ and O are unary operators, $\bot$ and $\top$ are constants, and $\varphi \to \psi =_{df} \neg \varphi \vee \psi$.

As usual, we define the Lindenbaum algebra of **KD**.

Definition 3.6: For $\Gamma \subseteq$ For(**KD**), the following equivalence relation is defined by $\equiv$:

$$\varphi \equiv \psi \Leftrightarrow \Gamma \vdash \varphi \to \psi \text{ and } \Gamma \vdash \psi \to \varphi.$$

The relation , more than an equivalence relation, is a congruence relation, since by rule **Rom**: $\varphi \equiv \psi \Leftrightarrow \Gamma \vdash \varphi \leftrightarrow \psi \Rightarrow \Gamma \vdash \mathrm{O}\, \varphi \leftrightarrow \mathrm{O}\, \psi \Leftrightarrow \mathrm{O}\, \varphi \equiv \mathrm{O}\, \psi$.

If $\Gamma \cup \{\psi\} \subseteq$ For(**KD**), we denote by $[\psi]_\Gamma = \{\sigma \in \text{For}(\mathbf{KD}) : \sigma \equiv \psi\}$ the equivalence class of ψ module $\equiv$ and Γ.

Definition 3.7: The Lindenbaum algebra of **KD**, denoted by $\mathbf{A}_\Gamma(\mathbf{KD})$, is the quotient algebra defined by:

$$\mathbf{A}_\Gamma(\mathbf{KD}) = (\text{For}(\mathbf{KD})|, 0, 1, \wedge|, \vee|, \neg|, \mathrm{O}|), \text{ such that:}$$

(i) $[\varphi] \wedge|_\equiv [\psi] = [\varphi \wedge \psi]$

(ii) $[\varphi] \vee|_\equiv [\psi] = [\varphi \vee \psi]$

(iii) $\neg|_\equiv [\varphi] = [\neg \varphi]$

(iv) $\mathrm{O}|_\equiv [\varphi] = [\mathrm{O}\, \varphi]$

(v) $0 = [\varphi \wedge \neg \varphi] = [\bot]$ and

(vi) $1 = [\varphi \vee \neg \varphi] = [\top]$.

In general, it is not important to indicate the index $\equiv$ of these operations.

When $\Gamma = \varnothing$ we denote the Lindenbaum algebra of **KD** by $\mathbf{A}(\mathbf{KD})$.

Proposition 3.8: In $\mathbf{A}_\Gamma(\mathbf{KD})$ holds: $[\varphi] \leq [\psi] \Leftrightarrow \Gamma \vdash \varphi \to \psi$.

Proof: $[] \leq [\psi] \Leftrightarrow [\varphi] \vee [\psi] = [\psi] \Leftrightarrow [\varphi \vee \psi] = [\psi] \Leftrightarrow \Gamma \vdash \varphi \vee \psi \leftrightarrow \psi \Leftrightarrow \Gamma \vdash \varphi \to \psi$. ■

Proposition 3.9: The algebra $\mathbf{A}_\Gamma(\mathbf{KD})$ is a D-algebra.
Proof: **ON**: $O \top \Rightarrow [O \top] = 1 \Rightarrow O[\top] = 1$.

OD: $\neg O \perp \Rightarrow [\neg O \perp] = 1 \Rightarrow O[\perp] = 0$.

Since $\vdash O(\varphi \wedge \psi) \leftrightarrow (O \varphi \wedge O \psi)$, then $O[\varphi \wedge \psi] = [O \varphi] \wedge [O \psi]$. ■

Definition 3.10: The algebra $\mathbf{A}_\Gamma(\mathbf{KD})$ is the canonical model of $\Gamma \subseteq$ For(**KD**).

We denote a valuation on the canonical model by v_0: For(**KD**) $\to \mathbf{A}_\Gamma(\mathbf{KD})$. When $\Gamma = \varnothing$ we have v_0: For(**KD**) $\to \mathbf{A}(\mathbf{KD})$.

Corollary 3.11: Let $\varphi \in$ For(**KD**) and $\mathbf{A}(\mathbf{KD})$ be the canonical model for **KD**. If φ is a theorem of **KD**, then $[\varphi] = 1$, and if is irrefutable, then $[\varphi] \neq 0$.
Proof: If $\vdash \varphi$, as $\mathbf{A}(\mathbf{KD})$ always has an identity element 1, then:

1. $\vdash \varphi$	Hypothesis
2. $\vdash \varphi \to (\psi \to \varphi)$	CPC
3. $\vdash \varphi \to ((\varphi \to \varphi) \to \varphi)$	Substitution in 2
4. $\vdash (\varphi \to \varphi) \to \varphi$	MP in 1 and 3

Hence: $1 = [\varphi \to \varphi] \leq [\varphi]$, that is, $[] = 1$.

On the other hand, if $[\varphi] = 1$, then $[\ \to \varphi] \leq [\varphi]$ and so $\vdash (\varphi \to \varphi) \to \varphi$. Since $\vdash \varphi \to \varphi$, by MP, it follows that $\vdash \varphi$.

Now, φ is irrefutable iff $\nvdash\ \ \varphi$ iff $[\ \varphi] \neq 1$ iff $\neg [\varphi] \neq 1$ iff $[\varphi] \neq 0$. ■

From the preceding proposition and the definitions of 0 and 1 in the Lindenbaum algebra, it results that for each formula φ:

$$[\varphi] = 1 \text{ iff } \vdash \varphi \text{ and}$$
$$[\varphi] = 0 \text{ iff } \vdash \neg \varphi.$$

Theorem 3.12: (Soundness) The D-algebras are correct models for the logic **KD**.
Proof: Let $\mathbf{D} = (D, 0, 1, \wedge, \vee, \sim, \pi)$ be a D-algebra. It remains to prove that the axioms **ON**, **OC**, **OD** are valid and the rule **Rom** preserves validity:

ON: $v(O \top) = \pi\, v(\top) = \pi\, 1 = 1$.

OD: $v(\neg O \perp) = \sim v(O \perp) = \sim \pi\, v(\perp) = \sim \pi\, 0 = \sim 0 = 1$.

OC: $v(O \varphi \wedge O \psi) = v(O \varphi) \wedge v(O \psi) = \pi\, v(\varphi) \wedge \pi\, v(\psi) = \pi\, (v(\varphi) \wedge v(\psi)) = v(O\, (\varphi \wedge \psi))$.

Rom: $v(\varphi \to \psi) = 1 \Rightarrow v(\varphi) \leq v(\psi) \Rightarrow \pi\, v(\varphi) \leq \pi\, \mathrm{v}(\psi) \Rightarrow v(O \varphi) \leq v(O \psi) \Rightarrow v(O \varphi \to O \psi) = 1$.

■

Corollary 3.13: The logic **KD** is consistent.
Proof: Suppose that **KD** is not consistent. Then there is $\varphi \in$ For(**KD**) such that $\vdash$ and $\vdash \neg \varphi$.

So, by the Soundness Theorem, φ and $\neg \varphi$ are valid. Let v be a valuation in a D-algebra with exactly two elements $2 = \{0, 1\}$. Since φ is valid, then $v(\varphi) = 1$ and $v(\neg\varphi) = \sim v(\varphi) = 0$. But this contradicts the fact of $\neg \varphi$ is valid. ■

Theorem 3.14: For $\varphi \in$ For(**KD**), the following assertions are equivalent:

(i) $\vdash \varphi$

(ii) $\vDash \varphi$

(iii) φ is valid in every D-algebra of sets $\mathbf{B} = (B, , \cap, \cup, {}^{C}, \pi)$

(iv) $v_0(\varphi) = 1$, for the canonical valuation in $\mathbf{A}(\mathbf{KD})$.

Proof: (i) $\Rightarrow$ (ii): from the Soundness Theorem.

(ii) $\Rightarrow$ (iii): is immediate.

(iii) $\Rightarrow$ (iv): as every D-algebra is isomorphic to a D-algebra of sets $\mathbf{B} = (B, \varnothing, \cap, \cup, {}^{C}, \pi)$ and $\mathbb{A}$(**KD**) is a D-algebra, the result follows.

(iv) $\Rightarrow$ (i): if $\in$ For(**KD**) and it is not derivable in **KD**, by Corollary 3.11, $[\varphi] \neq 1$ in $\mathbb{A}$(**KD**) and then $v_0(\varphi) \neq 1$. Therefore is not a valid formula. ■

Corollary 3.15: (Completeness) For each $\varphi \in$ For(**KD**), if φ is valid, then φ is derivable in **KD**.

5 Strong completeness

In this subsection we show the strong adequacy of the algebraic models given by D-algebras.

As usual, $\Gamma \vDash \varphi$ denotes that every model of Γ is also a model of φ.

Proposition 4.1: (Strong soundness) For $\Gamma \subseteq$ For(**KD**), if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.
Proof: Let v: Var(**KD**) $\rightarrow$ **D** be an algebraic model for Γ. As in Theorem of Soundness, the rules of **KD** preserve validity and if $v_{\mathbf{D}}(\gamma) = 1$, for every $\gamma \in \Gamma$, then $v_{\mathbf{D}}() = 1$. ■

Proposition 4.2: Let $\Gamma \subseteq$ For(**KD**) and **D** a D-algebra. If there is a model v: For(**KD**) $\rightarrow$ **D** for Γ, then Γ is consistent.
Proof: Suppose that Γ is not consistent. Then $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\, \varphi$. Hence there is a valuation v such that $v_{\mathbf{D}}(\varphi) = 1$ and $v_{\mathbf{D}}(\neg\, \varphi) = 1$. If $v_{\mathbf{D}}(\neg\, \varphi) = 1$, then $\sim v_{\mathbf{D}}(\varphi) = 1$ and $v_{\mathbf{D}}(\varphi) = 0$. This way, we have a contradiction. ■

Definition 4.3: A model v: For(**KD**) $\rightarrow$ **D** is strongly adequate for Γ when:

$$\Gamma \vdash \varphi \text{ iff } \Gamma \vDash \varphi.$$

Proposition 4.4: If $\Gamma \subseteq$ For(**KD**) is consistent, then the canonical valuation is an adequate model to Γ.
Proof: Considering the canonical valuation v_0: For(**KD**) $\rightarrow$ $\mathbb{A}$(**KD**), that maps $v_0(\varphi) = [\varphi]$, by Corollary 3.11 and Proposition 4.1, $v_0() = 1$ if $\Gamma \vdash \varphi$. Therefore, we have that v_0 is an adequate model for Γ.

Theorem 4.5: For $\Gamma \subseteq$ For(**KD**), the following conditions are equivalent:

(i) Γ is consistent

(ii) there is an adequate model to Γ

(iii) there is an adequate model to Γ in a D-algebra of sets

$$\mathbf{B} = (B, \varnothing, \cap, \cup, {}^{C}, \pi)$$

(iv) there is a model for Γ.

Proof: (i) $\Rightarrow$ (ii) It follows of the previous proposition.

(ii) $\Rightarrow$ (iii) As $\mathbb{A}$(**KD**) is a D-algebra and every D-algebra is isomorphic to a D-algebra of sets $\mathbf{B}$ = (B, , $\cap$, $\cup$, ^{C}, π), then the result follows.

(iii) $\Rightarrow$ (iv) Immediate.

(vi) $\Rightarrow$ (i) It results directly by Proposition 4.2.

Corollary 4.6: (Strong adequacy) Let $\Gamma \cup \{\varphi\} \subseteq$ For(**KD**). If Γ is consistent, the following conditions are equivalent:

(i) $\Gamma \vdash \varphi$

(ii) $\Gamma \vDash \varphi$

(iii) every model of Γ in a D-algebra of sets $\mathbf{B} = (B, , \cap, \cup, ^C, \pi)$ is a model for φ

(iv) $v_0() = 1$ for the canonical valuation v_0.

6 D-algebra of filters

Now, we show that the mathematical structure of proper filters characterizes D-algebras, and they are models for **KD**.

Definition 5.1: A filter on a set E is a nonempty collection $\boldsymbol{F} \subseteq P(E)$ such that:

(i) if A, B $\boldsymbol{F}$, then $A \cap B \in \boldsymbol{F}$

(ii) if A $\boldsymbol{F}$ and $A \subseteq B$, then B $\boldsymbol{F}$.

Since $\boldsymbol{F}$ is nonempty, then $E \in \boldsymbol{F}$.

For the set E, the collections P(E) and {E} are trivial cases of filters.

Proposition 5.2: Any intersection of filters over E is a filter over E.

Definition 5.3: A proper filter on E is a filter $\boldsymbol{F}$ such that $\boldsymbol{F} \neq P(E)$.

If $\boldsymbol{F}$ is proper, then $\varnothing \notin \boldsymbol{F}$.

An intersection of proper filters is a proper filter, for the set $\varnothing$ does not belong to any one of them.

Definition 5.4: The filter $\boldsymbol{F}$ is free if $\cap \boldsymbol{F} = \varnothing$.

Example 5.5: For each nonempty subset $A \subseteq E$, the set $[A] = \{C \subseteq E : A \subseteq C\}$ is a filter on E. This filter is called the principal filter generated by A, and it is the least filter on E that contains A.

If $A = \{x\}$, the principal filter generated by A is the set $[A] = \{C \subseteq E : x \in C\}$.

Example 5.6: If E is infinite, then a subset $A \subseteq E$ is co-finite if its complement on E, A^C is finite. The family of all co-finite subsets of E is then a filter on E. This filter is called the Fréchet filter on E.

Each Fréchet filter is an example of a free filter.

Proposition 5.7: If the filter $\boldsymbol{F}$ is free, then it is not principal.
Proof: Let $\boldsymbol{F}$ be a filter on E. If $\boldsymbol{F}$ is principal, then $\boldsymbol{F} = [A]$, $A \neq \varnothing$ and A $\boldsymbol{F}$. Thus $\boldsymbol{F} = A$ and so $\boldsymbol{F}$ is not free. ■

Proposition 5.8: If E is finite, then every proper filter on E is principal.
Proof: Let $\boldsymbol{F}$ be a proper filter on the finite set E. Thus P(E) is finite too and $B = \cap \boldsymbol{F}$ $\boldsymbol{F}$. Thus $\boldsymbol{F} = [B]$ and as $\boldsymbol{F}$ is proper, then $B \neq \varnothing$. So, $\boldsymbol{F}$ is principal. ■

Proposition 5.9: If $\boldsymbol{F}$ is a filter on E, then it is free if, and only if, it contains the Fréchet filter on E.
Proof: If $\boldsymbol{F}$ is free, the intersection of all sets in $\boldsymbol{F}$ is empty. Thus, for each $x \in E$ there is a set $B_x \in \boldsymbol{F}$ such that $x \notin B_x$, $B_x \subseteq E\text{-}\{x\}$ and $E\text{-}\{x\} \in \boldsymbol{F}$.

Considering that each co-finite set is an intersection of sets of the type E - {x}, then all co-finite sets are in $\boldsymbol{F}$. Hence, $\boldsymbol{F}$ contains the Fréchet filter on E.

In the other side, let $\boldsymbol{F}_f$ the Fréchet filter on E and for an aleatory filter $\boldsymbol{F}$, $\boldsymbol{F}_f \subseteq \boldsymbol{F}$. Then $\cap\, \boldsymbol{F} \subseteq \cap\, \boldsymbol{F}_f$ = , for $\boldsymbol{F}_f$ is free. Hence, $\boldsymbol{F}$ is also free. ■

Definition 5.10: A family $\boldsymbol{A}$ of subsets of E has the finite intersection property (fip) if every finite subfamily of $\boldsymbol{A}$ has a nonempty intersection.

The proper filters have the finite intersection property, for if:
$A_1, ..., A_n$ $\boldsymbol{F}$, then $\cap_1^n A_i \in \boldsymbol{F}$ and $\cap_1^n A_i \neq \varnothing$.

Proposition 5.11: If $\boldsymbol{A} \subseteq P(E)$ has the fip, then there is a minimal filter $\boldsymbol{F}_{\boldsymbol{A}}$ that contains $\boldsymbol{A}$.
Proof: Let $\boldsymbol{F}_{\boldsymbol{A}} = \{B \subseteq E : B$ includes all finite intersections of $\boldsymbol{A}\}$. So, B is closed for finite intersections and super sets. ■

Definition 5.12: The filter $\boldsymbol{F}_{\boldsymbol{A}}$ is the filter generated by $\boldsymbol{A}$.

Corollary 5.13: Let $\boldsymbol{A} \subseteq P(E)$. Then $\boldsymbol{A}$ is included in a filter on E if, and only if, $\boldsymbol{A}$ has the fip.

We can define a D-space by the following.

Definition 5.14: A D-space is a pair (E, $\boldsymbol{F}$) such that E is a nonempty set and $\boldsymbol{F}$ is a proper filter on E.

Theorem 5.15: Each D-space (E, $\boldsymbol{F}$) determines a D-algebra.
Proof: Given a D-space (E, $\boldsymbol{F}$) we define π as the characteristic function of $\boldsymbol{F}$, that is, π: $\boldsymbol{F}$ $\{0, 1\}$, such that $\pi(A) = 1$ if, and only if, $A \in \boldsymbol{F}$.

(i) Since $E \in \boldsymbol{F}$, then $\pi(E) = 1$.

(ii) But, as $\boldsymbol{F}$ is proper, then $\pi(\varnothing) = 0$.

(iii) If $\pi(A) = 1$ and $(B) = 1$, then $A \in \boldsymbol{F}$ and $B \in \boldsymbol{F}$. As $\boldsymbol{F}$ is a filter, then $A \cap B \in \boldsymbol{F}$ and $\pi(A \cap B) = 1$. In the other side, if $\pi(A \cap B) = 1$, then $A \cap B \in \boldsymbol{F}$, A, $B \in \boldsymbol{F}$ and $\pi(A) = 1$ and $\pi(B) = 1$.

Hence $\pi(A) \wedge \pi(B) = \pi(A \cap B)$. ■

In addition to the D-algebras that are general algebraic models for **TK**, the D-spaces, which are very simple spaces with filters, are also adequate strong models for Standard Deontic Logic (FEITOSA; SOARES; LÁZARO, 2019).

7 Final considerations

From a very general view of Modal Logic, we presented Standard Deontic Logic, **KD**, in three equivalent characterizations. Considering one of these presentations, we introduced D-algebras, which are Boolean algebras increased by one operator that formalizes the deontic aspects of the modal operator in that presentation. Then, we showed that the D-algebras are adequate strong models for **KD**. Finally, we demonstrated that each mathematical structure of proper filters characterizes a D-algebra and thus constitutes an adequate strong model for **KD**.

In the further steps, we plan to investigate the relations between these models for **KD** with the usual Kripke semantics for **KD**.

References

BELL, J. L.; MACHOVER, M. *A course in mathematical logic*. Amsterdam: North-Holland, 1977.

BLACKBURN, P.; RIJKE, M.; VENEMA, Y. *Modal logic*. Cambridge: Cambridge University Press, 2001.

CARNIELLI, W. A.; PIZZI, C. *Modalità e multimodalità*. Milano: Franco Angeli, 2001.

CHAGROV, A.; ZAKHARYASCHEV, M. *Modal logic*. Oxford: Clarendon Press, 1997.

CHELLAS, B. *Modal Logic:* an introduction. Cambridge: Cambridge University Press, 1980.

DUNN, J. M.; HARDEGREE, G. M. *Algebraic methods in philosophical logic*. Oxford: Oxford University Press, 2001.

EBBINGHAUS, H. D.; FLUM, J.; THOMAS, W. *Mathematical logic*. New York: Springer-Verlag, 1984.

ENDERTON, H. B. *A mathematical introduction to logic*. San Diego: Academic Press, 1972.

FEITOSA, H. A.; SOARES, M. R.; LÁZARO, C. A. Lógica deôntica básica e filtros. *Argumentos Revista de Filosofia*, ano 11, n. 22, p. 7-16, 2019. DOI: https://doi.org/10.36517/Argumentos.22.1

FITTING, M.; MENDELSOHN, R. L. *First-order modal logic*. Dordrecht: Kluwer, 1998.

MENDELSON, E. *Introduction to mathematical logic*. 3. ed. Monterey, CA: Wadsworth and Brooks / Cole Advanced Books and Software, 1987.

MIRAGLIA, F. *Cálculo proposicional: uma interação da álgebra e da lógica*. Campinas: UNICAMP/CLE, 1987. (Coleção CLE, v. 1).

RASIOWA, H.; SIKORSKI, R. *The mathematics of metamathematics*. 2. ed. Waszawa: PWN - Polish Scientific Publishers, 1968.

RASIOWA, H. *An algebraic approach to non-classical logics*. Amsterdam: North-Holland, 1974.

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