OPEN ACCESS

COGNITIO
Revista de Filosofia
Centro de Estudos de Pragmatismo
São Paulo, v. 25, n. 1, p. 1-11, jan.-dez. 2024
e-ISSN: 2316-5278

https://doi.org/10.23925/2316-5278.2024v25i1:e67779

Galois connections and modal algebras

Conexões de Galois e álgebras modais

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Abstract: We investigate the properties of a modal algebra, more specifically, a non-distributive lattice with operators via Galois connections. Pairs of Galois are very common in mathematical environments, and, in this article, they appear as unary operators in lattices even without the distributivity. In a previous paper, Castiglioni and Ertola-Biraben studied the meet-complemented lattices with two modal operators for necessary □ and possible ◊. We observed that this pair of operators determines an adjunction. Then, we used Galois pairs on the meet-complemented lattices, showing some properties of this structure that were already been proved in their paper, and some new laws non presented. Lastly, we define a new pair of operators that also constitute another Galois pair.

Keywords: Algebraic logic. Modal algebra. Galois connections. Non-distributive lattices.

Resumo: *Investigamos as propriedades de uma álgebra modal, mais especificamente, uma rede não distributiva com operadores via conexões de Galois. Pares de Galois são muito comuns em ambientes matemáticos e, neste artigo, eles aparecem como operadores unários em redes mesmo sem a distributividade. Em um artigo anterior, Castiglioni e Ertola-Biraben estudaram as redes complementadas por encontro com dois operadores modais para necessário □ e possível ◊. Observamos que esse par de operadores determina uma adjunção. Então, usamos pares de Galois nas redes complementadas por encontro, mostrando algumas propriedades dessa estrutura que já haviam sido provadas em seu artigo e algumas novas leis não apresentadas. Por fim, definimos um novo par de operadores que também constituem outro par de Galois.*

Palavras-chave: *Algebra Modal. Cocexões de Galois. Lógica Algébrica. Redes Não-distributivas.*

Recebido em: 01/08/2024.
Aprovado em: 17/08/2024.
Publicado em: 11/09/2024.

1 Introduction

Castiglioni and Ertola-Biraben (2017) investigated the meet-complemented lattices with two modal operators. In this structure, since it is a lattice, we have a notion of order and, additionally, a kind of pseudo-complement operator defined, which is named meet-complement. Even with few elements, the tradition and the authors could introduce the modal operators and show a few numbers of laws to this structure.

As original contribution, we observed that this pair of modal operators determines an adjunction, a case of Galois pair. Since we have already developed logical investigations using the Galois pairs, we could apply this theory on the meet-complemented lattices, showing some of the properties already presented in Castiglioni and Ertola-

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Biraben (2017), but also new ones. Furthermore, we could define a new pair of operators that also constitute another Galois pair.

Thus, in a first moment, in Sections 2 and 3, we remember some algebraic concepts about lattices and Galois pairs, since these concepts are going to constitute the base of our developments.

In Section 4, we take two modal operators in a meet-complemented lattice with 0 and 1, as introduced by Castiglioni and Ertola-Biraben (2017), and show that they compose a type of Galois connection. Using this connection and results about the theory of Galois pairs, we show some properties of the limited meet-complemented lattices.

At last, Section 5, considering this same algebraic structure, we define new operators and show that these operators constitute a new Galois pair.

2 Lattices

In this section, we present some notions about lattices, some of which can be found in Birkhoff (1948), Dunn and Hardegree (2001), Miraglia (1987) and Rasiowa and Sikorski (1968).

Definition 2.1. A partial order on L is a binary relation $\leq$ which is reflexive, anti-symmetric and transitive.

Definition 2.2. A partially ordered set (poset) is a pair $\langle L, \leq \rangle$ such that L is a non-empty set and $\leq$ is a partial order on L.

Definition 2.3. If $\langle L, \leq \rangle$ is a poset and $a, b \in L$, then the supremum of $\{a, b\}$, case it exists, is the element $c \in L$ such that:

(i) $a \leq c$ and $b \leq c$

(ii) $a \leq d$ and $b \leq d \Rightarrow c \leq d$.

Definition 2.4. If $\langle L, \leq \rangle$ is a poset and $a, b \in L$, then the infimum of $\{a, b\}$, case it exists, is the element $e \in L$ such that:

(i) $e \leq a$ and $e \leq b$

(ii) $f \leq a$ and $f \leq b \Rightarrow f \leq e$.

It is usual to denote the supremum of $\{a, b\}$ by $sup\{a, b\}$ or $a \curlyvee b$, and the infimum by $inf\{a, b\}$ or $a \curlywedge b$.

The supremum of $\{a, b\}$ is the *least upper bound of* $\{a, b\}$ and the infimum is the *greatest lower bound of* $\{a, b\}$.

Definition 2.5. A *lattice* is a mathematical structure $\mathbf{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ formed by a partially ordered set $\langle L, \leq \rangle$ in which for any $a, b \in L$ there is $a \curlyvee b$ and $a \curlywedge b$ in L.

Proposition 2.6. If $\mathbf{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ is a lattice, then for all $a, b, c \in L$:

L_1 $(a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$ and $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c)$ [Associativity];

L_2 $a \curlywedge b = b \curlywedge a$ and $a \curlyvee b = b \curlyvee a$ [Commutativity];

L_3 $(a \curlywedge b) \curlyvee b = b$ and $(a \curlyvee b) \curlywedge b = b$ [Absorption];

L_4 $a \curlywedge a = a$ and $a \curlyvee a = a$ [Idempotency];

L_5 $a \curlywedge b = a \Leftrightarrow a \leq b \Leftrightarrow a \curlyvee b = b$ [Order].

An equivalent way to define a *lattice* is by the following one.

Lattice is an algebraic structure $\mathbf{L} = \langle L, \wedge, \vee \rangle$ such that L is a non-empty set, and $\wedge$ and $\vee$ are two binary operations defined on L satisfying:

ℓ_1 $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$ [Associativity];

ℓ_2 $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ [Commutativity];

ℓ_3 $(a \wedge b) \vee b = b$ and $(a \vee b) \wedge b = b$ [Absorption];

In the lattice $\mathbf{L} = \langle L, \land, \lor \rangle$ we can define a partial order $\leq$ such that: $a \leq b$ if, and only if, $a \lor b = b$, or equivalently, $a \leq b$ if, and only if, $a \land b = a$.

The mathematical structure $\mathbf{L} = \langle L, \leq, \land, \lor \rangle$ thus obtained is a lattice in the sense of the Definition 2.5 above, with $\land$ being the infimum and $\lor$ the supremum.

Proposition 2.7. If $\mathbf{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ is a lattice and $a, b, c, d \in L$, then:

L_6 $a \leq a \curlyvee b$ *and* $b \leq a \curlyvee b$;

L_7 $a \curlywedge b \leq a$ *and* $a \curlywedge b \leq b$;

L_8 $a \leq c$ *and* $b \leq c \Rightarrow a \curlyvee b \leq c$;

L_9 $c \leq a$ *and* $c \leq b \Rightarrow c \leq a \curlywedge b$;

L_{10} $a \leq c$ *and* $b \leq d \Rightarrow a \curlyvee b \leq c \curlyvee d$;

L_{11} $a \leq c$ *and* $b \leq d \Rightarrow a \curlywedge b \leq c \curlywedge d$;

L_{12} $a \leq b \Rightarrow a \curlywedge c \leq b \curlywedge c$;

L_{13} $a \leq b \Rightarrow a \curlyvee c \leq b \curlyvee c$.

Proposition 2.8. If $\mathbf{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ is a lattice, then:

L_{14} $(a \curlywedge b) \curlyvee (a \curlywedge c) \leq a \curlywedge (b \curlyvee c)$;

L_{15} $a \curlyvee (b \curlywedge c) \leq (a \curlyvee b) \curlywedge (a \curlyvee c)$.

Definition 2.9. Homomorphism between the lattices $\mathbf{L_1} = \langle L_1, \leq, \curlywedge, \curlyvee \rangle$ and $\mathbf{L_2} = \langle L_2, \leq, \curlywedge, \curlyvee \rangle$ is a function h from L_1 into L_2 such that:

$$h(a \curlywedge b) = h(a) \curlywedge h(b) \text{ and } h(a \curlyvee b) = h(a) \curlyvee h(b).$$

Every homomorphism of lattices preserves ordering, that is:

$$a \leq b \Leftrightarrow a \curlyvee b = b \Rightarrow h(a \curlyvee b) = h(b) \Leftrightarrow h(a) \curlyvee h(b) = h(b) \Leftrightarrow h(a) \leq h(b).$$

Definition 2.10. An isomorphism of lattices is a bijective homomorphism of lattices.

Definition 2.11. A lattice $\mathbf{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ is distributive if the following distributive laws are valid for all $a, b, c \in L$:

L_{16} $(a \curlywedge b) \curlyvee c = (a \curlyvee c) \curlywedge (b \curlyvee c)$ and $(a \curlyvee b) \curlywedge c = (a \curlywedge c) \curlyvee (b \curlywedge c)$.

These are distributive laws at right-hand side and, due to the commutative property, also the distributive laws at left-hand side are valid. Besides, only one of these two distributive laws would be enough to characterize the distributive property (Miraglia, 1987).

Definition 2.12. If the lattice $\mathbf{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ has the least element by the ordering $\leq$, then this element is the zero of **L** and is denoted by 0. If **L** has the greatest element relative to the ordering $\leq$, then this element is the one of **L** and it is denoted by 1.

If the lattice **L** has the elements 0 and 1, then for every $a \in L$:

L_{17} $a \curlywedge 0 = 0$ and $a \curlyvee 0 = a$;

L_{18} $a \curlywedge 1 = a$ and $a \curlyvee 1 = 1$.

We denote a lattice with 0 and 1 by $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee \rangle$. Such lattice is limited by 0 and 1.

Definition 2.13. If $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee \rangle$ is a poset with 0 and 1, then **L** has an involution ‘ if for all a, $b \in L$:

L_{19} $a = a''$;

L_{20} $a \leq b \Rightarrow b' \leq a'$.

We denote a lattice with 0, 1 and involution $'$ by $\mathbf{L} = \langle L, 0, 1, \leq, ', \curlywedge, \curlyvee \rangle$.

Proposition 2.14. If $\mathbf{L} = \langle L, 0, 1, \leq, ', \curlywedge, \curlyvee \rangle$ is a poset with involution, then the De Morgan's laws hold:

L_{21} $(a \curlywedge b)' = (a' \curlyvee b')$;

L_{22} $(a \curlyvee b)' = (a' \curlywedge b')$.

Indeed, in the presence of L_{20} in **L** = ⟨L, 0, 1, ≤, ′, ⋏, ⋎⟩ the conditions $L_{20,} L_{21}$ and L_{22} are equivalents. Besides, in this case, *sup*{*a*, *b*} is defined if, and only if, *inf*{*a*, *b*} is also defined.

Definition 2.15. If **L** = ⟨*L*, 0, 1, ≤, ⋏, ⋎⟩ is a lattice with 0 and 1 and for *a* ∈ *L* there exists the element ∼ *a* = *max*{*y* ∈ *L* : *a* ⋏ *y* = 0} in *L*, then ∼ *a* is the pseudo-complement of *a*.

Definition 2.16. A lattice **L** is pseudo-complemented if every element *a* ∈ *L* has a pseudo-complement ∼ *a* ∈ *L*.

The ps0eudo-complement comes from the tradition of intuitionistic logic and algebra (Rasiowa, 1974). This pseudo-complement characterizes a type of negation in intuitionistic logic and algebra.

Of course, this negation is different from the classical negation.

Definition 2.17. Let **L** = ⟨*L*, 0, 1, ≤, ⋏, ⋎⟩ be a lattice with 0 and 1. If *a* ∈ *L*, then the complement of a *a* in **L** is an element −*a* ∈ *L* such that:

L_{23} *a* ⋏ −*a* = 0 and

L_{24} *a* ⋎ −*a* = 1.

Every complement −*a* is a pseudo-complement ∼*a*, however, for example, the intuitionistic pseudo-complement of Intuitionistic Logic is not a complement.

The complement is an involution, but the pseudo-complement is only an almost involution, because for this concept it holds only *a* ≤ ∼∼ *a* and L_{20}.

Definition 2.18. The lattice **L** = ⟨*L*, 0, 1, ≤, ⋏, ⋎⟩ is complemented if every element of *L* has a complement in *L*. If every element of **L** has exactly one complement, then the lattice **L** is uniquely complemented.

If the complement of *a* is unique, we denote its complement by −*a*. This case we denote this lattice by **L** = ⟨*L*, 0, 1, ≤, −, ⋏, ⋎⟩.

Lemma. 2.19. If **L** = ⟨*L*, 0, 1, ≤, ⋏, ⋎⟩ is a distributive lattice with 0 and 1, and there exists a complement of *a*, then it is unique.

Proof. If *y* and *z* are two complements of *a*, then *a* ⋏ *y* = 0, *a* ⋎ *y* = 1, *a* ⋏ *z* = 0, and *a* ⋎ *z* = 1. As *z* = 0 ⋎ *z* = (*a* ⋏ *y*) ⋎ *z* = (*a* ⋎ *z*) ⋏ (*y* ⋎ *z*) = 1 ⋏ (*y* ⋎ *z*) = *y* ⋎ *z*, that is, *y* ≤ *z*. Analogously, *z* ≤ *y* and, hence, *z* = *y*.

Definition 2.20. A Boolean algebra **B** is a distributive and complemented lattice.

As is well known, the Boolean algebras are model for classical propositional logic, Brouwer (Heyting) algebras are models for intuitionistic propositional logic.

We are interested in a type of algebraic structure less specific than these algebras.

3 Galois pairs

In this section, we present the Galois connections and other pairs of functions associated with the Galois connections, named in general as Galois pairs.

We mainly follow the book of Dunn and Hardegree (2001), but there are so many variations on the definitions and notations about the Galois pairs. We also suggest the Herrlich and Husek (1990), Ore (1944) and Orlowska and Rewitzky's (2010) and Smith's (2010) papers.

We start with some preliminaries notions about functions that will be used in the development of theory.

Definition 3.1. Let *f* : $(A, \leq_A) \to (P, \leq_P)$ a function between two partial ordered sets. Then:

(i) the function *f* preserves the orders if: $a \leq_A b \Rightarrow f(a) \leq_P f(b)$;

(ii) the function *f* inverts the orders if: $a \leq_A b \Rightarrow f(b) \leq_P f(a)$.

Sometimes these functions are called increasing and decreasing, monotone, isotone and antitone.

Definition 3.2. If $f : (A, \leq_A) \to (A, \leq_A)$, then:

(i) the function f is idempotent if $f \circ f = f$;

(ii) the function f is extensive or inflationary if for every $a \in A$, $a \leq f(a)$;

(iii) the function f is deflationary if for every $a \in A$, $f(a) \leq a$.

Definition 3.3. If $f : (A, \leq_A) \to (A, \leq_A)$, then:

(i) the function f is a Tarski operator (deductive closure) if f is idempotent, inflationary, and preserves orders;

(ii) the function f is an interior operator if f is idempotent, deflationary, and preserves orders.

Definition 3.4. Given two partial ordered sets $(A, \leq_A)$ and $(P, \leq_P)$, and the functions $f : A \to P$ and $g : P \to A$, then the pair (f, g) is a Galois connection if, for every $a \in A$ and every $p \in P$:

$$a \leq_A g(p) \Leftrightarrow p \leq_P f(a).$$

From this definition, we have that if (f, g) is a Galois connection for the partial orders $(A, \leq_A)$ and $(P, \leq_P)$, then the pair (g, f) is also a Galois connection for the partial orders $(P, \leq_P)$ and $(A, \leq_A)$.

Example 3.5. Considering $A = P = \mathbb{Z}$, with the natural order of integers, $g = f$, and the function $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = -x$, then (f, g) is a Galois connection for $(\mathbb{Z}, \leq)$, because $a \leq g(p) \Leftrightarrow a \leq -p \Leftrightarrow p \leq -a \Leftrightarrow p \leq f(a)$.

Example 3.6. If $A = P$ is a set of mathematical sentences, the order is given by the classical conditional $\varphi \to \psi$, and $f = g$ is the operator of negations, that is, $f(\sigma) = g(\sigma) = \neg\sigma$, then (f, g) is a Galois connection, $\varphi \to \psi \Leftrightarrow \neg\psi \to \neg\varphi$.

We quote without proof the following proposition.

Proposition 3.7. Let $(A, \leq_A)$ and $(P, \leq_P)$ two partial orders, $f : A \to P$ and $g : P \to A$ functions. The pair (f, g) is a Galois connection if, and only if, for every $a, b \in A$ and for every $p, q \in P$ are valid the items:

(i) $a \leq b \Rightarrow f(b) \leq f(a)$;

(ii) $p \leq q \Rightarrow g(q) \leq g(p)$;

(iii) $a \leq g(f(a))$;

(iv) $p \leq f(g(p))$.

Thus, we have an alternative way to characterize any Galois connection. The pair (f, g) is a Galois connection if the functions f and g invert the orders, the composition $f \circ g$ is extensive, and the composition $g \circ f$ is inflationary.

When we look for the definition of Galois connection, four permutations can occur and theses cases generates other pairs of functions with some similitude, which are named, in the collective, as Galois pairs.

Definition 3.8. If $(A, \leq_A)$ and $(P, \leq_P)$ are partial ordered sets, $a \in A$ and $p \in P$ are random elements and $f : A \to P$ and $g : P \to A$ are functions, then:

(i) the pair $(f, g)^d$ is a dual Galois connection if: $g(p) \leq_A a \Leftrightarrow f(a) \leq_P p$;

(ii) the pair $[f, g]$ is an adjunction if: $a \leq_A g(p) \Leftrightarrow f(a) \leq_P p$;

(iii) the pair $[f, g]^d$ is a dual adjunction if: $g(p) \leq_A a \Leftrightarrow p \leq_P f(a)$.

The name adjunction comes from the theory of categories. Sometimes the pair $[f, g]$ is also called residuated.

As in the case of Galois connections, we have a proposition that give us equivalent conditions to describe an adjunction.

Proposition 3.9. Let $(A, \leq_A)$ and $(P, \leq_P)$ two partial orders, $f : A \to P$ and $g : P \to A$ functions. The pair $[f, g]$ is an adjunction if, and only if, for every $a, b \in A$ and for every $p, q \in P$, the following items are valid:

(i) $a \leq b \Rightarrow f(a) \leq f(b)$;

(ii) $p \leq q \Rightarrow g(p) \leq g(q)$;

(iii) $a \leq g(f(a))$;

(iv) $f(g(p)) \leq p$.

Now, we present some properties of adjunctions.

Proposition 3.10. If the pair $[f, g]$ is an adjunction, for the partial orders $(A, \leq_A)$ and $(P, \leq_P)$, then: $f(a) = f(g(f(a)))$ and $g(p) = g(f(g(p)))$.

Proposition 3.11. If the pair $[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P)$, then the compositions $g \circ f$ and $f \circ g$ are operators of Tarski and interior, respectively, over A and P.

Example 3.12. Given two sets E and F, let's consider a relation R between E and F, that is, $R \subseteq E \times F$ and the following order relations $(\mathcal{P}(E), \subseteq)$ and $(\mathcal{P}(F), \subseteq)$.

Now, we define the following functions: $f : \mathcal{P}(E) \rightarrow \mathcal{P}(F)$ by $A^f = \{y \in F : (\exists x \in A)(xRy)\}$ and $g : \mathcal{P}(F) \rightarrow \mathcal{P}(E)$ by $B^g = \{x \in E : (\forall y)(xRy \rightarrow y \in B)\}$.

We will show that $[f, g]$ is an adjunction. For this, we must observe that: (i) $A_1 \subseteq A_2 \Rightarrow A_1^f \subseteq A_2^f$; (ii) $B_1 \subseteq B_2 \Rightarrow B_1^g \subseteq B_2^g$; (iii) $(B^g)^f \subseteq B$; and (iv) $A \subseteq (A^f)^g$, in according with Proposition 3.9.

(i) Let $A_1 \subseteq A_2$. If $y \in A_1^f$, then $(\exists x \in A_1)(xRy)$. From hypothesis, $(\exists x \in A_2)(xRy)$ and, hence, $y \in A_2^f$. So $A_1^f \subseteq A_2^f$.

(ii) Let $B_1 \subseteq B_2$. If $x \in B_1^g$, then $(\forall y)(xRy \rightarrow y \in B_1)$. From hypothesis, $(\forall y)(xRy \rightarrow y \in B_2)$ and then $x \in B_2^g$. So $B_1^g \subseteq B_2^g$.

(iii) If $y \in (B^g)^f$, then exists $x \in B^g$ such that xRy. Now, if xRy and $x \in B^g$, then $y \in B$. So, $(B^g)^f \subseteq B$.

(iv) Let $A \nsubseteq (A^f)^g$. Then exists $x \in A$ such that $x \notin (A^f)^g$. From that, it is not the case that $(\forall y)(xRy \rightarrow y \in A^f)$. Thus, there exists y such that xRy and $y \notin A^f$. If $y \notin A^f$, then it is not the case that exists x such that xRy, and this contradicts the affirmation above.

Example 3.13. As a particular case of previous example, let's consider $E = F$ and R as a partial order $\leq$ on E.

Then $f : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is defined by $A^f = \{y \in E : (\exists x \in A)(x \leq y)\}$, that consists of all elements of A and elements of E that are greater than some element of A; and $g : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is defined by $B^g = \{x \in E : (\forall y)(x \leq y \rightarrow y \in B)\}$, that consists of all elements of B and elements of E that are not greater than any element of B.

Example 3.14. Let $h : E \rightarrow F$ a function and let's consider the order relations $(\mathcal{P}(E), \subseteq)$ and $(\mathcal{P}(F), \subseteq)$.

So we define the direct image via h by $f : \mathcal{P}(E) \rightarrow \mathcal{P}(F)$ with $A^f = \{y \in F : x \in A \text{ and } h(x) = y\}$ and the inverse image via h by $g : \mathcal{P}(F) \rightarrow \mathcal{P}(E)$ with $B^g = \{x \in E : y \in B \text{ and } h(x) = y\}$.

This way, we show that $[f, g]$ is an adjunction.

Of course the functions f and g preserve the orders.

If $y \in (B^g)^f$, then there exists $x \in B^g$ such that $h(x) = y$. Now, as $h(x) = y$ *and* $x \in B^g$, then $y \in B$. Hence, $(B^g)^f \subseteq B$.

If $x \in A$, like h is function, then for $y = f(x)$ we have that $y \in A^f$. So $x \in (A^f)^g$, that is, $A \subseteq (A^f)^g$.

Proposition 3.15. If the pair $[f, g]$ is an adjunction for the lattices $(A, \curlywedge, \curlyvee)$ and $(P, \curlywedge, \curlyvee)$ then:

(i) $f(x \curlyvee y) = f(x) \curlyvee f(y)$;

(ii) $g(x \curlywedge y) = g(x) \curlywedge g(y)$.

Proposition 3.16. If $[f, g_1]$ and $[f, g_2]$ are adjunctions for $(A, \leq_A)$ and $(P, \leq_P)$, then $g_1 = g_2$. If $[f_1, g]$ and $[f_2, g]$ are adjunctions for $(A, \leq_A)$ and $(P, \leq_P)$, then $f_1 = f_2$.

Proposition 3.17. If $[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P)$, then:

(i) $a \in g(P) \Leftrightarrow g(f(a)) = a$;

(ii) $p \in f(A) \Leftrightarrow f(g(p)) = p$;

(iii) $f(A) = f(g(P))$;

(iv) $g(P) = g(f(A))$.

This way, each point $a \in g(P)$ is a fix point of the function $g \circ f$ and each point $p \in f(A)$ is a fix point of the function $f \circ g$.

Considering that $g \circ f$ is a Tarski operator over A, then the set $g(P) = \{a \in A : a = g(f(a))\}$ contains all closed elements of A relative this operator. Dually, the set $f(A) = \{p \in P : p = f(g(p))\}$ contains all open elements of P relative to the operator of interior $f \circ g$.

Proposition 3.18. If $[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P)$, then:

(i) $f(a) = min\{p \in P : a \leq g(p)\}$;

(ii) $g(p) = max\{a \in A : f(a) \leq p\}$.

Proposition 3.19. If $[f_1, g_1]$ is an adjunction for $(A, \leq_A)$ and $(B, \leq_B)$, and $[f_2, g_2]$ is an adjunction for $(B, \leq_B)$ and $(C, \leq_C)$, then $[f_2 \circ f_1, g_1 \circ g_2]$ is an adjunction for $(A, \leq_A)$ and $(C, \leq_C)$.

Each Galois pair presents similar results to these associated with the adjunctions.

4 Adjunctions on meet-complemented lattices

In the work of Castiglioni and Ertola-Biraben (2017) there is a definition of meet-complemented lattices with several properties and examples of them. On this algebraic structure, the authors introduced two unary operators and showed so many properties of this new algebraic structure.

In the following, we detach these definitions and show that these unary operators there defined determine an adjunction.

So, considering the developments in Section 3, we establish many properties of these meet-complemented lattices with operators, many of them proved by Castiglioni and Ertola-Biraben (2017), but we show some others else.

Definition 4.1. If a is an element in a lattice $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee\rangle$, then its meet-complement, when it exists, is the element in L, denoted by $\sim a$, such that

$$\sim a = max\{b \in L : a \curlywedge b = 0\}.$$

The meet-complement is a pseudo-complement. A meet-complemented lattice is a lattice in which each element has a meet-complement. Also, in this structure, there is no distributivity for **L**. Indeed the most interesting cases are non-distributive.

Definition 4.2. Let $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee\rangle$ a meet-complemented lattice. The operator of necessity $\square : L \to L$ is defined by:

$$\square a = max\{b \in L : a \curlyvee \sim b = 1\}.$$

For any $a \in L$, as the maximum is unique and 0 always belong to $\{b \in L : a \curlyvee \sim b = 1\}$, then the operator $\square$ is well defined.

Besides, this definition is equivalent to say that, given $a \in L$, the $\square a$ is the unique element in L for which are valid the following conditions:

(1) $a \curlyvee \sim \square a = 1$;

(2) $a \curlyvee \sim b = 1 \Rightarrow b \leq \square a$.

The equivalence is trivial since the condition (2) tells us that $\square a$ is the greatest element satisfying the condition (1).

Definition 4.3. Let $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee\rangle$ a meet-complemented lattice. The operator of possibility $\Diamond : L \to L$ is defined by:

$$\Diamond a = min\{b \in L : \sim a \curlyvee b = 1\}.$$

Again the operator is well defined, since $1 \in \{b \in L : \sim a \curlyvee b = 1\}$.

Besides, this definition is equivalent to the following together conditions:

(3) $\sim a \curlyvee \Diamond a = 1$;

(4) $\sim a \curlyvee b = 1 \Rightarrow \Diamond a \leq b$.

The condition (4) tells us that $\Diamond a$ is the least element satisfying the condition (3).

Now, we can observe these two operators as an adjunction.

Proposition 4.4. For any $a, b \in L$: $a \leq b \Rightarrow \Box a \leq \Box b$.

Proof. Let $a \leq b$. From (1) we have $a \curlyvee \sim\Box a = 1$. Then $b \curlyvee \sim\Box a = 1$ and by (2) we have that $\Box a \leq \Box b$.

Proposition 4.5. For any $a, b \in L$: $a \leq b \Rightarrow \Diamond a \leq \Diamond b$.

Proof. Let $a \leq b$. Thus $\sim b \leq \sim a$. From (3) we have $\sim b \curlyvee \Diamond b = 1$. Then $\sim a \curlyvee \Diamond b = 1$ and by (4) we have that $\Diamond a \leq \Diamond b$.

Proposition 4.6. Let $a \in L$, then $\Diamond\Box a \leq a$.

Proof. From (1) we have $\sim\Box a \curlyvee a = 1$. Then using (4) we have $\Diamond\Box a \leq a$.

Proposition 4.7. Let $a \in L$, then $a \leq \Box\Diamond a$.

Proof. From (3) we have $\Diamond a \curlyvee \sim a = 1$. Then using (2) we have $a \leq \Box\Diamond a$.

Now, from these four propositions and with Proposition 3.9, we conclude that the pair $[\Diamond, \Box]$ is an adjunction as in Definition 3.8 (ii).

From the fact that $[\Diamond, \Box]$ is an adjunction, we can show some results over **L**.

Proposition 4.8. In the algebraic structure $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee, \Diamond, \Box\rangle$ are valid:

(i) $a \leq \Box b \Leftrightarrow \Diamond a \leq b$;

(ii) $\Diamond a = min\{p \in L : a \leq \Box p\}$;

(iii) $\Box p = max\{a \in L : \Diamond a \leq p\}$;

(iv) $a \in \Box(L) \Leftrightarrow \Box(\Diamond(a)) = a$;

(v) $a \in \Diamond(L) \Leftrightarrow \Diamond(\Box(b)) = b$.

Proof. From Definition 3.8, Proposition 3.18 and Proposition 3.19.

Proposition 4.9. In $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee, \Diamond, \Box\rangle$ we have:

(i) $\Box\Diamond\Box a = \Box a$;

(ii) $\Diamond\Box\Diamond a = \Diamond a$;

(iii) $\Diamond(a \curlyvee b) = \Diamond a \curlyvee \Diamond b$;

(iv) $\Box(a \curlywedge b) = \Box a \curlywedge \Box b$;

Proof. From Proposition 3.10 and Proposition 3.15.

From the Proposition 3.11, we can define one operator of Tarski $\triangledown$ and another of interior $\triangle$ as bellow.

Definition 4.10. Let $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee, \Diamond, \Box\rangle$ be a meet-complemented lattice with $\Diamond$ and $\Box$. The operators of Tarski (closure) $\triangledown$ and interior $\triangle$ are defined in the following way:

(i) $\triangledown a = \Box\Diamond a$;

(ii) $\triangle a = \Diamond\Box a$.

Proposition 4.11. In the algebra $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee, \Diamond, \Box\rangle$ we have:

(i) $\triangle a \leq a$;

(ii) $a \leq b \Rightarrow \triangle a \leq \triangle b$;

(iii) $\triangle a \leq \triangle\triangle a$;

(iv) $\triangle\triangle a = \triangle a$;

(v) $\triangle a \leq \triangle a \curlyvee \triangle b \leq \triangle(a \curlyvee b)$;

(vi) $\triangle(a \curlywedge b) \leq \triangle a \curlywedge \triangle b \leq \triangle a$.

Since $\triangledown$ is an operator of closure, then:

Proposition 4.12. In the algebra $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee, \Diamond, \Box\rangle$ we have:

(i) $a \leq \triangledown a$;

(ii) $a \leq b \Rightarrow \triangledown a \leq \triangledown b$;

(iii) $\triangledown\triangledown a \leq \triangledown a$;

(iv) $\nabla\nabla a = \nabla a$;

(v) $\nabla a \leq \nabla a \curlyvee \nabla b \leq \nabla(a \curlyvee b)$;

(vi) $\nabla(a \curlywedge b) \leq \nabla a \curlywedge \nabla b \leq \nabla a$.

Therefore, we have two more unary operators.

5 More operators for meet-complemented lattices

Besides the two operators of previous section, we can define more operators on meet-complemented lattices that act as a Galois pair.

In this section, we introduce this new pair that perform another adjunction.

Definition 5.1. Let $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee\rangle$ a meet-complemented lattice. The operator $\odot : L \to L$ is defined by:

$$\odot a = min\{b \in L : a \curlywedge \sim b = 0\}.$$

As the minimum is unique and 1 always belongs to $\{b \in L : a \curlywedge \sim b = 0\}$, for any $a \in L$, then the operator is well defined.

Besides, this definition is equivalent to the following together conditions:

(5) $a \curlywedge \sim\odot a = 0$;

(6) $a \curlywedge \sim b = 0 \Rightarrow \odot a \leq b$.

The condition (5) says that $\odot a$ exists and (6) that $\odot a$ is the minimum.

Definition 5.2. Let $\mathbf{L} = \langle L, 0, 1, \leq, \curlywedge, \curlyvee\rangle$ a meet-complemented lattice. The operator $\oplus : L \to L$ is defined by:

$$\oplus a = max\{b \in L : \sim a \curlywedge b = 0\}.$$

This operator is well defined too.

Besides, this definition is equivalent to the following together conditions:

(7) $\sim a \curlywedge \oplus a = 0$;

(8) $\sim a \curlywedge b = 0 \Rightarrow a \leq \oplus b$.

The condition (7) says that $\oplus a$ exists and (8) that $\oplus a$ is the maximum.

Now, we can observe these two operators forming an adjunction.

Proposition 5.3. For any $a, b \in L$: $a \leq b \Rightarrow \odot a \leq \odot b$.

Proof. Let $a \leq b$. From (5) we have $b \curlywedge \sim\odot b = 0$. Then $a \curlywedge \sim\odot b = 0$ and by (6) we have that $\odot a \leq \odot b$.

Proposition 5.4. For any $a, b \in L$: $a \leq b \Rightarrow \oplus a \leq \oplus b$.

Proof. Let $a \leq b$. Thus $\sim b \leq \sim a$. From (7) we have $\sim a \curlywedge \oplus a = 0$. Then $\sim b \curlywedge \oplus a = 0$ and by (8) we have that $\oplus a \leq \oplus b$.

Proposition 5.5. Let $a \in L$, then $a \leq \oplus\odot a$.

Proof. From (5) we have $\sim\odot a \curlywedge a = 0$. Then using (8) we have $a \leq \oplus\odot a$.

Proposition 5.6. Let $a \in L$, then $\odot\oplus a \leq a$.

Proof. From (7) we have $\oplus a \curlywedge \sim a = 0$. Then using (6) we have $\odot\oplus a \leq a$.

Now, from these four propositions and the Proposition 3.7, we conclude that the pair $[\oplus, \odot]$ is an adjunction.

So, every property established for an adjunction holds for the pair $[\oplus, \odot]$ and we have several new properties over these two new operators.

6 Final considerations

We have been interested in Galois pairs (connections) and its properties. From these concepts very frequent and useful in the logical and mathematical context, we could explicate some results involving operators that were introduced in the paper of Castiglioni and Ertola-Biraben (2017) and explore new results over them.

But, besides that, we could introduce new operators and show some of their properties too. As every property established for an adjunction holds for the new pair [$\oplus$, $\odot$], then we have several new properties over these two new operators.

We have also the interior and closure operators defined from the adjunctions [$\Diamond$, $\Box$] and [$\oplus$, $\odot$], with all consequences of them.

In this way, we have considered the Galois pairs as a methodological approach to explore mathematical and logical structures. And, with this, we were able to glimpse some new results, in a certain sense, for what was already in the literature.

Acknowledgements

This work has been sponsored by Coordination of Superior Level Staff Improvement (CAPES): process number 88887.959299/2024-00.

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OPEN ACCESS

COGNITIO
Revista de Filosofia
Centro de Estudos de Pragmatismo

São Paulo, v. 25, n. 1, p. 1-11, jan.-dez. 2024
e-ISSN: 2316-5278

https://doi.org/10.23925/2316-5278.2024v25i1:e67779