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DOSSIÊ PEIRCE E A LÓGICA / *DOSSIER PEIRCE AND LOGICS*

Topos of existential graphs over Riemann surfaces

Topos de grafos existenciais sobre superfícies de Riemann

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Abstract: Peirce's Existential Graphs provide a geometrical understanding of a variety of logics (classical, intuitionistic, modal, first-order). The geometrical interpretation is given by topological transformations of closed (Jordan) curves on the plane, but it can be extended to other surfaces (sphere, cylinder, torus, etc.) The result provides the appearance of new logics related to the shapes of the surfaces. Going beyond, one can draw existential graphs over general Riemann Surfaces, and, introducing tools from algebraic geometry (Sheaves, Grothendieck Toposes, Elementary Toposes), one can try to capture both the logics and the geometrical shapes through a new Topos of Existential Graphs over Riemann Surfaces, and through the classifier subobject of the topos. We offer new perspectives (concepts, definitions, examples, conjectures) along this road.

Keywords: Existential Graphs. Logic. Peirce. Sheaves. Topos.

Resumo: *Os Grafos Existenciais de Peirce oferecem uma compreensão geométrica de uma variedade de lógicas (clássica, intuicionista, modal, de primeira ordem). A sua interpretação geométrica é dada por transformações topológicas de curvas fechadas (Jordan) no plano, mas pode ser estendida a outras superfícies (esfera, cilindro, toro etc.). Além disso, é possível desenhar grafos existenciais sobre superfícies de Riemann gerais e, introduzindo ferramentas da geometria algébrica (Feixes, Toposes de Grothendieck, Toposes Elementares), é possível tentar capturar as lógicas nas formas geométricas por meio de um novo Topos de Grafos Existenciais sobre Superfícies de Riemann e por meio do subobjeto classificador do topos. Oferecemos novas perspectivas (conceitos, definições, exemplos, conjecturas) ao longo desse caminho.*

Palavras-chave: *Feixes. Grafos Existenciais. Logica. Peirce. Topos.*

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1 Introduction

This work[1] is situated in what we may call a "geometrization of mathematics", or more concretely, a **geometrization of logic** Such a proposal lies in a mathematical panorama governed by *Grothendieck toposes*, which, after their transit to *elementary toposes*, open up the study of many particular logics. Independently, on another side, Peirce's *existential graphs* offer a profound topological vision of logic, built on local transformations of Jordan curves over the complex plane. This

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1 This is a report of my Undergraduate Thesis, "Topos de Gráficos Existenciales sobre Superficies de Riemann", Departamento de Matemáticas, Universidad Nacional de Colombia, Sede Bogotá, 2022, 108 pp., under the orientation of Fernando Zalamea, whom I thank for his support in the construction and discussion of this thesis, not only in the ideas on which it was based, but also in his mathematical precision and accurate corrections.

situation has been extended in the last decade, with the emergence of existential graphs on nonplanar surfaces (sphere, cylinder, torus), where the geometrical tools acquire higher preponderance. Using then techniques from *sheaves* and *toposes*, applied to concepts on *Riemann surfaces*, we obtain a wide vision of alternative existential graphs, related to intrinsic and extrinsic properties of geometrical logics.

2 Existential graphs (EG)

When we study Existential Graphs (EG) on the plane, their basic constructions, their rules and the uniqueness of their axiomatizations, we obtain a variety of associated logics, by means of an entirely diagrammatic, coherent and unitary presentation (Peirce, 1903). The Alpha and Beta models of (EG) constitute by themselves a complete (Roberts, 1992) and consistent treatment of elementary logic (classical and first-order, respectively) (Roberts, 1963). Posterior developments propose a Gamma level associated with reasoning outside classical logic, covering not only the modal domain (Zeman, 1964), but also some higher-order logics. With certain additional transformations in the syntax, the (EG) have been extended also towards intuitionistic logic (Oostra, 2010), according to the fact that a natural semantics for intuitionistic logic is given by topological spaces (Tarski, 1938).

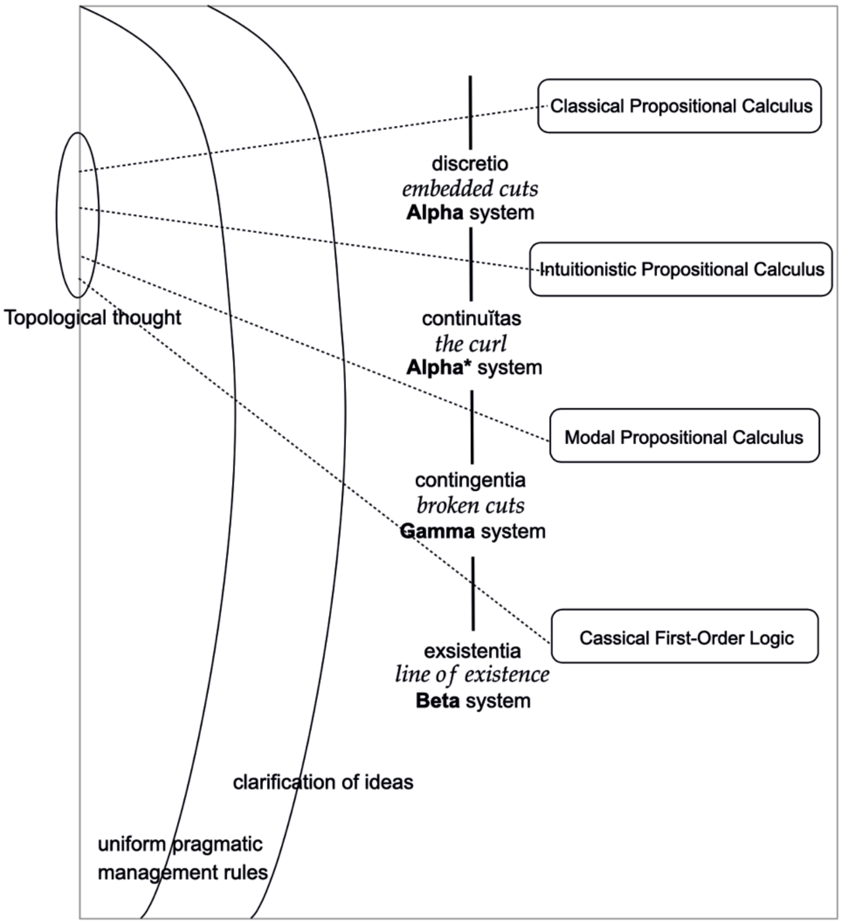


Figure 1: (EG) - archetypal roots of logical calculi

Thus, with common rules and with slight variations in the signs, a broad view of geometry-based logic is obtained. Within this (EG) system, deductive reasoning occurs through the insertion or omission of propositional letters (Pietarinen, 2021) together with closed curves (cuts) understood as negations. The deformation of the diverse elements can be asserted through a continuous background (initially, a complex plane) where the affirmations and modes of analysis are manifested under rules of writing or erasure, and dual pragmatic permissions such as iteration/deiteration or double cut, depending on the level and parity in areas associated to the cuts or enclosures present in the graphs. A precise form of reading provides an unambiguous interpretation, called *endoporeutic method* (Pietarinen, 2004), whose specificity and technical power are obtained thanks to the operation of (de)iteration.

Through topological thinking, the derived distinct Alpha, Alpha*, Beta, Gamma systems are guided by transversal conducts where the syntactic rules and their subsequent semantic clarification are involved. This unifies the diverse notions involved in the subsystems and gives them a universal significance in a logical system as a whole (EG), which is consistent and complete. The mediations are guided by an integral motive clarified in the signs involved (cut, curl, line of identity, and broken cut), following a uniform system of rules, which gives a coherent pragmatic sense to logical thought.

On the other hand, following the work of Oostra and his school in Ibagué, we obtain a description of the behavior of the inner logics associated to some nonplanar surfaces, through a technical unraveling of topological (geometric) properties, over and beyond negations as complex curves traced on the surface of each variety (Oostra, 2018). The unfolding of the formulas on two dimensions should be able to be completed, since 2-dimensionality is not reserved to the plane. For example, we may think of the sphere, the cylinder, the torus and, more generally, a Riemann Surface, over which we can elucidate new logical pathways, which would have remained invisible from the planar perspective.

Thanks to all this, a natural transit between two or more areas of mathematics is configured, and a fundamental question of dual character is generated that engages in the syntax, semantics and pragmatics of the logics associated to Existential Graphs on surfaces: How can we associate to known logics, the result of the mutation of logical systems through the surface where they develop? and, reciprocally, How can we make correspond to a given logic an adequate system in Existential Graphs over a concrete surface? (Oostra, 2022). Thus, in the space which mediates between the study of Logical Systems and the understanding of the Nature of Surfaces, emerge the many Variations of Existential Graphs.

By placing negations in complex environments, these negations capture the geometric nature of the manifold, and we will need in some cases additional restrictions or rules to maintain the sense and consistency of the new Alpha graphs. For example, to preserve the notions of parity and leveling in areas which allow (de)iteration and the use of the inference rules of the system, we will need the addition of new concepts to make their interpretations possible. In this sense, a notion of opposition allows to logically distinguish two regions separated by a curve on the surface, which in principle are topologically indistinguishable (but whose truth content is the negation of what is on the other side of the curve), since such a curve can be deformed smoothly on the surface and configure a new enclosure, as in the case of the sphere. We notice that, if we do not delimit the setting, we can arrive at contradictory deductions from any graph, a fact illustrated in the following figure:

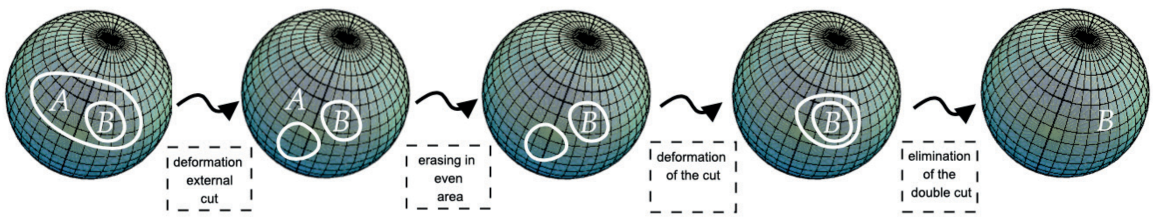


Figure 2: An example of (EG) on the sphere

Hence, by taking into consideration some necessary conditions, providing precise interpretations to certain classes of problematic curves, and adding new rules over surfaces (regarding non-contractible curves), the Alpha existential graphs can be modelled on the sphere, the cylinder, the Möbius band, or the torus. In this last case, for example, 3 different classes of curves can be drawn on its surface except isomorphism; only one of them will be contractible, i.e., the map from the unit circle to the surface, which draws a curve homotopic to a constant function; these curves are compatible with the usual (EG) transformations, generating an alternative and equivalent version of classical propositional logic, such as with Alpha existential graphs in the plane (Oostra, 2018). There are several classes of curves on each surface, depending on their geometric nature, and each of these classes corresponds to a *type of negation* on the surface, characterized in terms of their contractibility and their locality at a topological level. In particular, a contractible curve is apparently associated with the classical domain, and a non-contractible curve is associated with realms beyond classical logic. We can thus glimpse that, taking all negations (local and global, see below) associated to arbitrary curves present on surfaces locally homeomorphic to $\mathbb{C}$, we can open a way to new logics.

3 Riemann surfaces (RS)

To formalize these extensions it is useful to pass through Riemann Surfaces (RS) (Riemann, 1851), since these incarnate the settings of smoothness and continuity necessary for an expansion of our graphs. In addition, the use of (RS) will be consistent with both the classical definitions of Peirce, and the posterior non-classical bibliography. We arrive to a context that contains local environments isomorphic to the plane, but at the same time admits singularities where the surface extends through its ramification points, or poles. The geometric fullness and power of the complex variable incites then an intuition towards the possibility of potentiating logical concepts through new geometries.

Some of these geometries correspond to the archetypes of geometries discovered in the 19th century: parabolic geometry (Euclidean model), hyperbolic geometry (Poincaré's model) and elliptic geometry (Riemann's model), which come from the application of quotient spaces on the plane, the disk and the sphere respectively (Wegert, 2012). These models are collected in the description made previously by means of graphs, and thanks to the Riemann Surfaces Uniformization Theorem, we know that they correspond to an exclusive alternative, since every Riemann Surface simply connected ("without holes") arises, up to homeomorphism, from one of these three possibilities (Koebe, 1907; Poincaré, 1908).

A Riemann surface may be understood as the natural complexification of a topological surface (Zalamea, 2022), and defined through the following three steps:

I. A $2-dimensional\ topological\ manifold$ X is a Hausdorff topological space where every point has a neighborhood homeomorphic to an open set of $\mathbb{R}^2$.

II. A $complex\ structure$ on X consists of an "atlas", or family of "charts", $\{(U_i, \varphi_i): i \in I\}$, such that U_i is open in X, the atlas covers X ($\bigcup_{i \in I} U_i = X$), the maps are faithful (φ_i: $U_i \rightarrow \varphi(U_i)^{open} \subseteq \mathbb{C}$ is homeomorphism), and the transits are analytic (for each pair of maps (U_i, φ_i), (U_j, φ_j); $\varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$ is holomorphic);

III. A **Riemann Surface** (RS) is then defined as a 2-dimensional connected topological manifold X with a complex structure on it.

Since the transition maps between the charts are analytic, they must satisfy the Cauchy-Riemann conditions (Needham, 1997), and therefore their Jacobian is greater than zero. The fact that the charts are compatible forces then an orientation on the manifold: all (RS) result to be orientable (Kumaresan, 2002). Another fundamental notion of (complex variable) surfaces is the *genus*, defined as the minimum number of cuts (minus 1) that disconnect the surface (Ahlfors, 1953). The genus provides an intrinsic (geometrical-topological) invariant of the surface: it is obtained thanks to paths on the surface itself, without leaving it (Griffiths, 2014). A deep mathematical result captured through such a concept is the

Riemann-Roch Theorem, which allows to reconstruct the genus of a (RS) as an extrinsic (differential-complex) invariant of the surface: going out of the surface and comparing it in multiple ways (holomorphic and meromorphic data) with its environment (Roch, 1865).

Once the complex and differential panorama has been elucidated, through a dual characterization of logical and geometrical properties of surfaces, we may now propose a new logical distinction. The **local and global negations** on a surface (see Definition 2.1 below), will depend on the nature of the associated curves, since on arbitrary surfaces locally homeomorphic to $\mathbb{C}$, Jordan curves allow to capture forms of negation in the (EG).

Definition 2.1 (Local-Global Negations). *Given a Riemann Surface X, we will call a negation (associated to a Jordan curve C)* ***local*** *if there exists an atlas for X and a neighborhood V in the atlas such that $C \subseteq V$ and C is contractible (homotopically deformable to a point in the neighborhood). Otherwise, we will call such a negation* ***global***.

After applying this definition to some simple (RS) studied according to the models of graphs on surfaces, the following facts emerge:

a) In the plane and in the disk, the notions of locality and globality match, and every negation is both local and global.

b) In the cylinder, instead, the notions of locality and globality are separated; negations on the surface that do not surround the cylinder are local; a complete turn of the cylinder is instead a global negation, not a local one.

c) In the sphere, locality and globality are also distinguished; negations that are entirely embedded on one side of the atlas are local; on the other hand, a "maximal" negation (equatorial parallel type) is not localizable in a neighborhood, and is a global negation.

d) In the torus, of the three types of negations in scope, only one type is local, while the other two types (longitudinal and transversal cuts) are global.

With such notions of locality and globality, the paradoxical behavior of negation in the sphere can be partially explained (see Figure 2). Indeed, in the deformations of a Jordan curve on the sphere, a local negation and a global negation can be mistakenly identified: when a small Jordan curve on one side of the atlas moves to the other side of the atlas, the negation goes from local to global, and then back to local again. A logical system that restricts these steps (e.g., accepting deformations only within the local) could then help to eliminate the contradictory behavior of the negation.

We introduce now the concept of **local and global logics** (see Definition 2.2 below), to capture a variety of perspectives: classical, intuitionistic, or paraconsistent. Further, the appeararance of new logics related to the their number of non classical, alternative, negations, may capture the genus of the surface.

Definition 2.2 (Local-Global Logics). *An (EG) logic over a (RS) is* ***local*** *if all of its negations are local, and the logic is* ***global*** *if there is at least one global negation in the system.*

From the facts recently considered on the negations in the various systems, we can formulate the following conjecture:

Conjecture 1 (Oostra-Zalamea-Hugueth).

- *Every local (EG) logic over a (RS) is (super)intuitionistic.*
- *The global (EG) logic on the sphere is paraconsistent.*
- *The global (EG) logics over a (RS) of genus $n \geq 1$ are nonclassical, with exactly $3n$ associated notions of "negation".*

In the genus 0 case (plane, disk or sphere, by the Uniformization Theorem), local logics contemplate negations always reducible to a point on the surface, as in the intuitionistic case. In the genus 1 case (torus) there appear exactly 3 notions of negation, 2 of which (longitudinal cuts) cannot be reducible to combinations of local negations.

A brief sketch of this situation is also visualized in the following Figure 3. The example situates us better in the proposed environment and may perhaps serve as motivation for future work. We notice that on a two-hole toroid (genus 2 surface), there are 6 classes of negations, none of them deformable into each other, of which 5 will not be reducible to a point (since they are of non-contractible type), and therefore will not only be bound to combinations of local negations.

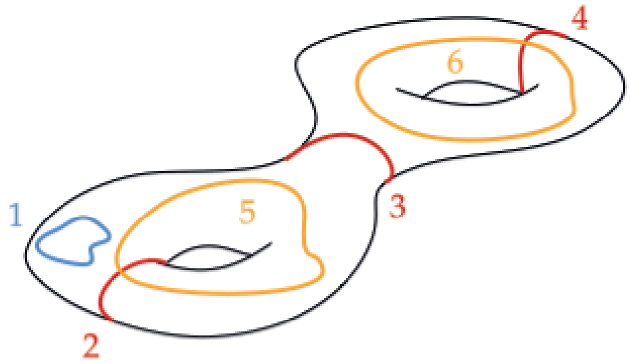


Figure 3: 6 classes of cuts over a (RS) of genus 2, the toroid with two holes

4 Toposes (T)

Analytical continuation in complex variables shows how the domains of a given analytic function are extended, defining additional values that allow to escape from possible divergences (Beltrametti, 2009). We have then a double dialectic between analytic univalued functions (in local neighborhoods) and analytic multivalued functions (in the global variety considered). This step can be solved if the gluing of the functions is independent of the paths by which they are glued, i.e., if one has an expectation of invariance beyond multiplicity, something which can be realized if the region is simply connected (Monodromy Theorem) (Zalamea, 2022). The sheaves (which are an abstract extension of analytical continuation), provide us with a ductile view to link the local and the global, traversing not only the precise foundations in topology required by our proposal, but also some needed tools of complex analysis and categories. This perspective reconstructs the understanding of a topological manifold or differential manifold on the basis of its projections on lower manifolds and the coherent information of its fibers (Morita, 2001).

It becomes then natural to transit to a Grothendieck Topos (GT) by taking all the sheaves over a site, where one obtains an abstract extension of the notion of topology, and therefore also of space (Caramello, 2018). A topos represents a strong connection between two tendencies of mathematical thought (space and number), delving into the oldest definitions of mathematics around the continuous and the discrete. A "sophisticated study of the interactions between space and number" appears, on the one hand the geometry that captures space, and on the other hand the arithmetic that captures number (Zalamea, 2024). Extending the classical situation of topology, a **Grothendieck topos** will be precisely, by definition, a category equivalent to a category of sheaves $Sh(C, J)$ over a site (C, J), where a site is a category C together with a Grothendieck topology J on it, that is, a synthetic rendering of covering properties of open sets in usual topology (the space covers itself, a covering of coverings is a covering, a pullback of a covering is a covering) (Artin, 1983). A sheaf over the site is, intuitively, a presheaf that "glues well" the overlapping sections (Mac Lane, S. and Moerdijk, I., 1992). This good gluing can be described in terms of covering properties on "locals" (complete Heyting algebras, which codify the properties of the lattice of open sets of a topology), coverings which, in turn, can be characterized by good extension properties.

Links between algebraic geometry and logic are also better captured through an Elementary Topos (ET), by considering the subobject classifier of the topos, a tool that describes, formally and precisely, the behavior of the logic inherent to the topos in question. A category $\mathcal{E}$ is an **elementary topos** if it possesses finite limits, is Cartesian closed (i.e., exponential objects exist, with good naturalness properties, or, what is the same, the product functor possesses right adjoint), and furthermore, the subobjects functor $Sub: \mathcal{E}^{op} \to Set$ is representable (or, what is the same, there exists a *subobject classifier* Ω such that $Sub(A) \approx \mathcal{E}(A, \Omega)$ is a natural isomorphism) (Lawvere, 1963). A fundamental intuition behind toposes is that they act at the intersection of geometric (Grothendieck) and logical (Lawvere) concepts.

In a topos of presheaves $\mathcal{E} = Set^{\mathcal{C}^{op}}$ the subobject classifier is forced via Yoneda: $Sub(h_A) \approx \mathcal{E}(h_A, \Omega) = Nat(h_A, \Omega) \approx \Omega(A)$. Therefore, $\Omega: \mathcal{C}^{op} \to Set$ can be defined by a correspondence (in objects) $A \mapsto Sub(h_A)$ and (in morphisms) $(A \to B) \mapsto (\Omega_f: Sub(h_A) \to Sub(h_B))$. A lemma on pullbacks (Mac Lane; Moerdijk, 1992) ensures that Ω_f is a functor, the transformations turn out to be natural, and Ω is well-defined. Since the Yoneda embedding is dense (i.e., every presheaf is a limit of representable functors, $F = lim_{\rightarrow} h_A$), the description $Sub(-) \approx Nat(-, \Omega)$ extends to all presheaves in the topos.

The internal logic of a topos is derived from certain elementary exactness properties in the topos. At the outset, everything depends on showing that, for all a in the topos, $0 \rightarrowtail a$ is monic. This follows from the following three facts (which hold in any closed Cartesian category with initial object 0): *(i)* $a \times 0 \approx 0$ (using the fact $\mathcal{E}(a \times 0, b) \approx \mathcal{E}(0, b^a)$, that is, the exponential property, it turns out that the unique emergent morphism of $a \times 0$ forces isomorphism with 0), *(ii)* if there exists $f: a \to 0$ then $a \approx 0$ (by properties of projection and composition in products), *(iii)* every arrow entering 0 forces an isomorphism (by *(ii)*), thus the arrow is monic.

Taking characteristic functions (χ) of monos (m), provided by pullbacks in the subobject classifier, from the fact that $0 \rightarrowtail 1$ is mono we can perform a natural construction of connectives on the topos. See items *(i)-(v)* below, constructed recursively over each emergent monic at each level:

- *(i)* *Falsity*: $\perp = \chi_{0 \rightarrowtail 1}$
- *(ii)* Negation: $\neg = \chi_{\perp}$
- *(iii)* Conjunction: $\wedge = \chi_{<\top,\top>}$
- *(iv)* Implication: $\Rightarrow = \chi_{eq(\wedge,\pi 1)}$
- *(v)* Disjunction: $\vee = \chi_{im[id \times \top, \top \times id]}$.

These constructions express, through morphisms, the usual constructions of connectives as operators on $\{0,1\}$. Moreover, in every topos we will always have that $Sub(\Omega)$ is a Heyting algebra, which points to an intuitionistic underlying logic. It can be proved in fact that deducibility in intuitionistic propositional calculus is equivalent to validity in every Heyting algebra, which is equivalent to validity in every topos.

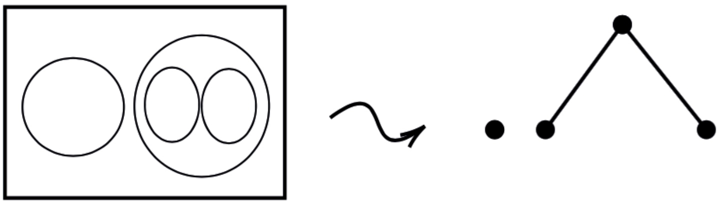
Sheaves are found at the intersection of a multitude of mathematical concepts: complex variable (coming from Riemann's analytical continuation), differential geometry (coming from Leray's work on differential equations), abstract algebra (coming from the formalizations of the French school, via Cartan, Lazard, Godement, Serre), algebraic geometry (coming from Grothendieck's work on abelian categories, Riemann-Roch, schemes, topos), logic (coming from Lawvere's first-order axiomatizations, up to Caicedo's logic of sheaves), etc. Sheaves constitute therefore a truly central and primordial concept for our current understanding of the mathematical world. In what follows in this paper, we will now think of existential graphs as sheaves, not only on the plane but also on other Riemann surfaces, and we will delve into the collection (topos) of all these various sheaves of existential graphs.

5 Combinatorial: Categorical Vision

The advantage of a categorical reading of any mathematical subject lies in the extension of the range of possibilities offered by abstractness. In an elementary topos the notions of localization, topologization and intuitionistic systematization coincide. So, aiming at a structural translation of our system, which is the central proposal of our work, we appeal to some recent developments by Gangle and collaborators, whose recent contributions allow us to take the last step towards a consistent axiomatization of our model (Gangle, 2022).

Gangle's approach provides a combinatorial reading of Alpha-slice embeddings (model of classical logic) via trees, a view of trees as presheaves, and an intertwining of static and variable marks to capture occurrences of propositional letters. It allows to obtain a category of objects that syntactically represents Alpha existential graphs. A sharp logical notation, such as (EG), organizes relations between the components of its syntax, so that the logical properties and relations under a concrete interpretation, governed under simple processing rules, are precise and unambiguous. The diagrammatic syntax offered in Peirce's system provides a natural environment for distinguishing the relations between syntactic combinations of elements in an argument. Reconstructing this diagrammatic reasoning by means of an appropriate category of functors (more precisely, a category of presheaves), we can diversify the richness of the structure lying at the Alpha level. Here, the morphisms obey hierarchies in the structure according to the origin and destination of the arrows of the directed graphs which represent nests of cuts from the (EG). So we can view each element in the class of structures as a concrete contravariant functor that collects relations between cuts, and represents them conjunctively.

Precisely (Gangle, 2022) observe that embeddings (nests) of Alpha cuts can be put in correspondence (*) with finite trees, associating a node to a cut, and a branch to an embedding (following the endoporeutic method). An example of this correspondence (*) is shown in the following figure:



If $\mathcal{C}_{(\mathcal{N},\leq)}$ denotes the category corresponding to the partially ordered set $(N, \leq)$, let $\mathcal{F} = Set^{\mathcal{C}^{op}_{(\mathcal{N},\leq)}}$ be the associated presheaf category of forests. A subcategory $\mathcal{E} \subseteq \mathcal{F}$ can then be defined, whose objects are the functors $F\colon \mathcal{C}^{op}_{(\mathcal{N},\leq)} \to FinSet$ (i.e., finitary presheaves) that "terminate in finite steps" (i.e., for which there exists n , with $Fn = \emptyset$). Thanks to the correspondence (*), $\mathcal{E}$ correctly models Alpha-slice embeddings. The morphisms of $\mathcal{E} = \mathcal{E}\mathcal{G}_{\alpha*}$ are monic morphisms between the same objects in $\mathcal{F}$.

Going further, (Gangle, 2022) reconstruct an Alpha graph as a nests of cuts with additional variables ("marks"). This addition is obtained by a new functor that records these variable occurrences. Formally, if F $\in \mathcal{E}$ is a nests of cuts, a "distinguished" Alpha graph over the "skeleton" F (our terminology) is defined as a pair $(F, \hat{F}\,)$ where $F, \hat{F} \in \mathcal{E}$ are such that $F \rightarrowtail \hat{F}$ and one has an iteration and embedding control condition that captures the occurrence of distinct propositional letters, which are replaced by empty Alpha cuts in the areas where the variables appear (= pseudographs, in Peirce's terminology). Comparing the resulting embeddings of both processes, when factored through the functor $\hat{F}$, we notice that F is canonically injected into $\hat{F}$. We can thus make correspond a graph with marks (letters or variables) to an ordered pair $(F, \hat{F})$, with F and $\hat{F}$ cut-only graphs, described as some functor of $\mathcal{C}^{op}_{(\mathcal{N},\leq)}$ on $FinSet$ such that for some natural n, its image is empty. In order to be able to represent Alpha graphs in their entirety (i.e., with repetitions of eventual letters), Gangle and collaborators introduce a monoidal action T on the distinguished graphs (the orbits then give rise to a letter identification). This finally gives rise to a category $\mathcal{EG}_\alpha$ formed by triples $(F, \hat{F}, T_{(F,\hat{F})})$, a construction which fully captures, in a categorical language, the syntax of Alpha graphs.

6 The topos of (alpha) existential graphs over Riemann surfaces (TEGRS)

The Gangle-Caterina-Tohme construction of $\mathcal{EG}_\alpha$ relies heavily on a property of "normality" (our terminology) of planar slice nests, where a leveling of areas (with well-defined notions of evenness and oddness) is recorded. This corresponds to a branching of the associated trees: leveling, not only the levels (of the cuts), but also the branches (of the trees), shows an increasing progression in the finite segments of $(\mathbb{N}, \leq)$, indicators of $FinSet^{\mathcal{C}^{op}_{(\mathcal{N},\leq)}}$. However, the situation can become entirely different in the case of (nonplanar) graphs on general (RS). An example of this is how a double cut on the sphere, represented by a tree of type 2, can be deformed into two separate cuts on the sphere, represented by a tree of type 1 + 1. In that case, the notions of progression, leveling, branching are lost. These notions are only preserved in the case of systems restricted to local negations, and in that case the logics of $\mathcal{EG}_\alpha$ (EG over the plane) and $\mathcal{EG}_{\mathcal{RS}}$ (EG over RS) coincide.

Thanks to the notions of locality/globality and linearity/nonlinearity, the transit from Existential Graphs on the Plane to Existential Graphs on Riemann Surfaces, can be understood as a complex/differential/homological transit between the local linear and the global nonlinear, which emerges naturally in differential equations, complex variable and sheaf theory. Here we record this transit, for the first time, with logical tools and by means of purely structural properties (generations and adjunctions) of the categories and immersions at stake.

To attempt to describe then, in general, an (EG) environment over a (RS), one must modify the construction $\mathcal{E}$ formed by finitary presheaves $F: \mathcal{C}^{op}_{(\mathcal{N},\leq)} \to FinSet$. The embedded records of cuts captured by the order $(N, \leq)$ may not work in a general (RS), when global negations (cuts) appear.

To allow for "alternative deformations" that break the linearity (and its consequents: progression, leveling, branching), a new binary relation R must be introduced on a set A that needs not be a linear type order. The category of (EG) over (RS) would then arise from a new construction consisting of (not necessarily finitary) presheaves $F: \mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})} \to Set$, where the algebraic properties of the relation R would capture the logical properties of (EG) over (RS).

The combinatorial construction of the category $\mathcal{EG}_\alpha$ (Gangle-Caterina-Tohme), and the conceptual view of its eventual extension $\mathcal{EG}_{\mathcal{RS}}$ to nonplanar (RS) environments (Zalamea-Hugueth), emphasize an extrinsic descriptive character, in order to syntactically capture (EG) via subfunctors of presheaves. From the point of view of the natural environment of toposes, in which these constructions are immersed, we have that:

$$\mathcal{EG}_\alpha \hookrightarrow FinSet^{\mathcal{C}^{op}_{(\mathcal{N},\leq)}} \times FinSet^{\mathcal{C}^{op}_{(\mathcal{N},\leq)}} \times Set^{T} \; (*)$$

Where the three categories on the right are toposes of presheaves. Similarly, one would have another immersion for the case of our proposal:

$$\mathcal{EG}_{\mathcal{RS}} \hookrightarrow Set^{\mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})}} \times Set^{\mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})}} \times Set^{T} \; (**)$$

Where the three categories on the right are still toposes of presheaves.

An intrinsic study of (EG), on the plane or on an arbitrary (RS), should be able to characterize $\mathcal{EG}_\alpha$ and $\mathcal{EG}_{\mathcal{RS}}$. In that sense, we postulate that the general construction corresponds to an **intermediate topos** $T(EGRS)$ associated to Existential Graphs on Riemann Surfaces (closed under limits and exponentials), in such a way that one has an injection of the logical system into the topos, described by a nonlinear universal product of categories (toposes of not necessarily finitary presheaves). These are endowed with an algebraic action T, not necessarily linear, which allows deformations necessary to preserve the leveling, progression and branching in the leveling of the graphs, unlike the Topos of Alpha graphs on the plane, which is described under the topos of the linear universal product of Finite Sets. In this general setting, we may then present the following conjecture:

Conjecture 2 (Zalamea-Hugueth).

- *There exists an intermediate topos* $T(EGRS)$ *associated with the category* $\mathcal{EG}_{\mathcal{RS}}$*, such that one has* $\mathcal{EG}_{\mathcal{RS}} \hookrightarrow T(EGRS) \hookrightarrow Set^{\mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})}} \times Set^{\mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})}} \times Set^{T}$*, and it "characterizes" in a sense the category of Existential Graphs on Riemann Surfaces. The topos could be the "generated topos" by* $\mathcal{EG}_{\mathcal{RS}}$ *(closure under limits and exponentials) within the "nonlinear universal" product topos* $Set^{\mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})}} \times Set^{\mathcal{C}^{op}_{(\mathcal{A},\mathcal{R})}} \times Set^{T}$.
- *The* $T(EGRS)$ ***classifier****, to be described via products and Yoneda (presheaves, and ideal subfunctors of a monoid), encodes the logic of (EG) over (RS).*
- *Local and global logics over a (RS) (Section 2) can be characterized via exactness properties on the topos* $T(EGRS)$.
- *In the case of graphs over the plane, one would obtain a topos* $T(EG_{\alpha})$ *generated by* $\mathcal{EG}_{\alpha}$ *inside the "linear universal" topos product* $FinSet^{\mathcal{C}^{op}_{(\mathcal{N},\leq)}} \times FinSet^{\mathcal{C}^{op}_{(\mathcal{N},\leq)}} \times Set^{T}$.

Exploring the situation further, it is to be hoped that the local and global logics over (RS) may be characterized through good properties of the associated **fundamental group** (FG) emerging in the topos (Artin, 1983). In this way, relations between the genus of (RS), the homological/homotopical properties of the fundamental group (FG) (Grothendieck, 1968), the internal logical properties of the classifier (T), and the local/global characteristics of the external logics involved (EG), would provide new and profound bridges between many central areas of mathematics (logic, algebra, complex geometry, topology, categories), offering fertile ground for future research.

References

AHLFORS, L. *Complex Analysis.* New York: McGraw-Hill, 1953.

ARTIN, M. A. *Théorie des Topos et Cohomologie Etale des Schemas.* Seminaire de Geometrie Algebrique du Bois-Marie 1963-1964 (SGA 4): Tome 1. Springer Berlin Heidelberg, 1983.

BELTRAMETTI, M. *Lectures on Curves, Surfaces and Projective Varieties:* A Classical View of Algebraic Geometry. European Mathematical Society, 2009.

CARAMELLO, O. *Theories, Sites, Toposes:* Relating and Studying Mathematical Theories Through Topos--theoretic 'bridges'. Oxford University Press, 2018.

GANGLE, G. C. A Generic Figures Reconstruction of Peirce's Existential Graphs (Alpha). *Erkenntnis,* v. 87, p. 623-656, 2022. https://doi.org/10.1007/s10670-019-00211-5.

GRIFFITHS, P. A. *Principles of Algebraic Geometry.* Wiley Classics Library, 2014.

GROTHENDIECK, A. *Classes de faisceaux et théorème de Riemann-Roch.* Institut des hautes études scientifiques, 1968.

KOEBE, P. Über die Uniformisierung der algebraischen Kurven. *Göttinger Nachrichten*, p. 191-210, 1907.

KUMARESAN, S. *A Course in Differential Geometry and Lie Groups.* New Delhi: Hindustan Book Agency, 2002.

LAWVERE, F. W. *Functorial Semantics of Algebraic Theories.* Phd Thesis, Columbia University, New York, 1963.

MAC LANE, S.; MOERDIJK, I. *Sheaves in Geometry and Logic.* A First Introduction to Topos Theory. New York: Springer-Verlag, 1992.

MORITA, S. *Geometry of Differential Forms.* American Mathematical Society, 2001.

NEEDHAM, T. *Visual Complex Analysis.* London: Clarendon Press, 1997.

OOSTRA, A. Los gráficos alfa de Peirce aplicados a la lógica intuicionista. *Cuadernos de Sistemática Peirceana,* v. *2*, p. 25-60, 2010.

OOSTRA, A. *Notas de Lógica Matemática.* Ibagué, Tolima: Universidad del Tolima, 2018.

OOSTRA, A. Intuitionistic and Geometrical Extensions of Peirce's Existential Graphs. *In*: ZALAMEA, F. (Ed.). *Advances in Peirce Mathematics.* The Colombian School. Berlin: De Gruyter, 2022.

PEIRCE, C. S. *Syllabus*: Syllabus of a course of Lectures at the Lowell Institute beginning 1903, Nov. 23. On Some Topics of Logic. MS [R] 478, 1903.

PIETARINEN, A.-V. The Endoporeutic Method. *In*: PIETARINEN, A. -V.; QUEIROZ, M. B. (Eds.). *The Commens Encyclopedia:* The Digital Encyclopedia of Peirce Studies. Commens, 2004.

PIETARINEN, A.-V. *2 The 1903 Lowell Lectures* (Vol. 2). Berlin, Boston: De Gruyter, 2021.

POINCARÉ, H. Sur l'uniformisation des fonctions analytiques. *Acta Mathematica,* v. 31, p. 1-63, 1908. https://doi.org/10.1007/BF02415442.

RIEMANN, B. *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen komplexen Größe.* Phd thesis, Universität Göttingen, Göttingen, 1851.

ROBERTS, D. D. *The Existential Graphs of Charles S. Peirce.* Phd Thesis, University of Illinois, Urbana--Champaign, 1963.

ROBERTS, D. D. The Existential Graphs. *Computers Math. Applic*., v. *23*, n. 6-9, p. 639-663, 1992. https://doi.org/10.1016/0898-1221(92)90127-4.

ROCH, G. Ueber die Anzahl der willkürlichen Constanten in algebraischen Functionen. *Journal für die reine und angewandte Mathematik,* v. 64, p. 372-376, 1865. https://doi.org/10.1515/9783112368527-027.

TARSKI, A. Der Aussagenkalkul und die Topologie. *Fundamenta Mathematicae*, p. 103-134, 1938. https://doi.org/10.4064/fm-31-1-103-134.

WEGERT, E. *Visual Complex Functions:* An Introduction with Phase Portraits. Springer Basel, 2012.

ZALAMEA, F. *Modelos en haces para el pensamiento matemático.* Bogotá: Editorial Universidad Nacional de Colombia, 2022.

ZALAMEA, F. Grothendieck: A Short Guide to His Mathematical and Philosophical Work (1949–1991). *In*: SRIRAMAN, B (Ed.). *Handbook of the History and Philosophy of Mathematical Practice*. Springer International Publishing, 2024. p. 1257–1296.

ZEMAN, J. J. *The Graphical Logic of C. S. Peirce.* Phd Thesis. Chicago: University of Chicago, 1964.

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