Learning Mathematics, Doing Mathematics: A Learner Centered Teaching Model

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Abstract
Learning mathematics, Doing Mathematics is a learner-centered teaching proposal under development since 1998 at the Colegio de Ciencias y Humanidades (CCH), a high school system dependant on the National Autonomous University of Mexico. It looks for the creation of a Teaching and Learning Environment (TLE) in which students work and learn collaboratively, with tolerance, respect and responsibility. The main features of this teaching proposal and its theoretical and philosophical foundations are presented.

Resumen
Aprender Matemática, Haciendo Matemática es una propuesta de enseñanza en desarrollo desde 1998 en el Colegio de Ciencias y Humanidades, un subsistema de bachillerato dependiente de la Universidad Nacional Autónoma de México. La propuesta busca la creación de un Medio Ambiente de Enseñanza-aprendizaje (MAE) en el cual los estudiantes trabajen de manera colaborativa, con tolerancia, respeto y responsabilidad. En el artículo se presentan las principales características del este modelo de enseñanza y sus fundamentos teóricos y filosóficos.

Introduction

In the last few years, in many countries, there have been changes in the Mathematics curriculum aimed at a more effective teaching. As a consequence, most of the mathematics curricula are focused on the development of student knowledge and competencies. The trend in these curricula is to put the focus of attention in the student in what is called a learner-centered perspective.

The learner-centered approach has been under development since the decade of 1950, first as a person-centered education, and then as learner-centered education in the 1990s. Person-centered education emphasizes teacher empathy, unconditional positive regard, genuineness, non-directivity, and encouragement of critical thinking. While learner-centered education couples a focus on individual learning with a focus on learning. It is rooted in four domains: metacognitive and cognitive; affective and motivational; developmental and social; and individual differences factors (Cornelius-White, 2007, pp. 113-115).

In a Mathematics learner-centered teaching model, learners are who learn mathematics by doing mathematics; they are an active part in the acquisition of their own mathematical knowledge. The model is based on a pragmatic basis of "hands on" –

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something very similar to the process of learning a trade: a shoemaker apprentice learns working in a shoemaker's workshop not in a classroom. Thus, students are like mathematician apprentices that learn applying their previous knowledge in a sort of mathematics workshop. Learning arises in a social context of collaboration and harmony, and the teacher is a guide that leads the process, designs the teaching activities and monitors the whole process.

This paper has the aim to characterize the Learner-centered Teaching Model called, *Teaching Mathematics, Doing Mathematics* (2006, 2007a) and the theoretical principles on which the model is based.

1. The Teaching Model

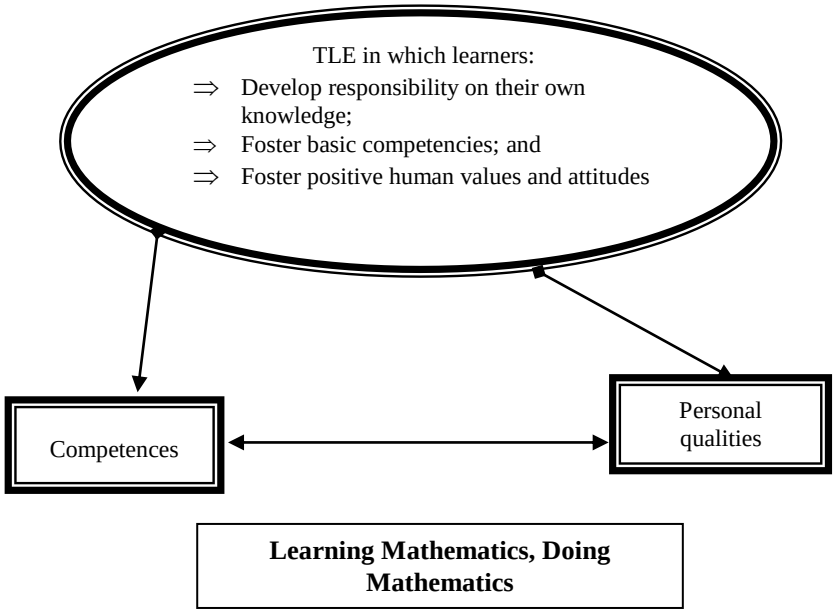
Learning Mathematics, Doing Mathematics is a teaching proposal in which students are the main characters on the learning process. It arose as a need to improve the teaching of Mathematics taking care of the educational foundations of the CCH –created in 1971. All the educational goals of CCH rest on three principles: Learn to Learn, Learn to Do, and Learn to Be (CCH, 1996). It is aimed to foster in students a Basic Culture, defined as the basic knowledge and attitude that allow the acquisition of further knowledge and new attitudes toward it and its applications.

We say that a person with a Basic Culture in Mathematics is the one that possesses:

- A mathematical thinking that allows to distinguish patterns and generalize, justify results with mathematical arguments, and use several representations of a same mathematical object.
- Problem solving skills that allow him/her to pose and solve problems inside and outside a mathematical context.
- Technology competence that allows him/her to use the technology at hand in order to facilitate problem solving and the acquisition of knowledge.
- Positive attitudes toward mathematical tasks that allow him/her to pose and solve problems as his/her responsibility that will abound in his/her own benefit and the benefit of others.
- Human values that allow him/her to have a better coexistence with other individuals and with the surrounding environment.

Thus, the achievement of these five issues is sought through the acquisition of Competences and Personal Qualities. The three first ones, Mathematical Thinking, Problem Solving and Technology, constitute the Competence aspect of the teaching model; and the two remaining ones, Positive Attitudes toward Mathematics and Human Values, are the Personal Quality aspect of the model.

Thus, in *Learning Mathematics, Doing Mathematics* it is proposed a teaching-learning environment (TLE) that foster knowledge and allow the acquisition of a Basic Culture. TLE is shaped by all things that influence the learning process in a classroom: from furniture and its arrangement to learning materials as textbooks and the kind of activities students do in a classroom, and attitudes of teacher toward students, students toward teacher and among students.



2. Competences

Classroom activities aimed to foster learner's competences are classified in three types:

Exploration. In this kind of tasks, learners explore a specific situation in order to answer a question o solve a problem. During the exploration there could arise some conjectures that is necessary to validate. With this kind of activities it is possible to foster the ability to generalize and develop a deductive thinking, at the same time that learners acquire new knowledge on how to solve problems.

As an example we have the following *Sketchpad* exploration activity:

In a equilateral triangle find a point such that the sum of the distance from this point to each of the triangle sides is minimum. Before you start your construction and explorations, try to figure out where this point could be. Explain your findings.

The main guess when students face this activity is that the point should be at the center of the triangle. They are encouraged to advance some explanation of this fact. After some exploration and the use of the drag function of the software, they realize that the point could be anywhere inside the triangle. They have, then, the task of explain this new fact, using the knowledge they already have or seeking new one.

Exploration activities are often used in Euclidean geometry topics, but it is possible to extend them to almost any mathematical topic in the curriculum.

Modeling. In *Learning Mathematics, Doing Mathematics* modeling is defined as the process of finding a mathematical model that best reproduces the data obtained in the study of a phenomenon or situation from any field of knowledge or daily life. Mainly, a model could be a mathematical function, but we could also have geometrical, numerical or other kind of models.

We define two kinds of modeling activities:

- Think and Act Situations in which learners should figure out how to obtain the data needed to build the model. Here it is an example:

  In a parking lot drivers are charged $3.00 for the three first hours and then $4.00 for any additional 15 minutes. Find a mathematical model that best represents this situation. How much do you have to pay in this parking lot if you park your car for 4:30 hours? How long could you park your car in this parking lot if you have only $15.00?

  Here, it is easy for the student to calculate the figures that answer the questions. The relevant discussion arises when they try to find the model. Our experience tells us that this kind of problems facilitates the understanding of mathematical model concept and that of function.
- Curve Fitting Situations in which learners start with a set of data taken from some given situation and must find a model that best approximate the data values. The idea here is that learners plot the data and the curve that best fit those data. The teacher can give learners the data or learners could take

measurements from some experimental setting. See, for instance, the following activity:

The data on the table belong to the weight of female turkeys and food shares depending on their age in weeks in a turkey farm at Santa Catarina, Brasil (Adapted from Biembengut y Heins, 2000)

Age (weeks)	***Weight (grams)***	***Share (grams)***	***Age (weeks)***	***Weight (grams)***	***Share (grams)***
1	*107*	*104*	*10*	*4194*	*1568*
2	*222*	*230*	*11*	*4870*	*1710*
3	*423*	*340*	*12*	*5519*	*1957*
4	*665*	*470*	*13*	*6141*	*1969*
5	*971*	*700*	*14*	*6732*	*2093*
6	*1466*	*922*	*15*	*7290*	*2115*
7	*2079*	*1146*	*16*	*7813*	*2165*
8	*2745*	*1270*	*17*	*8299*	*2160*
9	*3495*	*1396*	*18*	*8744*	*2180*

From these data, find the right time to sell the turkeys. Explain your answer. Plot the weight and age data and find the equation of the curve that best fit the data points. With this mathematical model can you reasonably estimate the weight of a 12.5 weeks turkey, 20 weeks turkey? Explain.

The first try is to fit a line, but the linear model in far from suitable. So they try a 2^{nd} grade function. In order to find a suitable model, students put into play some of the most used mathematical concepts, namely, systems of linear equations. Besides, the activity could be used in higher courses with matrices and further fitting a logistic function.

With modeling activities we have a twofold goal: for learners is to find a mathematical model; and for the teacher is to make their students put into action the mathematics they already know in order to find the model, and learn new mathematics.

Non-routine problems. This kind of problems should be solved by non conventional ways and there is no a fixed method to solve them. The following is an example of a non-routine problem (NCTM, 1991):

There are 13 houses at one side of a road. Where should you locate a bus stop so the sum of the distances from each house to the bus stop is a minimum.

Here the students could try some problem solving heuristics like studying an easier situation, for instance, with two or three houses, and then adding more houses in order to find a pattern.

With this kind of problems it is possible to explore, have conjecture validation situations, and foster a mathematical thinking and problem solving skills.

For all the teaching activities, it is possible to increase its learning potential if we use technological tools as information and communication technologies and dynamic software. In this respect it is fruitful the use of *The Geometer's Sketchpad* (KCP, 2003), because of its versatility in the study of almost any topic in Mathematics. We propose as well the use of other technology as CAS calculators.

3. Personal Qualities

Personal qualities are positive attitudes towards mathematical tasks, and human values that allow a collaborative and harmonic coexistence in the classroom. In order to construct a TLE that fosters the development of personal qualities in learners and teachers, it is necessary that teachers acquire a real leadership in the class. Leadership based on teacher's work and knowledge, and a humanitarian relationship with learners; a leadership that has nothing to do with the authoritarian attitude that most mathematics teachers adopt in a traditional TLE.

In consequence, the proposed TLE is the proper place where learners will learn mathematics, doing mathematics. It is the right place for discussions and exchange of ideas and opinions; the open place where a learner will find the help of their classmates and teacher in order to learn what it is difficult for him or her, and where he or she could help if it is the case. It should be the place where learners do what they should do with self confidence; the place where learners acquire self esteem and learn to accept other learners as they are.

In *Learning Mathematics, Doing Mathematics*, it is proposed a TLE which fosters, mainly, three basic human values that permit learners coexistence and knowledge development (2008).

Cooperation. This is the action of working in harmony with other persons to achieve the same goals. The permanent proposal of cooperation is a mutual benefit in human relations; it is grounded on mutual respect, tolerance and responsibility. Regarding learners, it implies the ability of putting aside their own ideas and proposals, when it is necessary, in order to achieve common goals.

Tolerance. Is the skill of taking into account and accept other people ideas. It leads to a peaceful coexistence. When learners reckon and accept individuality and diversity of others, it is possible that they get rid of disagreements and relieve tensions. It offers the chance of discover and remove prejudices in regard of gender, ideology, religion, or race. It implies knowing each other and a respect for others' ideas and situations.

Respect. It is the recognition of rights and dignity of other individuals. It implies an attitude of no offense and no violation of such rights. It is possible to have a respectful attitude towards individuals, society and nature. This respect must begin with one self, with the recognition of our own individuality that grows and develops among other individualities. In a classroom, respect allows a learner to express freely, with confidence and without fears. This will raise his or her self-esteem.

If we have a TLE in which learners work cooperatively, with tolerance and respect it is very likely to foster a responsibility for their own mathematical work and attitudes, and those of the others.

One of the first requisites for the development of the proposed TLE is that teachers hold these values and act in consequence.

4. Assessment

Finally, a central part of *Learning Mathematics, Doing Mathematics* is the assessment of the processes that take place in the TLE. Assessment is useful to feedback the whole teaching and learning process in ways that improve learners and teachers performance.

Assessing is not synonymous of grading. We take grading as the process of quantification what students have learned on a give scale with the exclusive purpose of determining if those students deserve to pass the course or not.

In many mathematics classrooms assessment and grading are synonyms. And the main grading/assessment instrument is a questionnaire (backed up by home assignments). The point here is that the information got in this way is not representative of the

knowledge acquired nor gives clues about student attitudes. At most, it gives clues about student performance in answering questionnaires under stress conditions within a short period of time.

Assessing must be understood as the process of recovering appropriate and relevant information about the TLE that gives the teacher the needed feedback to determine if the course is achieving its goals with the class. Assessment must give the student the necessary information about the quality of his performance and the extent of his advance in the study of the matter. Finally, assessment must be a sign on the effectiveness of curricula, about its weakness and strengths; it should be a guide for the changes and reforms on the curricula.

In Learning Math, Doing Math, we begin from the idea that the ultimate goal in teaching is that learners learn. So, assessment must contribute to that learning, giving evidence of how good is teacher performance, and how much students advance in acquiring the knowledge and the attitudes intended in the curriculum and the teaching model.

Teacher performance has to do with the TLE he allows inside the classroom. In case the TLE is not the appropriate for the learning of his students, he must take the necessary measures to achieve it. So teacher must monitor the social interactions among learners and assure that their relationship between each other is positive.

In order to asses if teacher activities are suitable and students learning is advancing, it is proposed that in some activities, called Assessment Activities, teacher does not take part in guiding students actions or giving them clues to the solution. They must solve the problems or go through the activities by their own. At the end of the activity they should deliver a writing report with their findings and results. Sometimes these reports must be written in couples and sometimes individually. It is also useful ask them a written reflection about their own performance, what do they learned, the quality of the activities and the attitudes of teacher.

There should be clear for learners that all these activities have the exclusive goal of improve teacher performance, their own performance, and class planning on an everyday basis.

Now, in Learning Math, Doing Math, the responsibility to get a good grade must be, on a great extent, on the student. He must convince his teacher that he had learned the

intended knowledge. In order to do that, he must prepare a portfolio with a sample of solved problems or activities on each topic he feels reflect better this learning. And must be prepared to orally defend his claim if he is asked to do it.

The information gathered in teaching activities is processed through some assessment instruments such as rubrics, checking lists, results matrices, Gowin's V, and classroom journals. For more details about the assessment aspect of the model see (2009).

5. Theoretical Background

The promotion of a Basic Culture in students depends on the TLE built in the classroom. The main features of a *Learning mathematics, Doing Mathematics* TLE are the following:

- Learners work in couples but in no circumstances free communication among couples is forbidden, on the contrary, communication is encouraged during the whole process.
- Learners are responsible for the acquisition of the knowledge their teacher, and the school institution, want them to acquire.
- Assessment activities should take into account the explicit and non explicit knowledge of learners.
- Inside the classroom there is tolerance, respect and cooperation.
- Most teaching and assessment activities are mathematical modeling activities, in order to contextualize the Mathematics learners are learning.
- Most activities use technology.
- Teachers are guides on the acquisition of knowledge; they should have the appropriate knowledge to assess and validate their students learning.

These features are based on the nature of knowledge as a socio-cultural production. We assert that knowledge is situated and depends on the context in which it is acquired (Vigotsky, 1978; Kuhn, 1970; Kitcher, 1984; 2007b), and is dependant on the actions learners take in some context, that is, it is mediated by action as stated by Dewey (1989). According to Dewey, intellectual organization has its origins and part of its development when humans organized the necessary actions aimed to the attainment of a goal. That is, human nature (and therefore knowledge) is expressed through an

intentional, reflective and meaningful activity. Such activity is situated in dynamic historical and cultural contexts that model and limit it. The action that characterizes the human existence is marked by an interaction between the ways in which history and culture model people, even while people are making that history and culture.

The previous paragraph gives us the basis for the construction of a TLE as the one proposed in *Learning Mathematics, Doing Mathematics*. The whole class should be like a community that looks for the same mathematical knowledge. In doing do, they work collaboratively in a environment of harmony and tolerance.

Now, mathematical knowledge has been understood as a collection of validated propositions (a set of theorems along with its proof, based on a set of axioms and postulates and definitions). But not all the knowledge can be made explicit in this way; *know-how*, and *tacit* knowledge are relevant in all areas of human thinking, including Mathematics. Know-how uses human understanding, activity and experience to make mathematics and validate it. Thus human knowledge, particularly math knowledge, has two aspects: a propositional aspect that consists of the knowledge that an individual says he knows and that can be justified or validated through community accepted methods (in math these methods are mathematical proofs); and a tacit aspect whose validation is on the individual *know-how* and his/her attitude toward problem situations (Ernest, 1999).

In our model we assess both types of knowledge with alternative assessment tools, other than tests and homework.

Knowledge is also mediated by the use of tools, physical tools or artifacts, and psychological tools or signs (Vigotsky, 1978; Wertsch, 1985, 1993; Noss and Hoyles, 1996). According to this approach, technological tools are a crucial element in the relationship between theory and practice. Tools have a function aimed to do an action, and other aimed to control that action; the first function has an external orientation (the use of the tool as artifact), and the second one has an internal orientation (the use of the tool to attain meaning).

Artifacts and signs are product of human creation and part of its cultural inheritance. Physical tools have the purpose of dominate nature, and psychological tools have the purpose of dominate oneself. Both functions are intimately linked together because alteration of nature by man changes the nature of man. A physical tool can, through a

process of internalization, become a psychological tool. Psychological tools will produce new meanings. Thus, in Mathematics Education we must to take into account the use of tools and signs as mediators of knowledge.

The mathematical knowledge that an individual has must be accepted by a *competence judge*. In the case of a student mathematical knowledge this judge can be her teacher or her classmates. Consequently, validation knowledge, tacit or not, depends on the ability of the individual to convince other individuals that he or she has that knowledge, and on the ability of the other people to understand what the individual wants to justify. That is, validation of knowledge is a social action (Ernest, 1999).

In order to validate students knowledge, it is necessary to take into account not only their propositional knowledge, but their tacit knowledge. Thus, it is necessary to assess their performance in mathematical tasks and activities, the ways in which they solve problems, and how they justify its solution, how they communicates their findings in oral and writing forms, and how they uses the technology at hand. All these knowledge assessment tasks are the ones a teacher should do in his/her daily teaching activities. According to Bloom's taxonomy (Bloom, 1984) assessment is at the highest level that a human been could attain, above abilities such as analysis and synthesis. Assessment is the ability of doing critical considerations of knowledge productions both own and others productions.

Consequently, a mathematics teacher, in order to assess the learning of his/her students and become *competence judge*, should have assessment abilities. Thus, in *Learning Mathematics, Doing Mathematics*, is important that teachers have the knowledge about Mathematics and about how to best teach it.

According to Brousseau (1997), in a classroom, teacher transfers or delegates to students the responsibility to solve the teaching activity he gives them, and students take this responsibility as theirs, because they know that in doing them, they are going to learn the mathematics they need to learn. In solving a given problematic situation or doing an activity, students put into play three patterns. An action pattern that arises when the students know what to do to solve the problem. A communication pattern that arises when one student knows what to do and communicates his solution path to others. And a validation pattern that comes out when two or more students have different solution paths to the same situation and they try to convince each other of the validity of their proposals.

Final Considerations

Learning Mathematics, Doing Mathematics is an attempt to improve teacher's practice taking into account the findings of some educational researches and considering some philosophical and epistemological issues about the social character of learning.

The teaching model has been put into practice in high school math classes (mainly in CCH) and in some education master degree programs with great acceptance. Assessment activities and information gathered in class journals during the courses show evidences that the model has some degree of success, mainly regarding the raising of learner's self esteem and motivation.

Because of these evidences of the acceptance and success of the teaching model, is why we propose further research regarding some specific aspects of *Learning Mathematics, Doing Mathematics*. Some of the research lines that arise here are: the effect of assessment feedback on student and teacher performance, factors that promote a positive change of attitude towards mathematical knowledge and towards classroom coexistence.

Finally, we believe that the foundations of this teaching model could be easily transfer to the teaching of mathematics in other levels as Secondary and Primary Schools and Higher Education, as well as in programs for teachers training.

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