Numerical representations and technologies: possibilities from a configuration formed by teachers-with-GeoGebra

Representações numéricas e tecnologias: possibilidades a partir de uma configuração formada por professores-com-GeoGebra

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Abstract

This paper reports a qualitative research whose subjects were a group of public basic education teachers who took part in a workshop approaching, as its main subjects, primality of positive integers and the Fundamental Theorem of Arithmetic, relevant topics in the number theory, dealt with from different technological perspectives and analysed under a theoretical proposal connected to the concepts of transparency and opacity of numerical representations and to the construct humans-with-media. The session where interactions occurred took place in a computing lab, in the framework of the Study Program for Post Graduates in Mathematics Education at PUC university, in São Paulo, and it consisted of two activities: during the first one, participants had to tell, from a specific representation, if a given positive integer would be prime or not; during the second one, teachers used an application from GeoGebra to determine, through different representations, which numbers in a random list would be prime. The analysis showed that participants had difficulties in the mobilization of knowledge related to the FTA, which led them to adopt strategies with a high cognitive cost and make mistakes. Likewise, data showed that similar hindrances were overcome on the basis of the educational proposal planned from a configuration of human-beings-with-GeoGebra.

Keywords: mathematics education; fundamental theorem of arithmetic; numerical representations; humans-with-media; digital technologies; GeoGebra.

Resumo

Este artigo relata uma investigação qualitativa que teve como sujeitos um grupo de professores da educação básica pública, participantes de uma oficina cujos temas principais foram a primalidade de inteiros positivos e o teorema fundamental da aritmética, tópicos relevantes da teoria dos números, tratados sob diferentes perspectivas tecnológicas e analisados sob uma proposta teórica ligada aos conceitos de transparência e opacidade das representações numéricas e ao constructo seres-humanos-com-mídias. A sessão na qual aconteceram as interações foi realizada em um laboratório de informática no âmbito do Programa de Estudos Pós-Graduados em Educação Matemática da PUC/SP e foi composta por duas atividades: na primeira delas, os participantes deveriam indicar, a partir de uma representação específica, se determinado número inteiro positivo seria primo ou não; na segunda, os professores utilizaram uma aplicação do GeoGebra para determinar, por meio de diferentes representações, quais números de uma relação aleatória seriam primos. As análises indicaram que os participantes apresentaram dificuldades na mobilização do conhecimento relativo ao TFA, o que os levou a adotar estratégias de alto custo

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cognitivo e a cometer erros; da mesma forma, os dados indicaram que semelhantes percalços foram superados a partir da proposta didática planejada a partir de uma configuração de seres-humanos-com-GeoGebra.

Palavras-chave: Educação Matemática; Teorema Fundamental da Aritmética; representações numéricas; seres-humanos-com-mídias; tecnologias digitais; GeoGebra.

Introduction

To recognize a positive integer as a prime or a composite number may seem a simple task, not very interesting in terms of teaching or researching in the field of Mathematics Education. Once stated that a natural number is prime if, and only if, it can be divided uniquely by itself and by one, we might think that there are no associated complexities. Ultimately, it just seems to be a question of finding a divisor between 2 and the square root of the number tested for primality; or of applying the so-called 'divisibility rules'. Such a search, however, can have quite a high operational cost. For example, if the candidate number is 30847, we must consider that its first divisor is 109 and that there is only another one apart from the number tested for primality itself and number 1, which is 283.

In other words, concepts related to primality are usually considered trivial; so the questions arising from these concepts are also considered non-important, as they are apparently devoid of complexity (RESENDE, 2007). However, this is not what we can see in the papers of Zazkis and Campbell (1996), Zazkis and Gadowsky (2001), Zazkis and Liljedahl (2004), Oliveira and Fonseca (2015), for example. Such researches show that teachers that are being trained at the beginning of their studies as well as in further training, do not always know in depth about subjects that are essential to the mathematical understanding of the Number Theory, which is expected in basic education, as, for example, the Fundamental Theorem of Arithmetic (FTA)[2] and its didactic consequences.

This article, based on the abovementioned researches, describes a research whose participants were a group of basic education teachers from public schools, involved in the projects "Mathematics Teaching and Learning Processes in Technological Environments PEA-MAT/DIMAT"[3], constituted through a partnership between the universities PUC/SP and

[2] *Any integer n, n > 1, either is prime or it can be decomposed as a product of prime numbers; this decomposition is unique, apart from the order.*

[3] This project received financial support from CNPq and FAPESP and it is connected to research group PEA-MAT (PUC/SP).

PUC-PERU, and "Technologies and Mathematical Education: researches on fluency in devices, tools, artefacts and interfaces"[4], also linked to the PUC/SP.

In the proposed activities, teachers had to identify whether several numbers were prime, in situations where divisibility rules, for example, were an inefficient strategy and where FTA knowledge would be relevant. Through the chosen research instruments – a question about the subject 'primality of natural numbers' (interface: pencil-and-paper) and an application built by using the software GeoGebra (digital interface) – we sought to highlight the strategies used by the subjects to solve the problems, their ideas about the representation of natural numbers and the influence of the type of technology on the mobilization of the mathematical knowledge at issue.

In this respect, arguments related to the number representation arise, as well as arguments related to the use of technology in Mathematics Education, which leads to the following theoretical treatment.

1. Numerical representations

One of the core concepts addressed in this research refers to *transparency* and *opacity* of numerical representations. In this respect, the study of Zazkis and Liljedahl (2004) mentions the role of representations in the field of natural numbers. In their paper, the authors discussed data obtained from a research also conducted with basic education teachers in training, which focused on their understanding of prime numbers, so as to detect the factors that interfere with such understanding. The argument used in the analysis of collected data is that the lack of transparency of prime numbers representation is a hindrance to understand them.

This idea was taken from the paper of Lesh, Behr and Post (1987). When making reference to different representations of rational numbers, the authors show that they "incorporate" mathematical structures, meaning that they represent them materially. Thus, representational systems can be regarded as opaque or transparent. In this respect, according to the authors, a transparent representation has no more nor less meaning than the ideas or structures it represents, while an opaque representation emphasizes some aspects of the ideas or structures it represents while hiding some other ones. When having different representational possibilities, a didactic strategy should, for example,

[4] This project received financial support from CNPq and it is connected to research group PEA-MAT (PUC/SP).

capitalise on the strengths of a specific representational system and minimize its weaknesses. Such factors would be, according to the authors, extremely important to the acquisition and use of mathematical ideas.

From Lesh, Behr and Post's proposal (1987), Zazkis and Gadowsky (2001) introduce the notion of relative transparency and opaqueness, focusing on numerical representations. The authors suggest, on their paper, "that all representations of numbers are opaque in the sense that they always hide some of the features of a number, although they might reveal other, with respect to which they would be 'transparent'" (ZAZKIS; SIROTIC, 2004, p. 498). As an example, the authors provide a list with the following items:

(a) 216^2, (b) 36^3, (c) 3 x 15552, (d) 5 x 7 x 31 x 43 + 1, (e) 12 x 3000 + 12 x 888. The authors mention that such expressions do not seem to represent the same number, 46656, and they quote Mason (1998 *apud* ZAZKIS; GADOWSKY, 2001, p. 45) by stating that "each representation shifts our attention to different properties of the number".

With regard to representational transparency and opacity, the authors state that representation (a) is transparent in relation to the fact that the number in question is a perfect square, while representation (b) is transparent in relation to the fact that 46656 is a perfect cube. Representation (c), in its turn, shows with transparency that the number is a multiple of 3 and of 15552. In other words, although (a) and (b) also allow to reach the conclusion that 46656 is a multiple of 3, they do not reveal anything about 15552; that is to say, representations (a) and (b) are opaque with respect to the fact that 46656 is a multiple of 15552.

Another relevant example about the unvarying opacity of numerical representations refers to the extension of the concept brought about by Zazkis and Liljedahl (2004), from specific numbers to sets of numbers that share the same feature, verifiable through an algebraic notation. For example, *17k* is a transparent representation for a multiple of 17, in the sense that this feature is built-in (or "can be seen") in this form of representation; however, it is not possible to determine whether *17k* is a multiple of 3 only with this representation. In this case, it is stated that such representation is opaque with respect to divisibility by 3 (ZAZKIS, 2002).

So we can conclude that the paper by Zazkis and Liljedahl (2004) is an important framework in the concept of transparent and opaque representations used in this research. Based on this assumption for numeric representation, the authors claim that all representations of this nature are opaque, but they have transparent features: "for example, representing the number 784 as 28^2 emphasizes that this is a perfect square but de-emphasizes the divisibility of this number by 98. That is, in representing 784 as 28^2, the property of 784 being a perfect square is transparent and property of 784 being divisible by 98 is opaque" (ZAZKIS; LILJEDAHL, 2004, p. 166).

In relation to the work we describe here, activities involving the features of numerical representations were presented to the subjects by using different means and technological interfaces. It is precisely in this respect that it seems relevant to align arguments about the use of technologies in the field of didactic proposals for Mathematics Education, which is done as follows.

2. About the use of technologies in Mathematics Education

Technologies have generally been present in education processes from time immemorial, if we consider, like Lévy (1993), that transmutation of temporalities in Human History brought about different instruments; some of them prevailed over the others depending on the evolutionary character of the society in a given time. Thus, orality, writing and information technology are enrolled as technologies of intelligence. The ascendancy of information technology did not suppress previous technological proposals, but constituted, in relation to them, a feature of redefining functions and, in the last resort, of convergence. Thus, it is unavoidable to associate the use of some technology in processes of teaching or learning: a construction of knowledge and its forms of access, therefore, were always linked to more or less material tools of technological nature (OLIVEIRA, 2009). And this connection occurs somehow beyond the mere use of an extending support for a human ability. From the perspective of Noss and Hoyles (1996):

> Focusing on technology draws attention to epistemology: for new technologies - all technologies - inevitably alter how knowledge is constructed and what it means to any individual. This is true for the computer as it is for the pencil, but the newness of the computer forces our recognition of the fact (apud BORBA; VILLARREAL, 2005, p. 17).

Other authors agree with this vision, i.e. the process of constructing knowledge by people (and its meaning) is *conditioned* by technologies. For example, when explaining what he understands as conditioning in this respect, Castells (2002) states that “technology is society”, so much so that “society cannot be understood or represented without its technological tools” (CASTELLS, 2002, p. 25). The same author, on the other hand, makes clear, in his proposals, that conditioning is not the same as determining; i.e. it would be inconceivable to admit that technologies determine human activity. Equally, Lévy (1999) points out that “[saying that human activities are conditioned by technique] means that some possibilities arise, that certain cultural or social options could not be thought about seriously without its presence” (LÉVY, 1999, p. 25). Borba and Villareal (2005) agree with them:

> Therefore, any suggestion that human thinking could become standardized by a given media would not be defendable, as different and unpredicted uses of a given [technological] medium could always take place. Media, therefore, condition the way one may think, but do not determine the way one thinks (BORBA; VILLARREAL, 2005, p. 16).

Kenski (2003), in the same way, advocates that the use of technologies in educational environments goes beyond the condition of manipulating mere electronic media. For the author, “the human being passes by life culturally mediated by technologies which are contemporary to them. Such technologies change their way of thinking, feeling and acting. They also change their ways of communicating and acquiring knowledge” (KENSKI, 2003, p.21).

In relation with Mathematics learning, it is possible to consider the influence of technologies of all kinds, mainly when considering Lévy’s proposals (1993) about orality, writing and information technology as the predominant poles. The “novelty” of digital technologies, materialised in computers, cell phones and tablets, cannot hide the fact that the construction of mathematical knowledge has always counted on technological actors: “books, paper and pencil are media that allow mathematical learning and comprehension, but they are so incorporated into school activities that their influences on the construction of mathematical knowledge are almost imperceptible or invisible” (BORBA; VILLARREAL, 2005, p.92).

Thus, the perspective shown so far gets people and technologies involved in the construction of mathematical knowledge. However, technologies do not play a part in such association as human-capacity substitutes, not even as a supplement to them. Such a critics was stated by Tikhomirov (1981), specifically relating to computing systems. The Russian author states that the proposals of substitution and supplementation do not correspond to the role technologies have in human beings' activities, either because they use different (and less complex) procedures for the same activities, when compared to the way people carry them out; or because the influence of instruments like a computer cannot be limited , for example, to the quantitative dimension (processing more data in less time), and ignoring their qualitative aspect by just focusing on one of the many and complex areas of people's cognitive structure.

The alternative proposal by Tikhomirov (1981) is that computing technologies reorganize human thinking. The author vindicates that using computing applications allows unusual mediation forms, by delegating to the computer the role of tool for the human mental activity, detainer of similar functions to those carried out by language in the vygotskinian logics. Thus,

> The process of acquiring knowledge is changed (i.e., it is now possible to reduce the number of formal procedures to be acquired thanks to the use of computers). This gives us a basis for stating that as a result of computerization, a new stage in the ontogenetic development of thinking has also developed. […] Memory, the storage of information, and its search (or reproduction) are reorganized. Communication is changed, since human communication with the computer, especially in the period when languages that are similar to natural language are being created, is a new form of communication. Human relations are mediated through use of computers (TIKHOMIROV, 1981, p. 274).

Such considerations support the statement that, in Mathematics, learning is a process involving technologies which are somehow integrated in people; this allows intentionalities, strategies (didactic, for example), plannings and wills to come into play. According to Borba and Villarreal (2005), such integration must be of such an order that it excludes any attempt to see these items – people and technologies, as separate groups. So, for these authors, mathematical knowledge is formed from a collective of humans-with-media, considering, as already pointed out, that media reorganize people's thinking and that the presence of different technologies conditions the production of different forms of knowledge. As stated by Gonçalves (2014) in his research work,

> According to this approach, knowledge construction is the product of the relations between human cognitive structure and tools provided by media. Such definition interferes with the concept of using technology in the classroom, in the sense that media do not replace the objects of study, but, from its relation with human beings, they can promote a new kind of knowledge, different in quality from the one that is, somehow, imposed by other forms (GONÇALVES, 2014, p.60).

It is precisely in relation to the use of digital technology in the classroom that we must admit it that its mere introduction does not mean much in itself: a didactic conception needs to be figured out and it is in this regard that the approach proposed by Borba and Villarreal (2005) is most valuable, as it points out the need to assume "the idea that changes in educational practices should take into account this reorganization of thinking and the solution of problems by humans-computer systems" (p. 14). Thus, didactic interfaces that make these possibilities feasible can be applied based on discussions about modelling in Mathematics Education, for example, or on interventions of this concept through experimentation and visualization, which lead, in their turn, to research routes which use converging media (people-with-spreadsheets-and-with-pencil-and-paper, people-with-Internet, etc.).

So, what does this proposal depend on? i.e. which collectives of humans-with-technologies are to be mobilized? Or even:

> Why using computers instead of pencil and paper? Is it to motivate students, to fulfil a school requirement or to treat certain contents in a different way – maybe more appropriate? In terms of learning, what is the benefit obtained with the introduction of the computer? (BITTAR, 2000, p.92).

Thus, in the research here explained, the two activities described below tried to investigate the understanding of numerical representations related to prime numbers and the FTA presented by a group of teachers in continuous training, based on the mobilization of these collective of people-with-technologies at different moments, with different media: during the first activity, teachers-with-pencil-and-paper; in the second, teachers-with-computers-and-GeoGebra.

3. Methodological approach

The research we describe here has a qualitative nature, and is characterized by the presentation of descriptive data, by an open planning and by its focus on reality in a complex and contextual way (LUDKE; ANDRÉ, 1986). From the point of view of this modality, our research is characterized as a case study, as stated by Ponte (2006):

> A case study aims to know a well-defined entity like a person, an institution, a course, a discipline, an education system, a policy or any other social unit. Its goal is to understand in depth the 'how' and the 'why' of this entity, evidencing its identity and its own features, mainly those aspects which are of interest to the researcher. This research is assumed to be particularistic, i.e., it deliberately focusses on a specific situation which is supposed to be unique or special, at

> least in certain aspects, attempting to unveil what is there in it of most essential and unique and, in this way, contribute to the global understanding of a certain phenomenon of interest (PONTE, 2006, p.2)

The same author states that case studies have been used, among others, to analyse initial or continuous training of teachers, which is precisely the scenario for this research.

The participants of this research are eight teachers from basic education public schools in São Paulo (six) and Pará (two)[5], all of them voluntary in workshops carried out in the framework of the projects "Mathematics Teaching and Learning Processes in Technological Environments PEA-MAT/DIMAT" and "Technologies and Mathematics Education: researches on fluency in devices, tools, artefacts and interfaces" by the group Teaching and Learning Processes in Mathematics (PEA-MAT according to the original), linked to the Post Graduate Study Program in Mathematics Education at PUC/SP. The research we describe here was conducted in one of the computing labs of the abovementioned institution, in a single session which lasted around four hours[6]. Among the subjects so described, five are in basic and secondary education while three are only in basic education. Moreover, all of them have completed their bachelor's degree in Mathematics, but three of them are attending an Academic Master in Mathematics Education while two have completed their Specialization in Mathematics Education.

Participants in this study were invited to working with two kinds of activities involving knowledge on primality in the framework of the Theory of Numbers. The first activity had a question, to be solved individually, by presenting the answer in writing. The formulation is as follows: "*Consider F = 151 x 157. Is F a prime number? Circle YES/NO, and explain your decision*" (ZAZKIS; LILJEDAHL, 2004, p.169).

To answer to the abovementioned question, students would have to write down the option 'No', since the representation in question, with transparent features, shows that F is composite, and it even relates the component prime factors. Students should resort to the Fundamental Theorem of Arithmetic (FTA) to conclude that the said decomposition is unique, apart from the order. (OLIVEIRA; FONSECA, 2015).

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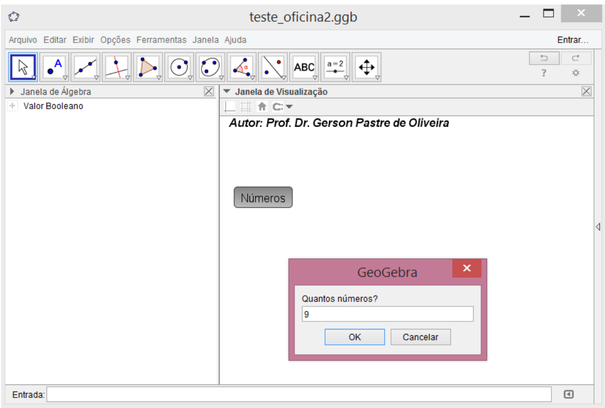
[5] São Paulo and Pará are states of Brazil, located at southeast and at north of country, respectively.

[6] The session described in this article is one among the four that were carried out about the subject "Theory of Numbers and Technologies" and it is part of the abovementioned projects, which involve, on the whole, around 200 teachers of Basic Education Schools.

Otherwise, subjects could perform the multiplication contained in the question, obtaining 23707, an opaque representation regarding detection of the composite character of the number. From this other representation, although it is not necessary in order to resolve the problem, they could make tests with several possible divisors to ponder whether this number would be prime or not. It can even happen that the subject tries to divide 23707 by the prime numbers from 3 onwards, giving up after some attempts, since natural numbers below 151 are not divisors of 23707; in this case, participant can even get to wrongly state that the number is prime (OLIVEIRA; FONSECA, 2015).

The second activity was carried out immediately after the first one. Individually, the eight teachers had in front of them a GeoGebra screen containing only one button with the word "Números"[7] on it. They were all informed by the researcher, the author of this article, that the application would raffle a quantity of numbers indicated by them, when they pressed the said button. It was suggested by the researcher that nine numbers were raffled to each, so they all had the same amount of numbers for the task.

Figure 1 – GeoGebra initial screen in application "Raffle numbers"



Source: developed by the author

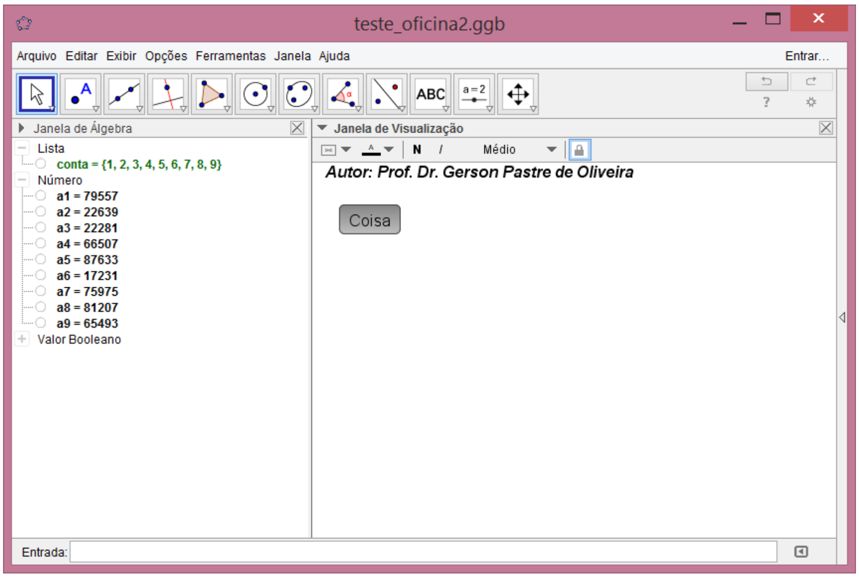
Then, researcher instructed subjects with regard to the result they would see at clicking the button: nine numbers would be shown on the Algebra View of the software. They had to state, by writing on a blank paper that had been handed out and in a lapse of 20 minutes, whether each number was prime or not. In the meantime, they should not click on the button with the label 'Coisa'[8] on it (Figure 2) shown after the raffle.

[7] *Numbers*
[8] *Thing*

Regarding numbers, it is worth noticing that the construction of the code in *javascript*[9], which carried out the raffling, took into account the random selection among numbers that, in such context, could be considered ‘big’, given that each one of them would be within the interval 1001 to 99999 (Figure 2). The goal was to restrict direct application of the divisibility rules and of division algorithms by the first known primes, in most cases.

Figure 2 – Raffled numbers and raffle code in JavaScript

```
var k = 1001
var q = 49999
var m = 0
do
{
	b=prompt('Quantos números?',9)
}while (b<=0)
for (i=1;i<=b;i++)
{
	ggbApplet.evalCommand("a" + i + " = 2 * RandomBetween["+ k + "," + q+"] + 1")
}
i=0
ggbApplet.evalCommand("conta = Sequence[" + b + "]")
ggbApplet.evalCommand("fatorar = true")
ggbApplet.evalCommand("num = false")
```



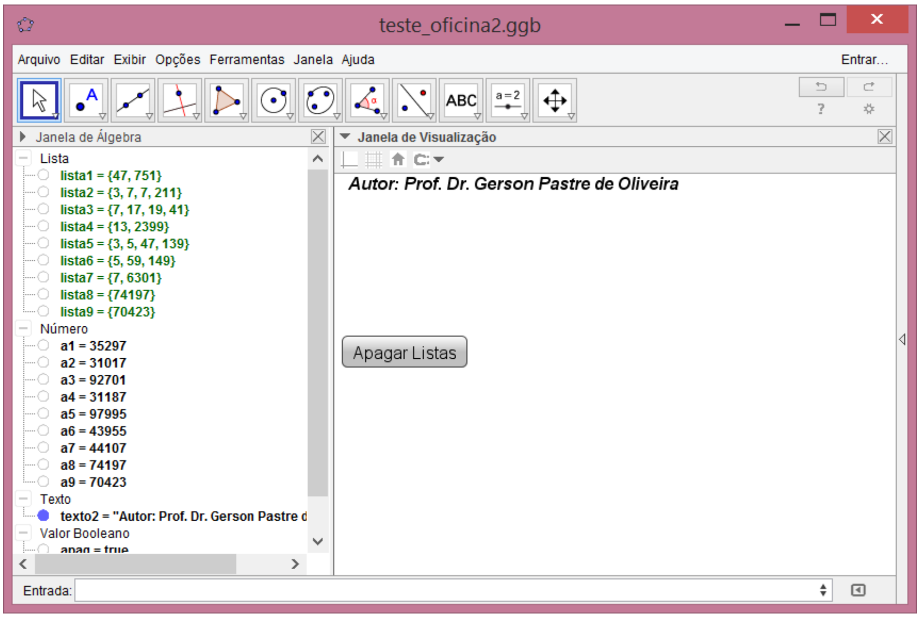
Source: developed by the author

[9] *JavaScript* is a programming language. GeoGebra allows constructing codes linked to objects, such as buttons, which are executed when clicked on and which can carry out actions on the software operating environment. Concerning this article, clicking on button ‘Numbers’ causes a certain amount *b* of numbers to be raffled, being *b* determined by the application user and bigger than zero. The code must be inserted in the “Programming” tab from the object’s option ‘Features’.

Regarding subjects' responses, although numbers could be considered "big", as it has been said, some of them were expected to be searched through the application of divisibility rules and/or algorithms of division by prime numbers, and that such procedures might bring success in some cases (as 47301, which is divisible by 3, just like the sum of its algorisms, which also is) and failure in some other (as 39203, which is not prime, but whose prime factors are 197 and 199, numbers that make unfeasible the abovementioned strategies). Attempts of decomposition in prime factors might possibly be carried out, with similar difficulties. In theory, representations provided by the software were totally opaque regarding numbers primality, at least, up to this point of the experiment.

Once the time allowed was over, regardless of the amount of resolutions among the nine proposals, subjects were invited to click on button "Coisa". After this action, the Algebra View of GeoGebra showed the decomposition of each of the nine numbers in prime factors, in the form of lists (Figure 3) – in the case of primes just the number itself was presented. The button title could then be 'Factorize', but the idea was to promote a research where subjects were the authors of the hypothesis formulated to solve the problem: the use of this title might imply that teachers needed, compulsorily, to do the factorization of the numbers, which would compromise their autonomy. The code in *javascript* for the factorization of the raffled numbers can be seen in Figure 4.

Figure 3 – Numbers and their decompositions in prime factors



Source: developed by the author

Figure 4 – Code in *javascript* of button 'Coisa'

```
b=prompt("Quantos números?")
ggbApplet.evalCommand("Delete[conta]")
for (i=1; i <= b; i++)
{
        ggbApplet.evalCommand(sorteia = "a"+ i )
        ggbApplet.evalCommand("PrimeFactors[" + sorteia +"]")
}
ggbApplet.evalCommand("apag = true")
ggbApplet.evalCommand("fatorar = false")
```

Source: developed by the author

Immediately after and still individually, participants were invited to revisit their previous answers in the light of the new data obtained with the factorization carried out in GeoGebra. At that occasion, subjects were expected to relate the lists obtained with the numbers raffled at the beginning and, when they observed their decomposition in prime factors, they should use the FTA so as to decide on the primality or not of each of the numbers. Ten minutes were allowed for this part of the activity, after which there would be a debate involving the subjects and the researcher. The button "Apagar Listas"[10] (Figure 3) could be used to go back to the initial screen, which would allow repeating the experience as many times as considered necessary by participants.

The session was carried out on the basis of the organization we describe here. For the analysis, which follows, the participants' names were omitted in order to keep their anonymity: they were replaced by **Teach1** up to **Teach8**. Data were collected by means of two protocols produced by the teachers, comments (with notes and pictures) and audio recordings of discussions.

4. Analysis

At the very beginning of first activity, **Teach2** asked the researcher if he could use a spreadsheet program, like Microsoft Excel, to answer. The participant told he intended to test all the numbers within the interval 2 to 23706 by division, placing them as divisors and having as a dividend number 23707. According to **Teach2**, as soon as he found out an operation whose remainder were zero he would stop, because the number would not be prime in that case. When questioned about the operational cost of such strategy, which, even in relation to the proposed automatization of the spreadsheet,

[10] *Delete lists*

might be very high, **Teach2** stated that this was the only way he knew to determine the primality of a number, i.e. by carrying out successive divisions.

It must be said that, in this case, the teacher-with-Excel's approach might be successful, once verification could be interrupted at number 151. Nonetheless, what must be taken into account is the efficiency of this strategy to solve the problem from the point of view of mathematical knowledge. With bigger numbers, as for example, 411998603 (which is not prime), such procedure would lead to, at least, 20197 tests. Although experimentation may be a possibility that digital technology has open, its use does not do without the search for more appropriate strategies. In other words, the learner is not exempted of the researching effort and of investing on conjectures from the interaction with the results of the experiments: the idea should not be experimenting to exhaustion. In this respect, Borba and Villarreal (2005) and Oliveira (2013) point out that adding digital technologies without a reflection on the strategies used in knowledge construction processes do not grant improvements or qualitative benefits.

In any case, as at this stage calculators were not allowed, Excel wasn't either, which frustrated **Teach2**, who, after trying some operations with pencil and paper stated that the number in question would 'probably' be prime. Similar strategies were used by other four participants (**Teach3, Teach4, Teach5** and **Teach8**), who also stated, wrongly, that the number in question would be prime. In these cases, some of the typical mistakes that had already been verified in Zazkis and Liljedahl (2004) and in Oliveira and Fonseca (2015) were also found here:

- **Teach3** does countless division operations and ends up stating that "23707 is prime, because it can be divided by itself and by one". In this way, he shows he is not aware that this criterion does not distinguish between prime and composite numbers;
- **Teach4**, after several attempts using division operations, concluded that 23707 would be a prime number because it "ends in 7 and 7 is prime";
- For **Teach5**, as the divisibility tests by 2, 3, 5, 7, 11 and 13 'failed' (divisions had a remainder other than zero), the number in question would be prime – in this case the teacher indicates he believes that "decomposition into prime factors means decomposition into small prime factors" (ZAZKIS; CAMPBELL, 1996, p. 215);

- In the case of **Teach8**, several aspects were raised in support of the affirmation of number 23707 primality: according to the participant, “it is prime, because it is an odd number and it cannot be divided by its square root or any other prime number”. Like **Teach5**, **Teach8** limited the universe of prime numbers to the interval within 2 and 13 and it evidenced several confusions involving the concepts of perfect square numbers and odd numbers.

The representation of F provided in the question formulation has transparent features in relation to primality, because it presents the number by means of its unique decomposition in prime factors, in the way shown by Zazkis and Liljedahl (2004). However, the abovementioned teachers did not use this idea as expressed in the FTA, which points out the fact that a numeric representation with mathematical features that make it transparent can be kept opaque when knowledge about it is not mobilized by the individual. The same authors, apart from Lesh, Behr and Post (1987), Zazkis and Gadowsky (2001) and Oliveira and Fonseca (2015), state that didactic strategies can be used for the construction of evidence that reinforces transparent features of a given representational system. This concept sustained the use of GeoGebra software, which is analysed later on.

Furthermore, participants correctly stated that F, the number being tested for primality, would not be prime. According to **Teach1**, “F is divisible by 151 and 157, which makes it non-prime”. Participants **Teach6** and **Teach7** said, in a similar way, that F had other divisors besides itself and 1, which would make it non-prime. Nevertheless, none of the three participants that answered correctly did show any sign of using the FTA in their conjectures: when questioned about the possibility of F having other divisors apart from the ones mentioned, the three of them said it was possible but that they would have to test numbers up to a certain limit (according to **Teach1** up to the number square root; according to **Teach6** and **Teach7**, up to half the number). In this respect, these participants would tend to adopt a strategy with quite a high operational cost, therefore with more chances of mistakes, if the question related to the problem were, for example, *determine all the divisors of the number resulting from 151 x 157.* Another aspect worth highlighting is that none of the teachers showed awareness of the fact that factors 151 and 157 represented prime numbers.

From a different perspective, the technological component of the collective humans-with-pen-and-paper, even if intensively used, does not seem to support cognitive movements related to a change of strategies – in the case of participants who dried up attempts with divisibility algorithms – or to the use of formal notions in Mathematics, such as the FTA – in the case of participants who stated that 23707 might have other divisors.

With regard to the second activity, carried out in GeoGebra, participants accessed the application which was available on the computers in the institution's lab and, without further questions, they chose the raffling of nine numbers, as shown on figures 1 and 2. From this moment, teachers had 20 minutes to decide which numbers would be primes. All participants alleged, initially, that time was too tight and that numbers would be too big (odd numbers between 1001 and 99999) to be able to provide an answer. The researcher said that they should also state as many results as possible in this lapse of time, which could not be extended.

Until that moment, the technological aspect in the collective teachers-with-GeoGebra did not have great influence on the issue of transparency of the numeric representation regarding primality, because the program interface in question only provided odd random numbers within the said limits. Thus, as raffled numbers were potentially different, the amount of correct or wrong results showed significant differences among the subjects. Those whose raffled numbers allowed the application of divisibility rules or tests with 'small' prime factors (3 to 13) could be more directly applied obtained more hits than those whose raffled numbers were 11741 (59 x 199) or 31753 (113 x 281), for example. Generally, such numbers were wrongly stated to be prime. Even when prime numbers were identified, as 17231, a doubt used to remain, as in the case of **Teach5**, who wrote, next to the said number, "I think it is prime. I tested up to 13"[11]. As we have already seen, this strategy, in the case of numbers whose prime factors are all higher than 13, is not efficient. Thus, what could be observed was the return to the strategies previously used when the completing of first activity, even regarding the collective of teachers-with-pencil-and-paper.

Among the teachers' talks during the 20 minutes allowed for the exercise, several references were made to the numbers 'difficult form', a clear attempt to refer to their representation, which is clearly opaque in terms of the feature 'primality'. **Teach6** even

[11] The participant meant that he divided 17231 by prime numbers from 3 up to 13.

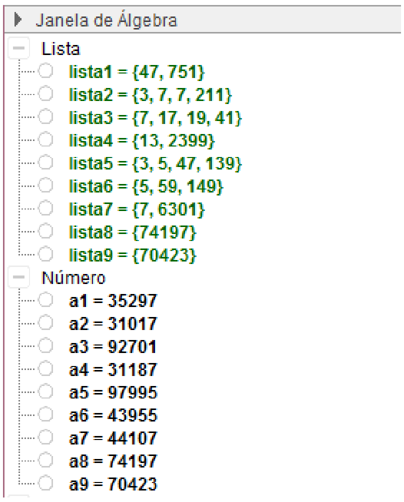
got to question the goal of GeoGebra in that context, as the application did not seem, according to him, “to make things easier for those who tried to solve the problem”.

Before the perplexity caused by the proposal, the researcher, after the time allowed, proceeded to coordinating a debate with the participants, whose main motivation was raising the conjectures and strategies that subjects had proposed in order to verify the primality of their nine numbers. None of the participants said to have tried nor even thought about obtaining the factorization of the numbers to be tested for primality in order to using knowledge on FTA.

Once the debate was ended, the researcher said that subjects could click on button ‘Coisa’, which would show, for each raffled number, the respective list of component prime factors (figure 3). In 10 minutes, then, teachers had to review their answers.

Immediately after clicking on the button, teachers had to construe the data showed on the screen. A visualization model available to them can be seen on figure 5.

Figure 5 – Algebra Window of GeoGebra: numbers and their respective prime factors



Source: developed by the author

When they realized the numeric equality and the correspondence between the lists provided and the raffled numbers (*list1* referred to number *a1*, *list2* referred to number *a2*, and so on), most teachers started searching for relations among the said components. Some of the talks in this respect are transcribed here below:

Teach6: *– Professor, I would like to review my answers.*

Researcher: *– Yes, why so?*

Teach6: *– Because I realized something that I had not seen before… the lists… they are the factors of each number...*

Teach1: *– By multiplying the numbers in the lists we obtain the raffled numbers, the exact same numbers!*

Teach4: *– True, but there are cases where only one number appears… these numbers are prime, as we can only multiply by one!*

Teach3: [does not seem convinced] *– Professor, can I raffle the numbers again?*

Researcher: *– Sure!*

Teach3: [after repeating the raffling and factorization] *– Gosh it's true! Prime numbers do not have factors, only composite numbers do.*

Teach4: *– A prime number's factors are itself and one...*

Teach7: *– Professor, I was thinking... In my case, one of the numbers is 88739... Factorization is shown as 7, 7 and 1811. I could as well write 49 and 1811, right?*

Researcher: *– What do you all think?*

Teach8: [after some discussion with pairs] *– I think that 49 can appear, but 49 is not prime, and lists show prime factors of numbers. The idea is that only prime numbers appear. We can see that all the factors, in all the numbers, are prime.*

Teach4: *– You are right. Any number can be written as a product of prime factors! This is it! Wow! First question was obvious! There is only one decomposition into prime numbers for each number.*

Teach8: *– It is the Fundamental Theorem of Arithmetic...*

After these observations, subjects could tell which numbers were prime and which were not, by repeating raffling and the whole procedure several times. As noted by Borba and Villarreal (2005), visualization and experimentation were important factors in the new strategy adopted by subjects from configuration humans-with-GeoGebra. According to these authors, such items allow, among other actions, for example, to invest in generating conjectures about the problems at issue (and testing them through countless examples), bring to light some results which were not known before the experiments and testing different ways of collecting results. The access to visual components, in the consolidation of the results of actions carried out by people-with-GeoGebra, became a way to transform the understanding they had about the problems at issue.

Another aspect that must be taken into account in the configuration teachers-with-GeoGebra, is the dynamism of digital technologies, that has been seen here as a possibility to manipulate the parameters, attributes or values which served the constitution and/or definition of a mathematical construct in a computerized context. Before the possibilities open by this resource, a fundamental investigative movement to mathematics finds consistent subsidies from the development, testing and validation (or refutation) of conjectures. This could be largely seen in the experiment we describe here, when teachers invested, through experimenting and visualizing, in the procedure repetition, using the regularities observed in factorizations – and dynamically obtained, as a means to support the reorganization of ideas on the primality of the presented numbers. All these factors collaborated to mobilise knowledge on FTA for the solving of the problem.

Generally, teachers looked quite enthusiastic about the conclusions reached and the successes they achieved. Other possibilities were pointed in dialogues with subjects, even about the way how the application was developed and how other interventions might take place based on this knowledge, linked to computers programming. However, such ramifications are out of the scope of this article, although they can be approached in other papers.

Final considerations

The discussion following the last activity was quite fruitful, with the intense participation of all the subjects, who expressed their observations regarding the route they had taken along the activities and the discoveries that they made. Briefly, participants stated that, in activity 1 they had not realized that the 'way' in which the number was written (its representation) allowed the question to be answered directly by mobilizing knowledge related to the FTA. After the two stages of the experiment, according to the teachers, number F would not be prime because it could be, itself, represented in prime factors. **Teach4** and **Teach8** spontaneously said that this idea would correspond to the 'wording' of the fundamental theorem of arithmetic, and the others agreed with them to this respect.

Regarding activity 2, participants said that numbers were not expressed in a 'convenient way' (transparent representation), i.e., according to them, they were 'big numbers' that were not decomposed in prime factors. Teachers mentioned the fact that they had spent the 20 minutes to try and say which of the numbers were prime, but that, if they had the appropriate representation in factors and had remembered the FTA, they would have done this much faster. This last feature was realized by them when they clicked on the second button (Coisa), causing the decomposition of the numbers in prime factors to appear. The participants concluded that, when there were other factors besides 1 and the number itself (trivial factorization), the number in question would not be prime. Moreover, teachers highlighted the importance of FTA knowledge and of the use of GeoGebra in the procedures, saying that this would be a good way to approach the subject in the classroom.

In spite of the somehow simple use of the software resources, such use can be seen as an alternative allowing to resume a typical feature of calculators in consistent didactic strategies, i.e. minimizing the complexity of operational tasks (calculations), allowing the effort to solve mathematical problems to focus on the mobilization/construction of relevant knowledge (SELVA; BORBA, 2010). Thus, in this context, the application would work as a 'calculator to decompose in prime factors' of the raffled numbers, turning a numeric representation –opaque in relation to primality, into a transparent one, provided that the knowledge about the FTA is mobilized. In this case, the configuration people-with-GeoGebra contributed in a more efficient way to direct the problem-solving effort towards a strategy that brings more chances of success. Dialogues show, although only to some extent, the renegotiation of meanings, the conjectural reformulations and, after a certain point, when they started using the digital resource as fluently as they used pencil and paper, the reorganization of thinking which allowed the right answer to come out.

What remains to be said, according to the theoretical base for this research, is that numerical representations of prime numbers are opaque, although they have transparent features when the FTA is taken into account for the numeric decomposition in prime factors. However, like in the works that served as a reference for this research, teachers described here did not recognize a representation where this knowledge would allow identifying a number as composite. Therefore, even if they are opaque, representations of prime numbers can give opportunity to transparent features, as soon as the appreciation of underlying meanings and concepts is taken into account. When this does not occur, the tendency is to call for large, expensive solutions in cognitive terms. Most mistakes observed arose from the difficulty to extend the concepts from FTA into a different context from the one they had seen during training. To investigate the nature of these processes and develop proposals in order to avoid such difficulties to remain among basic school teachers, seems to be an important challenge, open to new researches.

Finally, the fact that teachers realized that they had failed to recover the concepts explained by the FTA and they had finally recovered them spontaneously when using the interface, seems to point out that, although this does not close the inquiries on the subject, the use of didactic strategies with resources offered by dynamic technologies of digital nature, can be one of the ways of working with issues of this nature. From features such as visualization and experimentation, linked to the dynamism available thanks to the software, subjects led discussions to culminate in valid conjectures, thus reaching the conclusion that the use of the said theorem and of representations with transparent features, supplied by the factorization into prime numbers, could offer successful strategies to solve problems similar to those they had just faced. In this respect, the configuration teachers-with-GeoGebra seemed critical, according to the subjects, to the construction of correct answers and to the mobilization of the relevant mathematical knowledge.

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