Probabilistic thinking and probability literacy in the context of risk

Pensamento probabilístico e alfabetização em probabilidade no contexto do risco

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Abstract

The aim of this paper is to develop and synthesise ideas about probabilistic thinking and highlight the considerations by illustrating how the concept of probability is entrenched with the concept of risk. We use a hermeneutic way of argument, relate the ideas to the mathematical and philosophical background of probability, and illustrate our ideas by examples that relate probability considerations to risk. Special features of competing intuitions and strategies link probabilistic thinking to its roots in psychology, to the paradigm of causality, to its empirical expressions, and to thinking and decisions. Higher-ordered probabilistic thinking is described by the categories of the theoretical character of probability, conditional probability, and by the construct of probabilistic evidence. The robustness of probabilistic misconceptions is explained by an archetypical way of thinking. The peculiar logic of decisions adds to probabilistic thinking. Finally, the purpose of probability is declared as central issue for teaching and understanding probability. Throughout, the connection of probability to risk enhances probabilistic concepts and reveals a twin-character of probability and risk.

Keywords: *Probabilistic thinking, Probability literacy, Risk literacy, Mathematical thinking, Theoretical character of probability, Probabilistic evidence, Conditional probability, Archetypical strategies, Logic of decisions, Insurance contract.*

Resumo

O objetivo deste artigo é desenvolver e sintetizar ideias sobre o pensamento probabilístico e destacar as considerações apresentadas ilustrando como o conceito de probabilidade está enraizado no conceito de risco. Utilizamos argumentação hermenêutica, articulando as ideas com o quadro teórico matemático e filosófico da probabilidade, e ilustramos nossas ideas por meio de exemplos que relacionam probabilidade e risco. Características especiais de intuições e estratégias concorrentes ligam o pensamento probabilístico às suas raízes na psicologia, ao paradigma da causalidade, às suas expressões empíricas, ao pensamento e às decisões. O pensamento probabilístico de ordem superior é descrito pelas categorias de caráter teórico da probabilidade, probabilidade condicional e pela construção da evidência probabilística. A robustez dos equívocos probabilísticos é explicada por um modo de pensar arquetípico. A lógica peculiar das decisões é adicionada ao pensamento probabilístico. Finalmente, o propósito da probabilidade é declarado como questão central para o seu ensino e compreensão. Assim, a conexão da probabilidade ao risco realça conceitos probabilisticos e revela um caráter dialético da probabilidade e do risco.

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Palavras-chave: *Pensamento probabilístico; Letramento probabilístico; letramento sobre risco; Pensamento Matemático; Caráter teórico da probabilidade; Evidência probabilística; Probabilidade Condicional; Estratégias arquetípicas; Lógica de decisões; Contrato de seguros.*

Introduction

More than with other mathematical concepts, controversies play an essential role with probability; see, e.g., Batanero, Henry, and Parzysz (2005). There are several ways to conceptualise probability; from the diversity, only three have received a wider acknowledgement and outreach in applications. We will shortly describe these approaches and use acronyms to denote them with the intention to clarify and separate ideas (for the notation see also Borovcnik and Kapadia, 2014):

i. *Frequentist theory of probability (FQT).* The idea in the background is that relative frequencies converge (in a sophisticated way) towards the underlying probability if the random experiment is independently repeated under the same conditions infinitely many times. This perceived empirical regularity is mirrored in the Law of Large Numbers that can be derived from the mathematical setting of the theory, which is the usual axiomatic approach that goes back to Kolmogorov (1933).

ii. *Subjectivist theory of probability (SJT).* The lead idea in the background is a preference system of a person, from which a degree of confidence (a probability) may be derived for certain statements (the analogue of events); this is justified by the axiomatic approach of de Finetti (1937). Key is also the exchange of money for security as is done in an insurance treaty where the client can buy (financial) security from the insurance company. The central law in this approach is the Bayesian formula, which clarifies how data (FQT information) can be formally integrated to provide a new probability. SJT probability should not be confused with an arbitrary probability judgement; it is a qualitative form of knowledge on statements that has to fulfil certain conditions.

iii. *A priori theory of probability (APT).* The Laplace assumption of equal probabilities for all elementary events provides a unique probability measure on a finite sample space. The approach mirrors the idea of fairness; a decision is fair if it is made by an ideal chance device. Such devices may also be used to communicate values of probability and also to calibrate one's own scale of probability. There has been some debate whether APT as a priori theory may be confused with prior probabilities (used in the Bayes formula); alternative suggestions might be equiprobability theory (EQT) or Laplace theory (LPT). Yet, we advocate APT as it expresses the fact that within this approach *probabilities are given a priori*, which historically played a great role (one only has to know all possible cases).

The other basic concept we use in this paper is risk. By risk we understand a situation with inherent uncertainty about (future) outcomes, which are linked to an impact (cost, damage, or award). For the comparison of several options sometimes the expected value of the impact is taken as a criterion to evaluate the options, on other occasions other criteria are used for that purpose (like minimising the maximum cost). The usage of the term risk varies greatly between the two following extremes: Risk may refer to the probability of an adverse event without consideration of impact, or risk may only refer to the impact (damage) without any consideration of the related probability. The reader may find a detailed analysis of the logic of risk in Borovcnik (2015). With respect to the stakeholders that are involved in a risk situation, we see the following cases:

- The decision relates only to the person who decides. A person might consider going for a dangerous climbing tour.
- The decision is shared between two stakeholders situated at different levels. A person might consider taking out an insurance policy from an insurance company. Or, the doctor has to suggest therapies (an operation) to a patient.
- The decision relates to a societal risk. A screening scheme for early detection of a disease might be implemented or not.

The perception and judgement of risk might differ completely between various stakeholders and a decision that involves risk might have different implications to those who are involved. If, for example, a single person encounters an institution, the implications differ greatly. While a patient suffers personally from any treatment, a doctor or the institution may be liable for what they do. For details see Borovcnik and Kapadia (2011a, 2011b).

1. Mathematical thinking

Thinking is something human beings do all the time; of course, thinking is influenced by people’s experiences and the theoretical frameworks they have acquired in their schooling and elsewhere. Mathematical thinking is influenced by a mathematical background. It can be formal if the mathematical relationships are used precisely. Fischbein (1987) describes two ways of thinking that also apply for mathematicians: an intuitive, yet mathematical way of thinking and a way of thinking shaped by mathematical concepts they have acquired. In this latter case, thinking is guided by the so-called secondary intuitions that emerge from learning the mathematical concepts and replace the primary intuitions that existed already prior to formal education. Secondary intuitions form a strong tool that allows solving a mathematical problem without knowing all technical details (this aspect makes it also attractive for teaching as the technical details are known as obstacle to learning). Thinking mathematically comprises finding a suitable model to adequately represent the situation. Of course, in problems of applied mathematics, the constituents of the modelling process are wider and also comprise – beyond mathematics – knowledge of the context as well as criteria for assessing how well models match a situation. Literacy may be defined as the ability to find, read, interpret, analyse, and evaluate written information, and to detect inconsistences, errors, or biases within this information.

1.1 The role of probability

With respect to probability, Borovcnik (2011) states that randomness is a concept that allows one to think about the world. We have a strong affinity with other types of thinking, which might lead us to reinterpret the situation by concepts different from those predesigned by probability theory. This creates special challenges for teaching probability. Of the nine reasons in Borovcnik (2011) to support a stronger role of probability within statistics curricula, only one deals with statistical inference and random samples, which otherwise cannot properly be understood. There seems more on probability that points beyond its role as companion to statistics. We will develop ideas on probabilistic thinking and probability literacy and relate them to risk. Batanero and Borovcnik (2016, pp. 4) summarise the role of probability in a curriculum of probability and statistics in the following way:

> In recent years, there has been a shift in the way probability is taught at school level, from a classical Laplacean (or “axiomatic”) approach (common until the 1980s) towards a frequentist conception of probability, that is, an experimental approach where probabilities are estimated from long-range relative frequencies […]. Simulations and experiments are used to support students in understanding the Law of Large Numbers and grasping the sophisticated interaction between the notion of relative frequency and the *frequentist conception* of probability.
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> [Historically, t]he *subjectivist view* of probability, which is widely used in applied statistics, has been developed hand in hand with the frequentist view

> so that the two complement each other […]. Their interplay is relevant, especially for conditional probability. Bearing this in mind, we suggest a combination of both approaches in the teaching of probability.

1.2 Statistical literacy and statistical thinking

Statistics is generally associated with its role to generate evidence from data. Statistical knowledge comprises to apply proper models in specific situations, considering the validity and impact of assumptions, deriving the results and interpreting them in the context of the given task. As such a type of knowledge is essential for a modern data-driven society and science in the empirical paradigm (with evidence-based rules for generating new knowledge and insights), statistical literacy is highly esteemed. Consistently, there are many descriptions of statistical literacy and this literacy has found its explicit expression in curricula world-wide (see Batanero and Borovcnik, 2016, pp. 12). To be statistically literate, people need a basic understanding of statistics (including terms and symbols, graphs, and the basic logic of statistics) and the context. Batanero and Borovcnik (2016, p. 13) state that:

> Statistical literacy should also enable people to question the thinking associated with a specific method, to understand certain methods and their limitations, or to ask crucial questions to experts and understand their answers.

Gal (2002) identifies the ability to interpret and critically evaluate statistical information in diverse contexts and the ability to communicate opinions on such information. Of course, mathematical skills are required to process and communicate that information but may also hinder that people understand that kind of information. The ideas about statistical thinking are well-embedded in the statistical investigation cycle of Wild and Pfannkuch (1999) who – according to Batanero and Borovcnik (2016, p. 16) specify

> the complex thought processes involved in solving real-world problems using statistics. They used the modelling cycle described by the modelling group within mathematics education […] and filled its components with statistical ideas. They mainly focused on the process of empirical research with little reference to probability […]. Solving statistical problems involves a complete research cycle of *Problem, Planning, Data, Analysis*, and *Conclusion* (PPDAC).

2. Probability literacy and probabilistic thinking

Stochastic thinking has been delineated by different authors; as with statistical thinking, which are isolated within statistics with sparse connections to probability, stochastic thinking remained isolated on the probability side. For example, Heitele (1975) described a list of fundamental ideas related to understanding probability that reads like the headings of a mathematical textbook of probability. Other descriptions of stochastic thinking before the millennium followed Heitele closely. See also Borovcnik (1997) for a critical review of these ideas and an early attempt to focus on conceptions about the peculiar character of probabilistic information.

Gal (2005) expanded his view of statistical literacy also to probability. In his model, he includes the capability to interpret and critically evaluate probabilistic information and random phenomena and focuses on the importance of the context, in which such information is embedded. Essential for such literacy are the abilities to understand the meaning and language of basic probability concepts and to use probability arguments properly in private or public discussions. Gal also introduces dispositional elements in his model of probability literacy; appropriate beliefs and attitudes have to be controlled; personal feelings (such as risk aversion) should always be supported by defensible reasons. Again, mathematical competencies play neither an obvious nor a substantial role in Gal’s model of probabilistic literacy.

Beyond, little is published on an explicit description of probabilistic thinking and the mathematical competencies that are related to it. Borovcnik (2006) presents examples that illustrate probabilistic thinking, Borovcnik (2011) describes some aspects of it in more detail, Batanero and Borovcnik (2016) amend the list with higher-order thinking in probability. We will extend these considerations and try to synthesise them by relating the ideas to the context of risk.

2.1 Thinking probabilistically: competing intuitions and strategies

In this section, we synthesise ideas of probabilistic thinking. Borovcnik (2011) described five aspects of probabilistic thinking as

a. probability as index of surprise;
b. feedback from probabilistic situations is indirect;
c. the causal alternative to randomness;
d. the conflict between actions and reflections;
e. non-probabilistic criteria for decisions;

We comprise, rename, and reorder these categories of intuitive probabilistic thinking in the form of the following abilities:

1 the ability to balance between psychological and formal elements (a, e));

2 the understanding that direct criteria for success are missing (b);

3 the ability to separate between randomness and causality (c); and

4 the ability to separate reflecting on a problem and making a decision (d).

In the following, we describe these abilities and establish connections to risk to corroborate how interrelated probability and risk are and how literacy in probability is entrenched by literacy in risk.

1 *The ability to balance between psychological and formal elements arises when using a personal scale of probability*.

The psychological factors behind human behaviour seem to be of an archetypical character (see Section 3). Already Fischbein (1975) noticed that after schooling some raw strategies revert as pupils either did not understand the formal elements so that they ignore them or re-interpret them to fit to their deep-seated patterns of behaviour.

Randomness may be linked to surprise in the sense, the more surprising events are, the less probable they are judged. With highly surprising events, individuals are inclined to think about alternative explanations such as "God's interference" (the argument in statistical tests resumes this pattern). The formal assignments of probability and structural elements of dealing with probability are due to an intellectual approach to randomness. That can only provide unstable interrelations. Intuitive, simplifying short-cuts are much more convincing to the majority of people. With respect to risk, in many situations there is a combination of low probability and either high negative impact (negative implications) or high win (positive implication). The first part is associated with the idea of insurance and hedging, the last marks the ever-growing sector of games of chance and betting. In both cases, great emotions are involved, either fear or hopes. We know from psychology that there are deep-seated patterns of behaviour that let people flight or fight in such situations. There remains hardly a margin for reflective consideration of the situation under scrutiny. People tend to use idiosyncratic knowledge and personal views and re-interpret the given information and the problem situation using archetypical strategies (see Section 3).

2 *The understanding that there are no direct success criteria in random situations*.

The single case of applying any method is not simply repeated. Different from the standard situation in mathematics, people refuse (yes, it is emotionally laden) to perceive their situation as only one exemplar of many comparable cases. The single case is vividly described and it is part of real life. Any consideration of what else can or could happen is much less convincing. In such a single case, with chance, everything is possible. However, once people have decided about their approach and strategy, and the outcome is known to them, they start to re-interpret the course of action. Depending on their character, some claim that whatever they do, they get it worse (if they suffered from failure or negative impact), others claim that their specific strategy they had used made them win. Of course, a person with an idiosyncratic strategy for the decision can have success in a random situation. For example, dream of the numbers in the state lottery, tick them, and win. Who would be able to convince that person of the inadequacy of the used strategy? This personal re-interpretation of reasons for decisions and coping with the outcome of the decisions made is part of our life.

The theoretical answer usually is to face the single case with the repeated cases and compare the success of strategies in the long run. That does not work in most of personal situations. The main reason is that there is a different logic in the single case and in repeated decisions (see the variation of Kahneman and Tversky's task in Section 4) and the persons perceive their situation as unique. There is also a strong tendency to re-invent reality, which overrides monetary calculations for evaluating the rationality of the decision in favour of the insurance; Borovcnik and Kapadia (2017) state:

> A common example is the prevalent habit to take out an insurance policy on almost every aspect of our life. Part of the success criteria is that the person would say that the adverse event did not affect him as he has taken out the policy as if the policy would be a protection shield; it is like saying 'it did not rain today because I took an umbrella'.

3 *The ability to discriminate randomness and causality*.

The emergence of probability has been accompanied by the twin concepts of randomness and causality. Causal approaches deliver a simple mechanism: once it is known how a specific cause establishes the effect one can predict the future. In his definition of probability, Laplace restricted the field of applications to such cases where we do not know the exact causes. While his famous demon would know these causes and therefore not need probability, probability remains for those who remain ignorant about the true background of the phenomena. The Laplacean demon expresses a primitive determinism. Since Laplace, randomness has been introduced into thermodynamics of the 19th century as an irreducible element of physical views. The consequences of randomness in form of the laws of thermodynamics hold whether such a demon would still know everything and thus make causality obsolete. In fact, in the 20th century, theoretical physicists re-wrote physics from a pure random basis eliminating causality (Styer, 2000). However, causality comes back by more recent work on quantum mechanics (introducing hidden variables) so that a deterministic view on theoretical physics and the interplay between the twin concepts is revived (Dürr, Goldstein, Tumulka, and Zanghi, 2004).

Most people follow a causal approach to the world, based on simple interrelations between cause and effect. They have not understood all the changes of paradigms. Frequently, results (probabilities) are only acknowledged if a causal connotation can be attached to them. One specific example for that is conditional probability, which represents various meanings: it may represent timely forward and timely backward relations between the involved events (statements); it may represent causal and indicative relations. Timely forward may be interpreted causally, backwards in time cannot stand for a causal relation. A connection between a status of a disease and the result of a medical test has also two distinct directions: a disease may be seen to cause some changes in the metabolism so that some biometric values get higher or lower. However, there is no clear cause-effect relation between the biometric variable and the disease as the involved variables may vary in healthy persons and as other reasons might increase or decrease these values. Therefore, the probability to have this disease given that the medical test is positive (indicating the disease) lacks a causal interpretation and it can only be perceived as indicative. The simple reaction to that is – as may often be seen – either a neglect of indicative conditional probabilities or a transfer from the reverse direction, which is causally associated.

For risks, the situation would be simpler if hazards (exposition to potential dangerous situations) are causally related to outcome. Usually, there is also a considerable lag in time between the exposition to a risk and the occurrence of damage so that incidents of damage are not easy to link to the hazard. For the perception of risks, the extremes from above are to observe: either the expert's numbers are taken for granted and over interpreted or they are completely ignored as irrelevant and substituted by personal considerations. Already children learn fast if there is a direct connection between action and reaction so that – in the sense of causal relations – a connection is established and the action revised. Such learning follows a causal-like pattern.

4 *The ability to discriminate criteria for reflecting on a random situation from those, which may be applied for selecting a decision*.

For example, if the probabilities of two options are the same (and choosing the right option would be awarded by a prize), one might still have a strong preference for one.

Assume that there is a tendency for people to focus on such information from the context of tasks (or to add such information), which allows a direct choice of an action while neglecting other information. So far nothing is wrong with it. To conclude that this person has an inadequate perception of probability or of the situation is premature. To ask the person for reasons, to justify the first choice may confuse him completely as he would know that the *chances* are the same (at the level of reflection). As the situation leaves no suitable criterion for a good choice (level of decision), *there can be no justification* for his decision except his preference. Only if this person is prepared to accept a further "payment" to be able to freely choose (and select the preferred action), this introduces an irrational element into the situation.

To summarise: the criteria that are good to reflect on a probability value and those to select a decision what to do are different and may lead to irreconcilable results. To ask for a reason for something, where no reason can be given, may lead to an exchange of nonsensical statements (both think that the other one reacts irrationally). For risk, the probability might have an acknowledged small value and a decision has to be made (it does not suffice to have a reflective estimation of a probability). Yet, the person might opt for an inferior decision as for this decision the maximum loss seems much more important than anything else (see also Section 4) – any abstract probability cannot balance for the vivid impact.

2.2 Probabilistic thinking – secondary intuitions influenced by learning

We perceive probabilistic literacy as the ability to use relevant concepts and methods in everyday context and problems. We synthesise ideas of Batanero and Borovcnik (2016, pp. 20) who included the following components within probabilistic thinking:

a. Influence of prior probability judgement.
b. Asymmetry of conditional probabilities.
c. Theoretical character of independence.
d. The problem of small probabilities.
e. Correlation as probabilistic dependence.

Again, we combine some categories to facilitate an overview and to structure the components of probabilistic thinking in the following way:

1 Theoretical character of probability (c. and d.; combining SJT, APT and FQT aspects including the case of small probabilities);

2 Conditional probability (a. and b.) based on their dependence on prior judgments, as well as the asymmetry of probabilities relating to the direction of conditioning;

3 Concepts building on probabilistic evidence (e.) such as probabilistic dependence as conceptualised by the correlation coefficient or relative odds.

We expand on each of these aspects below and relate them to risk.

1 *Theoretical character of probability and independence*.

Steinbring (1991) characterises probability and independence as theoretical concept and elaborates on the development of suitable task systems to illustrate this theoretical character indirectly. Steinbring (pp. 142-144) states:

> As a basic concept, the concept of relative frequency is not free of such circularities either. Relative frequencies, or the limit of relative frequencies, can be interpreted as probability only if they refer to a stochastic object such as a stochastic collective. The definition of such a collective, however, assumes that concepts like chance and independence, for instance, are used simultaneously. […]
>
> The concept of probability is thus not just a result of abstraction and idealisation of empirical properties, […]. [We] use the term *theoretical character of mathematical knowledge* to denote this dependency of the concept and its meaning on the theory's level of development. In this view, the concepts are not simply the foundation of the theory, but conversely, it is only the theory which explains and develops the meaning of fundamental mathematical concepts.

Spiegelhalter (2014) speaks of probability as a metaphor. Probability is essentially non-empirical. To perceive the probability of a six on a die as an empirical property and keep rolling the dice expresses only a thought experiment. For the casino, the FQT conception gives more clues to understand the situation and design a game that will provide a long-term profit. However, for a player, often the situation is perceived as one-off (SJT, including the impact). Many chance situations and other more general situations (like that on health) are one-off so that a long-run interpretation as frequencies neither helps nor convinces people.

The repetition of the same experiment is less crucial in the context of games but all other incidents are questionable. For example, for a random sample (the basis of information in statistical inference situations), the elements have to be drawn independently and with the same probabilities from a population. Mostly, random sample is a wish and the term is used as a metaphor to indicate the care that has been taken to achieve a random sample. However, a check of this assumption can only be done by heuristic arguments such as there is no obvious bias to observe in known variables (all the quotas with respect to gender, age, etc. are well-represented).

In many other situations, applying independence (the other constituent of probability in an FQT conception) is an inherent requirement of probabilistic models. When, for example, two pieces of circumstantial evidence are applied before court, independence is often inappropriately applied (see, e.g., Gigerenzer, 2002).

For risk considerations in technical environments, an independence assumption is the only way for calculating the risk of failure for a system that is built up from several units. For example, to deal with redundancy (i.e., several units are built in a device in parallel so that the device fails only if all units fail), independence of the failure behaviour of the redundant items is a mere assumption (see also Borovcnik, 2006, for an example). Frequently, only with hindsight, one may recognise a violation of such an assumption as in the Fukushima plant accident of 2011 when all emergency systems were destroyed by one and the same cause (the tsunami) as they were hosted in the same building.

The quality of methods of statistical inference is usually measured by (conditional) probabilities: type-I or type-II errors of statistical tests, or confidence levels of confidence intervals. Basically, the inferential methods are intended to preserve an FQT conception of probability so that these entities are interpreted in a frequentist sense. For a frequentist interpretation, however, a repeatable situation of such tests (or application of confidence intervals) has to be assumed. A situation, which again reflects a thought experiment as such an inferential situation (comprising all steps of modelling including the errors to capture the real-world situation) is only repeatable on a production belt in mass production of items (where the concepts in fact have their origin). It might be better to leave these indices as they are as indices of *quality* that follow an ordinal (and not a metric) scale, void of a frequentist interpretation.

Connected to the theoretical character are small probabilities. Small probabilities are often related to risks and are calculated on the basis of mathematical models involving many assumptions and complex calculations. They have no empirical equivalent as, e.g., the independence between technical units of a complex system is a mere claim, not even an assumption, so that the whole model and the probabilities derived from it have more or less the character of a scenario in the sense of what-if. Furthermore, if we had data on small probabilities, they would not be sufficient. For a probability of 10^{-5}, a (random!) sample of 10000 is not enough to provide a sufficiently precise estimate of this probability so that is not possible to validate such small probabilities by data (see Batanero and Borovcnik, 2016). Historical approaches such as Borel in 1908 have been to ignore probabilities smaller than a threshold (such as 10^{-8}, see Borovcnik and Kapadia, 2014).

For risk, such small probabilities are not rare. For considering personal risks, a comparison of several actions might still orientate. If risks are shared among different stakeholders as is done in the health system, the difficulties to share the risk calculation increase considerably (see Borovcnik and Kapadia, 2011a, 2011b). If the risks are a projection into the future – something that *might* occur – considerations of risk in society get cumbersome. We illustrate matters by an example: The case of bovine spongiform encephalopathy (BSE) or mad cow disease is really serious and has aroused much public attention in the early 2000's. However, its background lacks any empirical justification; in fact, the prior probability of BSE is very small and it emerges simply from a speculation and not from data (see also Beck and Bornholdt, 2010). The difficulty with such societal risks is that small probabilities are combined with threats of an enormous damage. On the one side, the small probability has no empirical validation, on the other hand, the catastrophic scenario for the future raises fears and several stakeholders play with that but are also responsible in case that the negative outcome occurs. About the probability of many of the ubiquitous risks, we have no reliable data (that would emerge from independent experiments) so that we only can speculate about their relevance.

2 *Conditional probability.*

Conditional probabilities are not symmetric. The direction between the condition and the statement of interest is essential. While there is a qualitative symmetry in the sense "the more the statement *A* increases the probability of *B*, the more *B* increases the probability of *A*", the quantitative interrelation is not symmetric: there is no such relation as $P(A|B)=P(B|A)$; there is no general rule about the value of the reversed direction of the conditional probability $P(B|A)$ that would relate it to $P(A|B)$. For example, the probability of a positive result of the mammogram T^+ is high given that a woman has breast cancer (*C*): e.g., $P(T^+|C)=0.96$. However, neither is the reverse conditional probability $P(C|T^+)$ equal to 0.96 nor is there a fixed value for it; in fact, this probability depends on the so-called prevalence of the disease, i.e., on $P(C)$.

It is paradoxical that the probability for a person depends on the subgroup to which it belongs and may differ by the decision to which group the person is attributed so that there is no probability for that person at all. For example, the probability to have a specific disease after a positive medical test (indicating the disease) depends on the circumstance whether this person can be perceived as average (as if it had been selected randomly from the population for the medical test) or if it belongs to a subgroup with a special risk (age, gender, behavioural risk, genetic risk, etc.).

For many people, this dependence is motive to re-interpret the probability to conceive some disease as a consequence of something they do or are: a smoker might invoke that he belongs to a subgroup that is not affected as his grandfather was a heavy smoker who became very old and did not die from lung cancer.

Another consequence is that a systemic consideration whether to introduce a specific screening test for prevention of a disease (say mammography for breast cancer) can lead to an optimal solution that even would harm several individuals (by the collateral damage). What is good at the systemic level needs not to be optimal for individual persons. For example, the subgroups of 40 to 50 year old women as well as the 60+ women might not have a benefit of the screening: for the first subgroup, the false positives would lead to too much harm and damage while for the second group the detection of cancer at an early stage would lead to an unnecessary operation as the cancer in situ (sleeping) would not have grown so fast that it would have affected the woman's life (as cancer does not grow so fast in older people).

Furthermore, conditional probabilities can be interpreted as having causal and random perceptions. The direction between events in $P(A \mid B)$ may be interpreted by time (earlier, later) by cause (cause, effect), or by indication (symptom, status). Depending on such connotations, the interpretation and the acceptance of the values of these probabilities may vary and be unstable in a person's perception. It is a great step forward in statistical literacy to recognise that conditional probability covers all these aspects (for a thorough analysis of the concept of conditional probability, see Borovcnik, 2012). It is essential for probabilistic thinking to relate the reversed conditional probability to the prior probability of the various states (here to have or not to have breast cancer).

To evaluate risks, it is much easier to compare relative risks, or relative outcomes; in the case of mammography, to compare the scenario of introducing the scheme for as many women as possible with the alternative of not introducing it. On the other end, there is the decision of a woman (of a specific subgroup such as age) to undergo screening or not. However, all the collateral damage that is possible has to be included in such a balanced calculation: effect of biopsy, unnecessary operation, follow-up what would happen if a positive test would be neglected (would the cancer really grow, at which speed, etc.). Such data are not provided and partially will just never be available so that a rational deliberation of the risks seems difficult.

3 *Concepts such as correlation building probabilistic evidence*.

Causal interrelations are easy to understand and accept. However, in the real world the connections between variables are less clear and tight. Correlation (and association) have been developed to measure a kind of degree of dependence, a non-causal dependence. The relationships that are modelled are described by mathematical functions with no intention to state that there is a functional or a causal relation between the variables under scrutiny. The historical context of correlation was heredity and the desire to prove that intelligence is hereditary (see Batanero and Borovcnik, p. 152). If tall fathers have tall sons and shorter fathers have shorter sons and if such a relation is empirically found, then it remains to have a convincing concept to measure that co-relation. Obviously, Pearson and Galton had to investigate body length rather than intelligence as at 1870 there have not been such construct variables to conceptualise intelligence. Correlation was designed to measure the strength of the connection between heights of fathers and sons in order to prove heredity of intelligence.

If a correlation coefficient between variables is high, this does not show causal interrelations between these variables. It only indicates that connections between the variables may be found by a well-designed empirical study to detect either confounding effects (other variables blurring the co-relation) or to corroborate a hypothesis between the variables under scrutiny. It is very subtle how correlation (association) measures interrelations that are beyond causal or mathematical connotation. The complexity asks for simplification; rules of thumb would state that a correlation larger than 0.30 is at least intermediate, larger than 0.70 is strong. Thus, if a correlation coefficient is higher than 0.70, this is "strong evidence that …"? Furthermore, there are many investigations done, which are not well-controlled. They are prone to encounter confounding variables, for which no data have been collected. However, correlation can be increased, generated, or even changed in sign by other variables; one famous example for such phenomena is the Simpson paradox (see also Batanero and Borovcnik, 2016). If one has no data about such other variables (that are potential confounders), no such follow-up investigation is possible.

For risk issues, again, matters accumulate as generally hazards (potential causes of an adverse potential event) are not easy to measure and are often only open to investigation after the usually rare (!) adverse event has already happened. If the investigator would wait in an experiment for this event, it would never happen. That means, one has not the opportunity to design a balanced, randomised experiment (exposing the ones to the hazard and controlling the others not to be exposed to the hazard, which would also be precluded for ethical reasons). Remains that one has only a retrospective analysis by correlation or by different odds for the hazard that are – like reversing cause and effect – reversed to conclude from a higher exposition to a specific hazard (heavy smoker, e.g.) among those who suffer from the adverse event (lung cancer) that the hazard (heavy smoking) is "causing" lung cancer: "there is strong empirical evidence that heavy smoking causes lung cancer". That does not convince heavy smokers.

With risk, the complexity of analyses and involved concepts again asks for simplification and from the not well-understood results of empirical investigations only those are "accepted" that fit to the personal needs of the individual or the benefit of the institution. That is, in health issues, for example, the stakeholder of the medical institutions urges for the installation of screening schemes for various diseases especially early detection for cancer while the individuals react to fear or carelessness rather than a deliberation and weighing of options and if they start to weigh their risks they do not get the required information. Remarkably, the health institutions do not provide more sophisticated recommendations (different for the various subgroups) and still underplay the role of adverse side effects of the screening measurements as the effects will only turn out much later (they may turn out after more than twenty years).

3. Misconceptions and heuristics (strategies) in probability situations

We refer to milestones in research on misconceptions and classify misconceptions intending to relate them to archetypical general strategies of human beings.

Piaget and Inhelder (1951) is the earliest study on children's understanding of probability. Their work focused completely on the APT theory of probability using tasks with traditional devices (spinners, urns) with a small number of children in a longitudinal study. Young children, they found out, cannot distinguish between certainty and uncertainty and adequate handling with probability is possible only if the children have reached the stage of formal operations (after age 12). Fischbein (1975, 1987) based his approach on FQT and SJT and showed that positive incentives in the learning process would bring the pupils to a deeper understanding, especially when the barrier of verbal understanding is lowered. In general, Fischbein noticed that pupils may decrease in understanding probability in favour of a primitive causal understanding; he explained this phenomenon by the focus on causality in science and schooling.

The next major development was by Kahneman and Tversky (1972). Their extensive experiments with adults revealed serious deficiencies, which they explained by ad hoc heuristics. The initial work also showed a weakness in dealing with Bayes problems where people tend to ignore base-rates. Their research has found that private conceptions are fragile; people change their views by various cues in the tasks. Many researchers have connected to the paradigm of Kahneman and Tversky. In-line with Batanero and Borovcnik (2016), we describe some of these strategies (heuristics) and relate them to archetypical ways of behaviour. That may also explain why such strategies are so robust to teaching so that soon after leaving education these strategies dominate the way the majority of people think and act rather than they would use ways of thinking alongside the concepts that they have learned.

Availability. Probability is intuitively approximated to the ease of recalling relevant cases from memory. However, recording is biased (e.g., if adverse, an event is more likely to be registered) and the recall is biased (e.g., if emotionally laden, an incidence is more likely to be recalled). Archetypical here is the unstructured way of recording and recall, free of a neutral framework. We claim to use rational frames but is seems that we are much stronger associative thinkers than we would confess or like to be.

Equiprobability bias. According to this bias, people tend to judge cases as equally likely. Lecoutre (1992) was one of the first to name that tendency in people. The idea of fairness meets a deep desire in human nature (equity, equal chance); to attribute equal probabilities is another way to express fairness. The other way round, randomness is used to make a decision fair to all that are involved. From ancient times, we see a tendency to get rid of the responsibility for a decision by handing it over to a random device, which in turn corresponds to God’s decision (Borovcnik and Kapadia, 2014). Using randomness seems to have a great overlap with being fair so that randomness is often erroneously matched with equal probabilities.

Control of the future. Probability deals with uncertain situations relating to the past and to the future. To predict the future (and to decrease the uncertainty of it) seems to be a basic archetype of thinking that is met by models of causal connections as investigated in physics. Probability is a competing view on such situations, which is less direct and convincing. The history of science is signified by a mutual relation between causality and randomness. A different approach to foresee the future is to explore God’s will by divination. Konold’s (1989) *outcome orientation* fits to this deep human desire to control the future: people reinterpret given probabilistic information as a direct tool to predict (with certainty) the exact outcome of the next experiment.

Representativeness. The probability of a single outcome is equated to the probability of the group of similar outcomes (of which the specific outcome is a representative member). According to an archetypical human desire, group features are transferred to individuals that belong to that group. Fashion is one example; group features may let the individual member benefit from the group (team spirit, e.g.). This "group transfer" relates the representativeness heuristics to an archetypical strategy. The following example is from Batanero and Borovcnik (2016, p. 106)

> For the state lottery, some combinations of numbers have many similar combinations and "belong" to a large group; others have only a few and belong to a small group. In selecting one, some people believe that it increases their chances to win if they take a combination from a large group as if it would inherit the greater probability from the *group*, to which it belongs. […] However, in doing so they ignore that they win only if their *specific combination* is drawn.

Anchoring. Probability judgements are influenced by additional (possibly irrelevant) information given or highlighted recently. Anchoring is a fast-and-frugal way for processing information without analysis, which helps to react faster than others. And generally, the first determines the pace whether this is good or not. To be the motor of action seems to be more important than to analyse situations in a lengthy manner. This coincides with the capacity of leading a group and the leadership would get doubtful if a reaction takes too long.

Patterns. People perceive regularities, geometric or arithmetic patterns, or the cycle of the sun and generalise these patterns. It comes to no surprise that people focus so much on patterns of random sequences and draw awkward conclusions from them. If the sequence is random, any pattern may be seen if the series is short. The randomness means that specific patterns do not occur. With hindsight, a pattern may be recognised and taken as proof that randomness is violated. However, it is a difference to notice a pattern in given data and to formulate a pattern and then observe it in data.

Personal experience and information. People like to speak about individual cases and draw upon their personal experience. That anecdotal information is vivid compared to the *general* distribution of data about a (random) variable. The difficulty may be located in the point of time when to use personal information and when statistical information is better. There is no general rule for switching and, by no means, is statistical information always relevant for a person as it might belong to a special subgroup (e.g., low or high-risk group) or it might differ from the collective.

4. Probability and utility

The experiments of Kahneman and Tversky (1979) show how people are influenced in their perception of probability by the impact of the outcome. We discuss and re-interpret the experiments and add a long-run version to illustrate the logic of risk. We complete the exposition on risk by an example of Borovcnik (2015) to highlight the criteria that are useful for deciding between several options.

In experiment 1, Kahneman and Tversky (K/T) let people choose between options a_1 ($ 1000 for sure) and a_2 ($ 2500 or $ 0 with probability ½ each); experiment 2 compared the same amounts but with negative sign for losses. In the "winning" situation of experiment 1, people preferred the 1000 (a_1) while in the "losing" situation of experiment 2, they took (a_4). However, according to the expected value of the options, they should choose options a_2 respectively a_3 (cells in grey in Chart 1, with an expected value of 1250 and –1250).

Chart 1: Win and loss of the options in both Kahneman & Tversky experiments, its reformulations, and the repeated single decisions version of it.

	K/T view		A different view		Many single decisions	
	Experiment 1:		Experiment 1^*:		Experiment 1^s:	
Future [$]	Option a_1	Option a_2	a_1^* Do nothing	Option a_2^*	Option b_1	Option b_2
	1000	2500 with ½ 0 with ½	1000 For sure	1500 with ½ –1000 with ½	1	2.5 with ½ 0 with ½
	Experiment 2:		Experiment 2*:		Experiment 2^s:	
Future [$]	Option a_3	Option a_4	a_3^* Do nothing	Option a_4^*	Option b_3	Option b_4
	–1000	–2500 with ½ 0 with ½	–1000 For sure	–1500 with ½ 1000 with ½	–1	–2.5 with ½ 0 with ½

Source: Kahneman and Tversky (1979)

Kahneman and Tversky became famous for their interpretation of the deviation from the optimal choice: in winning situations, people are risk-aversive while in losing situations they are risk-seeking.

We explain the behaviour with a different view (experiments 1^* and 2^*): in experiment 1^*, the person already has the amount of 1000 (do nothing, option a_1^*). The additional amount to win is only 1500 but the person faces the risk to lose all. It may not pay for many persons to risk the fortunes (of 1000) for the little extra. A person that has already

1000 has to be paid more to seek the risk to lose all. That explains the observed behaviour much better by reference to utility. In experiment 2*, the person has debts of –1000; the person wants to get rid of these debts and, therefore, seeks the risk. It does not matter that by the end, he could have even –2500 as debts (this does not seem to be really worse than the initial situation); he seeks the chances to get to 0 balance. Again, this explains the observed behaviour much better as one can see that it makes sense to seek the risk in the second experiment but to avoid the risk in the first.

Much worse. If we split the decision in 1000 single decisions as is done in the s-version of the experiments (last column of Chart 1), the clear optimal decisions coincide with the best decisions in the original experiments according to the expected value criterion. Option b_1 delivers 1000 if applied a thousand times. With b_2 we can calculate the probability to get more than 1000 (i.e., win more than 400 times), which is $1–10^{-10}$, which is nearly certain. That means if a person makes comparable decisions repeatedly, the logic changes. What is even more striking, the separation between the options is much easier. There is nearly a probability up to certainty that the chosen decision is better than the other one while the one-off decision always makes difficulties.

The following task from Borovcnik (2015) highlights the risks inherent in a decision under uncertainty: Find an optimal number of copies to produce for a new weekly journal. The demand depends on random factors; the distribution of it is supposed to be sufficiently well known by market research and is displayed in Chart 2, which also refers to the production cost. The price of one issue is $ 1.60. How many copies should be produced if there is only an option between a_j = 1000, 2000, ..., and 5000?

Chart 2: Distribution of the demand and the cost of the several options related to the number of copies.

Demand d_i	1000	2000	3000	4000	5000
Probability p_i	0.40	0.30	0.20	0.06	0.04
No of copies a_j	1000	2000	3000	4000	5000
Cost $C(a_j)$	2000	2200	2400	2600	2800

Source: Borovcnik (2015)

For each decision, we can calculate the probability distribution for the profit. For a_j = 2000, we can sell all of them (per 1.60) if the demand is greater equal 2000 (which has a probability of 0.60; if we subtract the cost of 2200, we get a return of 1000; however, with 0.40 we sell only 1000 so that we lose –600. We take a risk of such a loss; in exchange for that, we have an *expected* return of 360. If we want to avoid the risk, we can investigate what happens if we only produce 1000 copies. However, this does not seem to be attractive at all as we would be able to sell only 1000 copies, which leaves us with a sure loss of –400 whatever the demand will be.

On the other hand, if we produce 3000, then we risk an even greater loss of –800 in case that only 1000 copies are sold, which happens with probability 0.40! Yet, in the other cases, we have a net return of 800 (2000 copies sold) and 2400 (3000 copies sold). That makes all in all an expected return of 640, which is much higher than for 2000. See Borovcnik (2015) for a matrix representation of all the calculations and the details. The decision for 3000 copies yields the maximum for the expected net return. The attempt to minimise maximum loss, on the other hand, leads to 1000 copies as the best decision; a decision that brings a loss whatever the demand will turn out.

There are several things to learn from this task: Firstly, the best decision depends on the criterion that is used. Useful criteria are the expected value of a decision and the maximum loss of a decision; the first is to maximise, the second to minimise. Secondly, sometimes, a decision is clearly inferior. Thirdly, to win more "on average" asks to accept a higher risk in the sense of facing a potentially higher loss; the probability of such losses need not be negligible as our task shows. The features of risk discussed here have a general validity.

5. Clarifying the purpose rather than the character of probability

To make probability meaningful is a challenging task for teaching. Too many diverging conceptions dominate the way students think, which may lead a deliberate discussion on the features of probability to contribute to confusion rather than it could clarify the nature of the concept. Spiegelhalter and Gage (2015) suggest using metaphors to carry the meaning. Borovcnik and Kapadia (2017) focus on paradigmatic issues where probability can be used rather than clarifying what probability really is. They state:

> The axiomatic foundation left the true character of probability open with regards its application, as in other areas of mathematics. For example, a line in geometry may be straight as in the plane or curved as in space. Perhaps it helps to understand the concept of probability and probabilistic thinking by analysing the purpose of applications of probability, i.e., highlighting the key features of specific situations where probability is used to solve a problem.

According to Borovcnik (2006), probability can serve the following five purposes.

1 *To make decisions under uncertainty transparent*.

To understand a simple probability statement $P(A)$, it takes quite a lot of conceptual understanding. To evaluate such a probability in a specific application, not only one has to understand the character of probability but also to find a value that fits to the situation. The information on this probability may be FQT-based, it may be derived on equally likely cases in the sense of APT, or it may be arbitrarily guessed (it should form some kind of knowledge so that it could be placed under SJT). Regardless of the character of probability, once one has a value for it, it could be used for making a decision transparent. For example, in taking out full-coverage insurance for the car, one has to evaluate the various possibilities (extent of damage) and their probabilities (see Borovcnik, 2015). If only no accident and total wreckage are considered in a first model and if these two possibilities are compared by probabilities, then an expected value of the decision of taking no insurance can be calculated. This value can then be compared to the premium in order to find the better (less costly) option. The benefit of this approach is that the influential parameters become transparent: if the probability of total wreckage is 0.05, then taking out the insurance is better, if it is 0.01, no insurance is better. The final decision gets *transparent*.

2 *To express qualitative knowledge by probabilities and update it by data.*

Probabilistic information is always temporary. If it is qualitative (SJT), then data should be sought and integrated into a revised judgement of the probability under scrutiny. The Bayesian formula is the proper algorithm for calculating such updates. The example of the medical diagnosis paradigmatically shows the inherent features of judgments under uncertainty: a prior knowledge in the form of a probability (for a disease) has to be updated to provide a posterior probability when the result of a medical diagnosis becomes known. For a decision what to do, it is essential that the errors are somehow controlled. To understand the rationale of diagnosing, to judge the used method of diagnosing, a deeper understanding of the formalism and the character of its components is essential. Only this gives a clear view on what empirical evidence is and that indicative knowledge can be used for improved judgements.

3 *To judge risks*.

To judge risks requires to combine the probabilities and the implications of the outcomes (be it measured in utility or in money) in order to derive an expected value of a risk. While it is difficult to attach a proper meaning to this absolute value, the comparison of the risk of two actions would reveal relevant differences. Anyway, for a decision, it suffices to know which one is better (relative to some further assumptions) regardless of the value of the risk (see Borovcnik, 2015). While it is easier to *compare risks* than to interpret a single risk, there are still more complications: first, comparisons need additional criteria (expected value is only one); second, people are not consistent in such comparisons as the experiments of Kahneman and Tversky have shown.

4 *To use resources better*.

If we use a technical device for a mission to a comet, a reliability calculation might reveal that its lifetime does not suffice for the whole time of the mission. We could think of taking two or even more. How many we take with the spacecraft depends on the weight we can allow for and on the cost. There should be a balance between a large reliability that the device is functioning throughout the mission and the cost of redundancy. We can use probability calculations based on qualitative initial values of the probability of one device to survive the mission to find a solution. Another example is to sell more tickets than one has places as airlines do; for details, see Borovcnik and Kapadia (2017). Borovcnik and Kapadia (2011c) give a series of examples where probability can be used to find a better solution for a practical problem by introducing a probabilistic perspective into the model.

5 *To fix prices in the exchange of certainty and uncertainty between two partners*.

The exchange of money and security seems to provide a basic paradigm for probability. Insurance is signified by the exchange of two different situations (for details see Borovcnik, 2015). The insurance company has no insecurity about the future (potential accidents) of a person's car while the person faces an unsecure future (such as an accident). The insurance contract implies that the two stakeholders change their position with respect to insecurity: the company takes over the monetary impact of a potential accident; the person is freed of the possible financial implications of an accident. The exchange, however, costs money. Using probability allows us to fix a price for the exchange of uncertainty for certainty by a contract.

Conclusions

Spiegelhalter (2014), a renowned specialist in risk, agrees that probability is difficult:

> I often get asked why people find probability so unintuitive and difficult. After years of research, I have concluded it's because probability really is unintuitive and difficult.

One can find guidelines for risk in Borovcnik (2015) or Borovcnik and Kapadia (2017):

> Historical struggles provide a valuable orientation. While empirical research about how people think and how successful teaching programmes have been helpful to improve teaching plans, we should not lose sight of key concepts and strategies from the past. One key lesson from history is that probability has always been a pluralistic concept and has drawn its meaning from the interplay

> of its interpretations. Simplification is a basic ingredient of teaching. However, sometimes it may undermine the complexities for certain concepts. The rather narrow focus on the frequency interpretation ignores some facets of probability. Simulation provides a *solving technique* but does not help to model a situation and discourages probabilistic thinking.

To consolidate probabilistic education, the ten assertions from Chance Encounters (KAPADIA and BOROVCNIK, 1991) still provide an essential orientation. From the perspective of risk, we should note that people tend to simplify matters once they get too complicated. Teaching approaches meet the challenge to build probabilistic literacy and stochastic thinking without going too far in the educational compromise to replace a theoretical concept of probability by material visualised icons. After all, probability literacy is tightly connected to literacy in risk.

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