

On an axiological dimension of rigour in school mathematics

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Abstract. In this paper, we seek to identify some elements of a professional teaching *logos* regarding rigour and point out an axiological dimension of this *logos*. In contrast to a more autonomous approach to rigour in the field of mathematical production, many teachers' justifications to their expectations about rigour are based on ethical or aesthetical grounds. The exploration of a possible link between this axiologisation of rigour and the social reproduction of the profession of mathematics teacher is also considered.

Résumé. Dans cet article, nous cherchons à identifier quelques éléments d'un *logos* professionnel des enseignants de mathématiques en relation avec la rigueur ; nous mettons en évidence une dimension axiologique de ce *logos* dans la mesure où nombre d'explications fournies par les enseignants concernant leurs attentes de rigueur sont fondées sur des considérations éthiques ou esthétiques – à la différence de ce qu'on peut observer dans le champ de production mathématique. Nous effectuons également une première exploration d'un lien possible entre cette axiologisation de la rigueur et la reproduction sociale de la profession de professeur de mathématiques.

O stern mathematics, I have not forgotten you since your learned teachings [...]. I made use of [them] when disdainfully rejecting the ephemeral joys of my short voyage, and in order to turn away from my door the sympathetic yet mistaken offers of my fellows. [...] I made use of them to outwit the pernicious wiles of my mortal enemy [...].

Lautréamont, *Songs of Maldoror*.

Rigour is commonly held as a defining feature of mathematical practice, and many works (see e.g. Balacheff, 1982; Balacheff, 1987; Duval, 1991, 1992-1993; and Gandit, 2004a; Gandit, 2004b for a perspective on teachers training) have considered the issue of teaching rigour at school: based on a thorough study of the possibly significant differences between argumentation and demonstration, between demonstration and proof, etc., these works often question the possible effects of didactical contracts on the learning of rigour (see esp. Gandit, 2004a; Gandit, 2004b). Paradoxically, little thought is given to the possible transpositive effects on the meaning and functions of rigour. The influence of a Weberian identification of ideal-types of rigour, apparent in many attempts to grasp the main features of demonstration or proof, induces a blind spot: rigour cannot only be defined, in a Carnapian way, by the collection of all situations where it is involved; the meaning of rigour, rather, could be the resultant of the different forces that oppose each other in the struggles at stake in the mathematical field. It is hardly conceivable that struggles about rigour in the school field exactly match those in the field of mathematical production. Therefore, meanings of rigour in both fields may differ significantly.

Our goal in this paper is twofold: perform a primary sketch of the possible meanings of rigour in both fields by exploring some of the functions of rigour within them; by exhibiting an

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axiological dimension of rigour at school, we also give a tentative account for the differences between both uses of rigour, as we emphasise on the heteronomy of the school field. This interpretation is supported by the careful examination of resources related to the selection of future mathematics teachers: they are shown to involve a wide range of ethical and aesthetical expectation that indicate both the dependence of the school field with respect to other social fields and the distance between these expectations and the actual functions of rigour in the mathematical field. We conclude by pointing out the possible didactical consequences of field effects, as in this case, where functionality of rigour is lost in the process of didactic transposition from one field to another.

1. Sociological fields, position-takings and the praxeological model

According to Pierre Bourdieu (1992, 2001), the structuration of a domain of social practice into a *field* is characterised by the fact that the agents involved in this practice are related to one another through a common share of important stakes, at the core of struggles that oppose them—as well as they are united in their belief in the importance of these stakes and struggles. In such a field, agents are located at different (social) *positions* according to their social trajectory and current social position—and this represents a first structuration of the field, which can oppose *e.g.* teachers originating from lower or higher social classes. A second structuration of fields can be found in the co-existence of different forms of *position-takings* (in French, *prises de position*: the agents' actions, which can be related to their position in the field) by the agents in their involvement in the struggles, in their beliefs or their practices, etc. A traditional assumption (supported by sociological investigations up to now) made in sociology of fields is the homology of both structures: the structure of *positions* can be used to predict, or understand, the structure of *position-takings*, which provides a means to explain some effects, specific to a struggle in a field, by relating them to social trajectories and positions of the agents in the field. Of course, in order to perform a rigorous analysis of what happens in a field, it is necessary to involve in the process of investigation a model of practice precise enough to allow subtle distinctions between position-takings. Our claim is that the praxeological model provided by the anthropological theory of the didactic (ATD, see Chevallard, 2007) gives important tools for the description of position-takings in general, and in the school field in particular.

In ATD, indeed, it is generally accepted that any human action is the realisation of a *type of tasks* T (blow one's nose, solve a second degree polynomial equation, ride a bike, etc.) by means of a *technique* τ (use a handkerchief, compute the discriminant, only get on the bike once it has reached a minimum speed, etc.); this technique can usually be justified (even –and often– in an incomplete way) by uttering a speech (*logos*) about the technique (*technè*), that is a *technology* θ (you have to use a handkerchief for sanitary reasons, the discriminant comes from the canonical factorisation of a second degree polynomial, you may fall from the bike if the velocity is not important enough, etc.); these justifications are always grounded on a common knowledge, often unconscious, a *theory* Θ . The *praxis* $[T/\tau]$ and the *logos* $[\theta/\Theta]$ constitute a *praxeology* $[T/\tau/\theta/\Theta]$. A way to analyse position-takings in a field is therefore to identify the many types of tasks the agents in the field have to realise, then to describe the actual techniques they use for this purpose and to exhibit elements of the *logos* they use to justify the efficiency of their techniques. One can assume that oppositions regarding the types of tasks, the techniques,

etc., will contribute to the structuration of a field. An important issue in modelling practice by means of this praxeological model is to avoid the production of artefacts by an unnatural separation between “know how” and “know what”, or between “knowledge” and “practice”. In the case we consider here, we can imagine that directly asking teachers what they think about what rigour is or should be at school, would not lead to the same material as the exploration of spontaneous speeches elaborated by teachers when they analyse or evaluate their students’ rigour.

In this work, we chose to identify elements of the *logos* of teachers, relative to rigour, by exploring two kinds of materials: an internet forum where teachers ask for advice for the evaluation of rigour; a “handbook of sound writing” written by a mathematics teacher for his students. In both cases, the didactical context favours the production of *logos* (and not only the exhibition of techniques) strongly related to the *praxis* teachers expect from their students. In order to emphasise on some characteristics of the teachers’ *logos*, we first sketch what could be a social history of the mathematical field with a focus on the emergence of rigour; in this perspective, we assume that further work in this direction could assess the relative importance of different position-takings about rigour and their relation to distinct positions in different historical states of the mathematical field: for instance, in the XIXth century, rigour was a means to tackle difficult mathematical problems related to mathematical analysis; it was also used in a didactical way for the production of handbooks, with the aim to establish the legitimacy of mathematical analysis by giving it algebraic “rigorous” grounds; the formalist standpoint on rigour was yet another position-taking that appeared progressively. We proceed to give a rough account of these historical position-takings in the mathematical field to allow a better understanding of the essential difference with position-takings about rigour at school.

2. Mathematical rigour and struggles in the mathematical field

It is commonly held that rigour is an intrinsic (though historically produced) feature of mathematics. The naïve vision of rigour is probably based on the idea of a directional history of mathematics that heads towards a specific kind of rigour, an achievement by itself which allows for a more precise and reliable practice of mathematics (the German mathematician Hermann Hankel said, “In most sciences, one generation tears down what another has built, and what one has established, the next undoes. In mathematics alone, each generation builds a new story to the old structure”, cited by Grabiner, 1974, p. 354). Even when this caricature is not supported, rigour is assumed to be a cultural standard that holds at school in the same way that it holds in the field of mathematical production. To put it differently, it is generally believed that the word “rigour” has a straightforward and unique meaning, or at least that the uses of rigour are the same wherever mathematics are being produced or used.

Our aim in this section is to give a brief account of the possible reasons why new standards of rigour appeared in the nineteenth century, and to understand their rise in relation to some of the main stakes in the struggles that animated the field of mathematical production at this time. This should enable us to understand the *functions* that were originally assumed by rigour –and, maybe, to measure the distance with the functions of rigour at school.

According to (Grabiner, 1974, p. 355), the “change [*in calculus, during the eighteenth and nineteenth centuries*] was a rejection of the mathematics of powerful techniques and novel

results in favor of the mathematics of clear definitions and rigorous proofs”; this change did not occur without pain and required that many positions taken by famous mathematicians be progressively abandoned “in favor” of other positions. This struggle between ancient positions (based on the reliance on the power of symbolism and the social need for efficient techniques) and new ones is reflected in what Grabiner calls a “change of attitude”: “there had to be a change in attitude. Without the techniques, of course, the change in attitude could never have borne fruit. But the change in attitude, though not sufficient, was a necessary condition for the establishment of rigor.” For instance, the transition from algebraic techniques based on inequalities, used for the approximation of polynomial roots (and the quantification of the error made in the process), towards the modern definition of the limit (where the choice of the Greek letter epsilon can be understood in reference to the word *error*), necessitated both the existence of these approximation results and an inclination to consider things on theoretical grounds. Sociological and historical conditions are put forward, such as the accumulation of isolated results, which made necessary an effort of systematisation, or the professionalisation of the mathematics field, with the growing importance of teaching and the emphasis on clarity and systematisation it entailed.

However, considering things at this level of generality strengthens the commonly held picture of a relatively united mathematical community; paradoxically, it also relies on an individualistic conception of history, where changes occur based on changes of “attitudes”. The expression “change of attitude” is misleading in that it is generally used to describe an individual psychological evolution. In this case, there may have been individual “changes of attitude”, but, more important, different attitudes must have co-existed and struggled against each other. The very fact that mathematics could grow as a highly demanding standard in the scientific field shows that it had to be crossed by high intensity struggles around important stakes, since such are the conditions for the emergence of proper rationality (Bourdieu, 2001).

In this sense, the stake of struggles such that the opposition between Kronecker and Weierstrass is not necessarily to lead someone else to “change his or her attitude”, but, rather, to *impose one’s attitude as the legitimate one*. Scientific choices (the choice to reject Lagrange’s approach to derivation, based on Taylor series, is a position-taking at the time of Cauchy, as can be understood in the mixture of proofs and arguments he gives in his *Avertissement*² to the *Résumé des leçons données à l’école royale polytechnique sur le calcul infinitésimal*) are position-takings in the mathematical field and legitimating such position-takings is an important stake. Investigations on problems related to trigonometric series could have led to abandon analysis as an ill-founded sector of mathematics and to repel it towards the borders of physics (which was, undoubtedly, an important position-taking in early nineteenth century mathematics)—the only way to make it tolerable in the field of mathematics was to give a precise explanation of some of the paradoxical issues raised by the study of trigonometric series (Grabiner, 1974). Choosing to study such problems in the frame of mathematical practice is a choice—it could have been avoided or neglected.

But the very tools and techniques used to tackle these difficulties are also the expression of a *choice*, of a position-taking. Infinitesimals, associated to a dynamic conception of limits that is still lively in Cauchy’s definitions of limits and continuity (if not in his proofs), gave way to an

² See <http://gallica.bnf.fr/ark:/12148/bpt6k90196z/f12.item.r=lagrange.zoom>.

arithmetic, static approach: that of Weierstrass—but why should arithmetisation of analysis appear as a satisfactory outcome? Weierstrassian rigour was certainly the most elaborate answer to the critics of analysis and it is not surprising that this answer was formulated in the language of arithmetic, probably because in the state of the mathematical field in the nineteenth century, the definition of admissible weapons in the struggle for the definition of acceptable mathematics gave a prominent part to arithmetic: symbolically dominant in the field, arithmetic was the common noble language to which any potential mathematical branch should first be translated. Following Gert Schubring (2001, p. 300), “communication constitutes the elementary act of science. On it are based not only scientific teaching and learning, but also scientific invention. Scientific invention is always an act of communication with a specific audience. For communication to be efficient, a common language and a common culture must be available.” The invention of a new language was, therefore, a necessary step in the emergence and autonomisation of mathematical analysis as a new subfield of the mathematical field; analysis, to be accepted as a legitimate mathematical practice, and not only as a necessary tool for dealing with physics problems, had first to impose its legitimacy in the very words of the dominant mathematics of the time (algebra, arithmetic, geometry) —as any literature from a dominated nation has to be translated into the contemporary dominant language in order to exist (Casanova, 2016). The impossibility to give a sound geometric proof of the consistency of mathematical analysis could only lead to the exploration of algebraic or arithmetical justifications; producing an arithmetical model of analysis was a way of imposing it as a legitimate tool to use in dealing with mathematical (even arithmetical) problems.

In this struggle, the production of didactical resources by mathematicians is one of the key weapons, and the least reason for this is not the difficulty to be understood by mathematicians holding positions spaced apart from theirs in the field. H el ene Gispert (Gispert-Chambaz, 1982) quotes a letter from Mittag-Leffler to Hermite in which the Swedish mathematician, a former student of Weierstrass, complains that:

Les³ allemands eux-m emes ne sont pas en g en eral assez au courant des id ees de Monsieur Weierstrass pour pouvoir saisir sans difficult e une exposition qui soit faite strictement d’apr es le mod ele classique qui a donn e [*sic*] le grand g eom etre. [...] Tout le mal vient de  a que Monsieur Weierstrass n’a pas publi e ses cours. C’est vrai que la m ethode de Weierstrass est enseign ee maintenant dans plusieurs universit es allemandes, *mais tout le monde n’est pas pourtant l’ el eve de Weierstrass ou l’ el eve de quelqu’un de ses  el eves* (Gispert-Chambaz, 1982, p. 33, my emphasis)

At some point, it became necessary to make the new language of analysis widespread. As Gispert puts it:

³ “German people themselves are generally not aware enough of the ideas of Monsieur Weierstrass to be able to grasp a presentation strictly made after the classical model given by the great geometer. [...] All the harm comes from that Monsieur Weierstrass did not publish his lecture notes. It is true that the method of Weierstrass is now taught in several German universities, but, nonetheless, *not every one is the student of Weierstrass or the student of one of his students*” (my translation, my emphasis).

Lorsqu'avait⁴ paru le livre de Dini de 1878, Cantor avait écrit à Dedekind dans une lettre du 18 janvier 1880 [qu']il faudrait “le traduire et l’adapter”, car “justement chez nous un ouvrage comme celui de Dini est nécessaire, et je le sens presque à chaque leçon que je donne à mes auditeurs.” (Gispert-Chambaz, 1982, p. 33)

Indeed, winning the struggle requires that the language of modern analysis be provided not only to all the strugglers, but also to potential followers: providing students with textbooks is a way to legitimate a new set of position-takings. On the other hand, writing textbooks requires that further attention be paid to details, technicalities, in one word: to rigour.

In this respect, the historical movement is conceivably a dialectical movement between research and teaching: while mathematical investigation has driven ahead some important formalisation techniques, which were subsequently introduced in the didactical exposition of mathematical knowledge, on the other hand, important questions relative to the rigorous foundations of mathematics arose in the successive attempts to produce a satisfactory didactical presentation of mathematics (Belhoste, 1998; Gispert-Chambaz, 1982; Schubring, 2001).

Therefore, the notion of rigour as it emerges in the XIXth century is both a set of mathematical praxeologies (the use of inequalities for the abstract definition of limits is at the core of a whole set of praxeologies for the study of convergence of sequences, but also a reversed viewpoint on the nature of approximation processes, now used to prove existence theorems, as pointed out by Grabiner, 1974) and a set of writing praxeologies (which are partially designed in order to convince algebraists and geometers of the soundness of Weierstrassian mathematical analysis).

Thus, the question “What exactly is rigour?” can only be answered by studying the history of the mathematical field and elaborating further on the structure of the position-takings of mathematicians about rigour: the formalist viewpoint on rigour is only one of the possible position-takings and many mathematicians would reject it as, for instance, partly sterile (Thurston, 1994; Zarca, 2012). A specific position taken about a topic such as “what exactly is rigour” can only be understood by building the universe of all compossible position-takings (Bourdieu, 1992). The meanings of rigour in both the school field and the academic field can only be understood by means of a systematic comparison of the structures of the position-takings of rigour within them, which can only be conveniently achieved by studying the specific functions of these position-takings: not only ‘technical’ functions, but all kinds of functions relevant to the specific struggles of the field.

The former brief analysis may suggest that important functions of rigour in the mathematical field are the following (at least in the crucial nineteenth century): building answers to technical issues in the study of trigonometric series; systematising relatively isolate results; upholding the soundness of mathematical analysis by developing and disseminating a ‘common language’ of analysis; etc. These functions do probably represent part of the actual “meaning” of rigour in the nineteenth century mathematical field.

⁴ “When the 1878 book by Dini was published, Cantor had written to Dedekind, in a letter of January the 18th, 1880, [that] it should be ‘translated, adapted’, for ‘such a work as Dini’s is necessary precisely to us, and I can feel it in almost every lecture I give to my audience” (my translation).

3. A problem of the profession: teachers position-takings about rigour

The unquestioned universality of mathematical rigour leads to the definition of mathematics as a school of rigour, as can be found in the very title of a book (Zarca, 2012), in which the nature of rigour is not really questioned. More important to us, the identity of mathematical rigour in the field of mathematical production and in school is implicitly assumed in such a vision of rigour. The aim of this section is to give a flavour of possible position-takings about rigour *at school*, in order to contrast them to position-takings in the mathematical field as they appeared in the former section. While (Zarca, 2012) has a rather direct way of questioning, at the risk of inducing idealised pictures of rigour, we choose to base our analysis on a fair use of ATD. By exploring situations where praxeologies fail, or are disseminated, one allows technological components of these praxeologies to be further exposed. This way, true functions (*raisons d'être*) of these praxeologies can be revealed.

3.1. Rigour and evaluation of rigour in high school

Unsurprisingly, the issue of evaluation is an important source of questions about rigour on forums⁵: “How would you define mathematical rigour? How do you see whether a pupil is rigorous or not? For instance, a pupil can be rigorous in his reasoning but not in his writing..... how can we evaluate him correctly then?” (Yaw83, 2013, May 31). The subsequent discussion shows that the question itself is not questionable (no one points the irrelevance of the question) though it has no obvious answer. Some indicate a doubt as to how to distinguish between “rigour in the reasoning” and “rigour in the writing”: “rigorous in one’s reasoning but not in one’s writing, I doubt it exists...”. Thus, “most of time, reasoning interacts with writing or at least with the means of the resolution of the problem” and, consequently, “the [pupil] is expected to be able to elaborate a thought *and to share about it*. Failing which, there is no possible interaction and therefore no possible building of knowledge” (original emphasis).

Therefore, rigour is intertwined with “writing”, but not necessarily with the aim of “sharing about [a thought]”, even though this aim is insistently put forward by one of the participants of the thread. Indeed, the recurrences of this point in the thread discuss further the fact that sound writing allows for a better elaboration of thought and, therefore, helps *the pupil him- or herself* in his or her work, which is conceived as a solitary activity, not a collaborative one: “one needs a medium to produce a reasoning. Language is needed in order to articulate a mathematical thought, to let it reach its full potential”, or: “language is the⁶ medium of thought elaboration. A faulty language can hardly support an elaborate thought”, and, to conclude with a tentative definition: “Rigour is the art of choosing among all those ideas, and to formulate them following the rules of writing specific to the mathematical language”.

Yet, such statements of good intentions could not utterly hide a second aim –which, on the contrary, they justify: “Even if reasoning, or simply intuition, prove satisfactory... how not to feel some concern for the student as regards his mastery of mathematical language and, therefore, his ability to apprehend more complex problems. [...] The consequence is, it is

⁵ Unless otherwise specified, my translation.

⁶ Incidentally, a solitary use of language appears as the only conceivable *milieu*. The communicative function of language (and its importance in the study of mathematical questions) is underrated. See the previous section about the importance of communication in the mathematical field.

unclear *how far can one gloss over form to evaluate content*. How can we urge the pupil to improve this form... if we do not *value it properly?*" (my emphasis) Evaluating rigour, by evaluating writing, supports the acquisition of "noble" praxeologies that will provide the pupil with "a consistent medium for an elaborate thought". So that, faced with an example of a student achieving the expected outcome, through a "non-rigorous" writing which, nevertheless, shows up an acceptable reasoning⁷, the same participant to the thread claims that "it might be comforting as regards the child's ability to reason... Though, nothing proves that the correct outcome was not obtained by a stroke of luck. Wouldn't the marker, since he obviously knows the correct result and reasoning, be inclined (consciously or not) to 'project' his own reasoning on an erroneous demonstration that leads to a correct result? Eventually, he may suppose that the child understood... whereas he definitely did not." At risk of pulling the student on a precarious path or, to the least, not engaging him towards the more demanding path of rigour (not "valuing it properly"), the benefit of the doubt must be superseded by a reasonable practice of the 'malefit' of the doubt: lacking evidence to the contrary, a faulty writing must be understood as the demonstration of a faulty reasoning. An immaterial and therefore unperceivable entity, the reasoning is only grasped in its embodiment in a concrete writing. The spiritual cannot be rated; in its material evidences, however piecemeal, will it be judged.

For now, let us point out that rigour is opposed to luck ("stroke of luck"), benevolence (the teacher "projects" his own solution), and is a means to evaluate mathematical reasoning: this set of oppositions is a means to sketch a definition of rigour that matches its actual functions in the school field.

3.2. A particular class of position-takings: teaching rigour in *classes préparatoires*

The *classes préparatoires aux grandes écoles* (CPGE, two-year preparatory classes for the top-ranking higher education establishments, such as engineering schools, students aged 18 to 19) provide an extreme example of position-takings about rigour in the French higher education system. Though we may assume that even in this "total institution"⁸ (Darmon, 2015) a range of positions might be met, the specific constraints imposed in this institution probably induce a higher homogeneity of practices among teachers. Indeed, students in *classes préparatoires* pass final competitive exams after a two-year training: these exams are extremely demanding and are said to require a high degree of "rigour" in mathematical writing. The professional question we have met in the previous section is therefore particularly acute in the CPGE institution: how to make sure one's students have reached the expected level of mathematical rigour? It could be expected that official curricula for the CPGE⁹ (Ministère de l'éducation nationale, 2013, May 30) give some precise information as to the writing praxeologies required for the final competitive examinations; yet, a rough comparison between secondary school and CPGE curricula shows no major difference: while secondary school (pupils aged 11 to 15) curricula (Ministère de l'éducation nationale, 2008, August 28) emphasize on the elaboration of proofs

⁷ « Question: If I buy three bags of candies, I have 10€ left. Initially, I had 17.5€. How much is a bag of candies? Student's answer: $17.5 - 10 = 7.5 \div 3 = 2.5$. A bag of candies is 2.5 euros. »

⁸ Following Erwin Goffman (1961), total institutions are characterised by the relative seclusion of their subjects (jails, psychiatric hospitals, monasteries, etc.).

⁹ In this work, we mainly focus on the curriculum of the *mathématiques, physique et sciences de l'ingénieur* (MPSI) section, where mathematics is dominant.

(formalisation of sketched proofs into demonstrations is a large range aim, that should be realized only at the end of high school), CPGE curricula insist on the paramount importance of good writing and clear (oral as well as written) communication; however, neither curricula provides teachers with a precise account of writing praxeologies (to build in high school, to enhance in CPGE).

In this respect, reading lecture notes from CPGE teachers may give a more precise picture of the way mathematics teachers in CPGE practically answer the difficulty of the “teaching of rigour”. A “small handbook of sound writing” (Bertault, n.d.) was successful in the CPGE universe: many teachers webpages or students forums refer to it as it fairly elaborates further on usually implicit expectations. In this sense, we take the handbook as an illustration of a typical position-taking about rigour in CPGE (and, as such, as a radicalised version of common position-takings in school in general).

At the beginning of the document, the author upholds the idea of a close relation between rigour and writing: in his opinion, “sound writing” means “present one’s thought with clarity, that is *in an orderly and rigorous way*” (my emphasis). The manifold rationales for rigour feature prominently writing issues—with that, the need of a clear communication: lacking rigour “you may not be understood”.

The whole of the “petit manuel” can be read as an effort in the dissemination of writing praxeologies conceived as “rigour” praxeologies. All in all, the handbook deals with writing praxeologies designed to achieve the following types of tasks: “start the resolution of a problem” (technique: introduce that of which is spoken; technology: once you have introduced f as the element of such and such functional space, “you have a fixed function f in the hand and therefore embarking on a serious thinking about it becomes possible”; theory: the way of writing an answer helps in the elaboration of a strategy of proof); “to quote a definition or a theorem” (technique: follow the example given in the handbook and keep in mind that absolute precision is required for the quotation of a definition or a theorem; technology “to quote a definition or a theorem requires perfect precision. Assumptions, notations and conclusions must be clearly exposed, and without fault”. Indeed, “*to know* a definition or a theorem is to be able to write them down this way [that is, with an extreme precision, illustrated on the example of the definition of the pointwise limit of a function]—and quickly, of course” (my emphasis); theory: one of the aims of sound writing is to provide the reader—the marker—with information on the depth of the student’s knowledge); “introduce a notation” (technique: if an object is to be met “several times in a reasoning”, choose a letter, say K , that “has not yet been used elsewhere in the reasoning”, then write a sentence of one of the two following forms: “We note K the...” or “We set $K = \dots$ ”, at the exclusion of the “banned” following form: “We note $K = \dots$ ”; technology: the letter must not be used elsewhere to “avoid confusion”; theory: notations are shortcuts, useful only when an object appears several times –they are used for communication purposes, not really for mathematical purposes); “make up a good balance between French and mathematics” (technique: “write in French or in mathematics, but not both at the same time!” except for the “most frequent allowed mixes”, as in “Let $x \in E$ ”; technology: about the exception, “this is only a conventional issue”); “add flair to a presentation” (technique: “riddle your text with those little words [“therefore”, “then”, “consequently”, “thus”, etc.] that will guide the reader and, if possible, varying: this will add flair to your presentation”; technology:

writing mathematics requires style, which can be achieved by a fair use of “little words”; theory: the use of logical connectors is not only a logical one); “avoid to let readers feel that you are blind about a formula” (technique: “you may have a mnemonic [here, the authors comments on the classical notation $\Delta = b^2 - 4ac$ for the discriminant of a second degree polynomial] to remember a formula, why not? But don’t you share it with every one, keep it for your own”; technology: “Otherwise, the feeling is that you have no perspective on the formula in question”; theory: not only is it important to show mastery of a given knowledge, you must prevent any demonstration of naivety about it); “writing an induction proof” (with the obvious technique: base case, induction step, conclusion) is surrounded by the following *logos* elements: “It is often enticing to replace an induction by ellipses ‘...’. [...] It is far less acceptable in a written paper than in an oral presentation –overall in a paper filled with mistakes, which could be suspected of *bluffing*” (my emphasis).

Propriety, decency, and honesty: the student is required to meet with ethical and aesthetical expectations. Using the notation “ $b^2 - 4ac$ ” when a , b and c have not been previously introduced gives a hint of the naivety of the author: he may be compared to a geologist who would use the mnemonic “stalactite–ceiling/stalagmite–ground” in an international conference (Bertault, n.d., p. 7). There are no mathematical grounds for the exclusion of the use of such a notation (the student may introduce properly a , b and c : the author of the handbook still rejects it), but that it makes the reader feel “that you have no perspective on the formula in question”. Let it be added that the whole discussion of this issue is at the core of a paragraph titled *La naïveté des notations classiques* (naivety of classical notations). Also, misuse of formalism (such as in induction proofs) will be interpreted on ethical grounds: far from projecting “his own reasoning on an erroneous demonstration” (Yaw83, 2013, May 31, see above), the marker will assume *bluffing*.

Consequently, rigour is primarily “norms”, a piecemeal evidence for reasoning, a base for elaboration of mathematical thought, is associated to flair and opposed to naivety, dullness, and dishonesty. This enriches our sketchy definition of school rigour.

4. The reproduction of a profession

We have seen that some of the primary concerns with rigour at school deal with aesthetical and ethical issues: avoid naivety, be honest, etc. A distinction concern (Bourdieu, 1984), the expectation of conformity to a definition of the craftsmanship of mathematician, to an *ethos*, serves to make a difference with philistines, who use “classical notation”, to draw a symbolic frontier between the *would-be* mathematicians and the *mathematicians-to-be* (Bourdieu, 1982). In this sense, competitive examinations play the role of institution rites: they *produce* the difference between mathematicians and non-mathematicians, and at the same time, they *produce* the legitimation of this frontier.

4.1. From logic to axiology: means to draw borders

The issue of turning would-be mathematicians into actual mathematicians, or, rather, of excluding those that do not match the implicit definition of the job, can be compared to a similar issue analysed by Bertrand Daunay (2002). The universal condemnation of paraphrase at school is both an effect and a means of the imposition of a classification of different practices of reading, with different degrees of legitimacy (of distinction). The impossibility to give a precise

and a (dia- or synchronically) non-varying definition of what paraphrase exactly is, together with the persistent and pervasive condemnation of its use, can be better understood if we consider that the difference between paraphrase and other (tolerated or valorised) tools is axiological rather than logical. Thus, after quoting Umberto Eco's attempt to distinguish between "creative" metaphors and "dead" metaphors, Daunay concludes that

"the question may finally be to know which axiology is built in such a discourse (with such words as 'original', 'creative', 'hazardous', 'burdensome', 'lengthy'), and how this discourse manages to *transpose the impossible into the category of the forbidden* ('cannot' must be understood as 'must not')" (p. 28, my emphasis)

The impossibility to give a precise definition of rigour or, rather, to give it a sound foundation with genuine rationales, is related to the *transposition of the "impossible" into the "forbidden" category*: to *forbid* students to use such and such wordings or notations or tools, rather than give a mathematical rationale for their *inappropriateness*. As a typical illustration of this drift from logic to axiology we mention an observation made in the class of a beginning teacher in France: as a student proposed to look for the "middle point [*sic*] of K , N and G " (the aim of the session was to introduce the notion of the centre of the circumscribed circle to a given triangle), the teacher replied: "that's it, but it's poorly said [*c'est mal dit*]". Whereas it could have been possible to ask the student how to construct the "middle point" of three given points, in order to make him and the class discover the consequences of this inadequate formulation (the fact that it is "*impossible*" to construct the "middle" point of three points), the teacher gives an aesthetic judgement ("poorly said"), at the risk of reinforcing the belief in the arbitrariness of mathematical expressions (why "centre" rather than "middle"?). Many such choices are supported by the drift from "it cannot be" (because it would be ambiguous, or it would not work, etc.) to "it must not be" (because it is "poorly said" or because it does not match mathematical conventions, etc.), from the "impossible" category to the "forbidden" category.

The most important point here is that the teacher's answer did not express the fruit of a careful reflexion but rather was a spontaneous reaction to the student's proposition: expression of a professional *habitus* rather than of a rational thinking. It could be argued that the position-takings of a new teacher are hardly the kind that must reflect a professional *habitus*; indeed, some aspects of the professional *habitus* of teachers are probably developed in the course of professional practice, after years of teaching. Though, the issue of rigour, of mathematical phrasing and writing, is attached to the very professional identity of mathematical teachers, and beginners in the profession are certainly (s)electd according to criteria that include a certain vision of rigour. In turn, the condemnation of some kinds of phrasing and the valorisation of others help the dissemination of a set of legitimate praxeologies, which in turn, define a legitimate professional *ethos*. What is at stake in the above intervention of the beginning teacher is the actualisation not only of mathematical expectations (it can be considered that expecting that students know the difference between 'centre' and 'middle point' is a genuinely mathematical expectation) but also of an axiological dimension of these expectations: the student must not only conform his or her practice to mathematically based expectations, but he or she must also accept that this practice be changed and adapted to the teacher's expectation *for ethical or aesthetical reasons* and, more generally, non-mathematical reasons. He or she must even accept that such non-mathematical (in the technical sense) reasons be indeed...

mathematical reasons. What is at stake here is a denial of the social nature of mathematical practice, which leads to the identification of social effects to logical effects (a typical example of symbolic violence, see Bourdieu & Passeron, 1990), but also to a play with the word “convention”: some conventions are not conventions in the mathematical sense, but are nonetheless introduced as such (for instance the *conventional* exclusion of “we note $K =$ ”).

4.2. Selection of teachers-to-be

At a higher level, while it could be expected that the difference between would-be mathematicians and mathematicians-to-be be made on mathematical grounds, it appears that a strong axiological dimension is also involved. Besides the acquisition of mathematical praxeologies, university and CPGE students are asked to give proofs of their acquisition of a mathematical *ethos* (Zarca, 2012), that is, of a set of inseparably mathematical and ethical praxeologies. The “small handbook” gives even more explicit hints of this expectation by providing many intimations and references to a mathematical community, and to the unquestionable fact that the students have to (and indeed have to wish to) belong to this community: “meeting with the notational conventions practiced in the community of persons you address” (p.1), “it is necessary to fix a notation for the sake of communication” (p.1), “each mathematician has his own little quirks. Yet, I know that most of my colleagues share the following little quirks” (p.1), “every student mathematician *worthy of that name*” (p.2, my emphasis), “just as a painter cannot paint without paint, brushes and canvasses, a mathematician cannot think without a medium for his thinking” (p.2), “costume bans” (p.3).

If we turn back to the situation of a new mathematics teacher, we must take into account the fact that he has been long trained in mathematics in institutions where the distinction between legitimate and illegitimate practices are the core of teaching. Whether they studied in CPGE and *grandes écoles* or at University, future high-school teachers have to pass a competitive examination (*agrégation* or *CAPES*—*certificat d’aptitude au professorat de l’enseignement du second degré*), by means of which they are “recognized and co-opted”, because they “recognize[d] the implicit laws of the system” (Bourdieu, 1993, p. 66)—mathematical laws, but not only. It can therefore be assumed that the position-takings of new teachers in their classes must, at least roughly, match the demands of the institution. Indeed, the examination of selection procedures highlights the main characteristics of the professional identity of teachers and reading reports of competitive exams such as CAPES (see <http://capes-math.org/index.php?id=archives>) and Agrégation (see <http://agreg.org/Rapports/>) gives hints on the admission fee to enter the school field as a mathematics teacher. Unsurprisingly, rigour is frequently mentioned in the reports, in relation with writing or presentation issues: “candidates cannot develop inductions rigorously” (CAPES externe 2008, p. 65), “manipulation of a variety of forms of reasoning (induction, contraposition, proofs by contradiction) is poorly managed and *above all, reasoning is poorly presented*” (CAPES externe 2009, my emphasis, p. 50), “If an effort in *presentation* is generally observed, a general lack of rigour must be reported: wordings such as ‘step by step’ or ‘and so on’ replace induction proofs which obviously *cannot* be avoided” (CAPES externe 2010, my emphasis, p. 57: *cannot* belongs apparently to the “impossible category”, whereas indeed it plays the part of *forbidding*).

Besides this confirmation of the link between rigour and writing, these reports also emphasise on the ethical dimension of rigour: “rigour is always necessary; we advise to avoid

paraphrase or bluff when the solution is not obvious” (Agrégation interne, 2012, my emphasis, p. 20); “it is more and more frequent to observe a will to *disguise* ill-based rationales, *with the hope that the reader will not notice and go ahead*. Do we have to recall that papers are read several times by two distinct markers? *Mathematics are a school of rigour and of intellectual honesty*, this data comes into consideration in the process of grading papers” (CAPES externe 2009, my emphasis, pp. 50); “let it be recalled that, beyond *rigour and precision*, a future teacher is expected to be *coherent and honest*” (Agrégation externe 2008, my emphasis, pp. 52-53). As it seems, a lack of rigour is more than often understood as a lack of honesty—rather than as a poor mastery of technical issues.

5. Conclusion

The following text was taken from a letter addressed, in 1999, by an *inspecteur général*¹⁰ de l'Éducation nationale to high school mathematics teachers: “A prominent part must be given to rigour. Students are constantly solicited [...] by *poor languages*, or even *corrupted languages*: commercials, more interested in *seduction* than in convincing on rational grounds; [...] language of games and variety shows, where all is *easy* and the value of *effort* fades away at the benefit of *luck* and skills. It is an essential mission of School [...] to speak a language [...] where effort and work are the key of success. This is why one *must* be rigorous” (Van der Oord, 1999, my emphasis). The above letter illustrates well the strong heteronomy of the school field: functions of rigour are closely related to the struggle against the influence of “corrupted languages”, “commercials”, “variety shows”, etc., all issues that can hardly be considered as *internal* or specific to the mathematical work.

Of course, this situation has didactic effects: the conditions and constraints under which the study of mathematical rigour is developed are not satisfactory for a *functional* building of praxeologies of rigour similar to those observed in the mathematical field. These conditions and constraints are the expression of a certain type of structuration of the school field, which does not allow for the required autonomy for the functional study of rigour: the influence of external issues prevents some position-takings to be outlawed as irrelevant (as they would probably be in the field of mathematical production).

References

- Balacheff, N. (1982). Preuve et démonstration en mathématiques au collège. *Recherches en didactique des mathématiques*, 3(3), 261-304.
- Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational studies in mathematics*, 18(2), 147-176.
- Belhoste, B. (1998). Pour une réévaluation du rôle de l'enseignement dans l'histoire des mathématiques. *Revue d'histoire des mathématiques*, 4, 289-304.
- Bertault, C. (n.d.). *Petit manuel de bonne rédaction*.
- Bourdieu, P. (1982). Les rites comme actes d'institution. *Actes de la recherche en sciences sociales*, 43(1), 58-63.
- Bourdieu, P. (1984). *Distinction: A social critique of the judgement of taste*. Cambridge: Harvard University Press.
- Bourdieu, P. (1992). *Les règles de l'art. Genèse et structure du champ littéraire*. Paris: Seuil.
- Bourdieu, P. (1993). *Sociology in question* (Vol. 18). London: Sage.

¹⁰ They are in charge of the application of official curricula, of the selection and initial and continue training of teachers, they represent the education ministry, organise the evaluation of CPGE teachers, etc.

- Bourdieu, P. (2001). *Science de la science et réflexivité*. Paris: Raisons d'agir.
- Bourdieu, P., & Passeron, J.-C. (1990). *Reproduction in education, society and culture* (Vol. 4). London: Sage.
- Casanova, P. (2016). *La langue mondiale: traduction et domination*. Paris: Seuil.
- Chevallard, Y. (2007). Passé et présent de la TAD. In L. Ruiz-Higueras, A. Estepa, & F. J. García (Eds.), *Sociedad, escuela y matemáticas. Aportaciones de la teoría antropológica de lo didáctico* (pp. 705-746). Jaén: Publicaciones de la Universidad de Jaén.
- Darmon, M. (2015). *Classes préparatoires: la fabrique d'une jeunesse dominante*. Paris: la Découverte.
- Daunay, B. (2002). *Éloge de la paraphrase*. Paris: Presses Universitaires Vincennes.
- Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational studies in mathematics*, 22(3), 233-261
- Duval, R. (1992-1993). Argumenter, démontrer, expliquer: continuité ou rupture cognitive? *Petit x*, 31, 37-61.
- Gandit, M. (2004a). Preuve ou démonstration, un thème pour la formation des enseignants de mathématiques: deuxième partie. *Petit x*, 66, 49-82.
- Gandit, M. (2004b). Preuve ou démonstration, un thème pour la formation des enseignants de mathématiques: première partie. *Petit x*, 65, 36-49.
- Gispert-Chambaz, H. (1982). *Camille Jordan et les fondements de l'analyse: Comparaison de la 1ère édition (1882-1887) et de la 2ème (1893) de son cours d'analyse de l'école Polytechnique*. (Thèse de troisième cycle) Université de Paris-Sud, Paris.
- Goffman, E. (1961). *On the characteristics of total institutions*. Paper presented at the Symposium on preventive and social psychiatry.
- Grabiner, J. V. (1974). Is mathematical truth time-dependent. *The American Mathematical Monthly*, 81(4), 354-365.
- Ministère de l'éducation nationale. (2008, August 28). Programmes du collège: programmes de l'enseignement de mathématiques. *Bulletin officiel spécial*, 6.
- Ministère de l'éducation nationale. (2013, May 30). *Programmes de la classe préparatoire scientifique Mathématiques, physique et sciences de l'ingénieur (MPSI) et programme de sciences industrielles de l'ingénieur de la classe préparatoire scientifique Mathématiques et physique (MP)*. Retrieved from http://www.education.gouv.fr/pid25535/bulletin_officiel.html?cid_bo=71622
- Schubring, G. (2001). Production mathématique, enseignement et communication. Remarques sur la note de Bruno Belhoste, "Pour une réévaluation du rôle de l'enseignement dans l'histoire des mathématiques" parue dans la RHM 4 (1998), p. 289-304. *Revue d'histoire des mathématiques*, 7, 295-305.
- Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the american mathematical society*, 30(2), 161-177.
- Van der Oord, É. (1999). Enseigner les mathématiques en classe de seconde à la rentrée 1999. Retrieved from http://pedagogie.ac-toulouse.fr/math/inspection/let_ipr/letipr3.htm#ig
- Yaw83. (2013, May 31). Rigueur mathématique. Retrieved from <http://www.ilemaths.net/sujet-rigueur-mathematique-559745.html>
- Zarca, B. (2012). *L'univers des mathématiciens: l'ethos professionnel des plus rigoureux des scientifiques*. Rennes, France: Presses universitaires de Rennes.