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# Detecting and sharing praxeologies in solving interconnecting problems: some observations from teacher education viewpoint

# Détecter et partager les praxéologies dans la résolution de problèmes d'interconnexion: quelques observations du point de vue de la formation des enseignants

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## Abstract

This paper discusses praxeologies available at different levels of schooling in view of a problem, which permits multiple solutions ranging from elementary to more advanced mathematical approaches. Solutions of the problem produced by mixed groups of K-12 teachers included numerical, pictorial and algebraic methods, and allowed observing possible paths within a finalized activity of study and research. They also gave some insights regarding teachers' readiness to support the continuity of students' praxeological development, and more generally, the potential within teachers' educational backgrounds to pursue the new paradigm of questioning the world.

**Keywords**: Teacher education, Praxeological development, Mathematical problems with multiple solutions.

## Résumé

Ce texte discute les praxéologies disponibles à différents niveaux de la scolarité pour résoudre un problème qui permet des résolutions multiples, depuis des approches élémentaires aux plus avancées. Les résolutions proposées par un groupe mixte d'enseignants de l'école élémentaire jusqu'au lycée ont employé des méthodes numériques, graphiques et algébriques, et permettent d'observer les parcours possibles

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d’une activité finalisée d’étude et de recherche. Elles nous laissent aussi percevoir la capacité des enseignants pour soutenir la continuité du développement praxéologique des élèves, et plus généralement le potentiel résultant de la formation des enseignants à poursuivre le nouveau paradigme du questionnement du monde.

**Mots-clés:** formation des enseignants, développement praxéologique, problèmes mathématiques aux solutions multiples.

## The paradigm of questioning the world.

While mathematical content is present at all levels of education, mathematics "suffers cultural rejection" and the majority of "people flee away from the subject as long as they no longer obliged to do it." (Chevallard, 2012). Chevallard proposed that the cause of this phenomenon is the dominance of the didactic paradigm of "visiting monuments", where curriculum is defined in terms of work $O$ that students need to study, while they often have little motivation to do so, particularly when they see no relevance of mathematics to real life and no connections within the subject. He put forward the new paradigm of "questioning the world", in which the work $O$, (and mathematical *praxeologies*, consisting of praxis blocks $\Pi$ and l*ogos* blocks $\Lambda$, that is, theory justifying related practices □), is studied in the consequence of inquiring into some deliberately selected questions Q. These questions produce motivation for the learner and thus generate *activities* and even programs of study and research within which their individual *study and research path* is not defined and not known ahead of time.

Chevallard (2011) distinguishes between *open* and *finalized* programs of study and research. In the latter case the questions $Q$ are selected in order to give students an opportunity to meet specific mathematical *praxeologies*, still leaving some freedom for the choice of possible study paths. While the idea of using guiding questions is not foreign for teaching of mathematics, Chevallard (2012) warns that "in too many cases, the so-called inquiry-based teaching resort to some form of 'fake inquiries', most often because the generating questions $Q$ of such inquiry is but a naïve trick to get students to meet and study work $O$, that the teacher will have determined in advance".

For the open programs of study and research, neither answers nor methods are known in advance. This vision requires of students "to be receptive towards yet unanswered questions", "be ready to study from scratch" and have "the capacity to locate resources and the knowledge necessary to take advantage of them". Since the studies may not be limited to a single domain of human knowledge, the following question is essential for the learning of mathematics: "What are the mathematics of the matter?"

Depending on mathematical background, available mathematical praxeologies and the degree of creativity of the inquirer, the answer to the above question could be attempted at different levels of sophistication and by various means including explicit observation, physical experimentations, pictorial, algebraic or pure logical reasoning. Several ideas may contribute to the answer and thus produce meaningful connections, including the ones within mathematics itself.

Consistent with the idea of gradual mathematics curriculum development, there exist different praxeologies within different educational institutions such as primary, secondary and tertiary schools. In principle, each level of education prepares the learner for the following one, however the links and relationships between them sometimes are not as obvious. For example, praxeologies that are present at the primary level are not necessarily a subset of the ones found at the university level, while the processes of explicit observation and physical experimentation occur at the levels beyond the elementary one. Despite the theoretical existence of links between different educational institutions, teachers practicing at just one level may forget about various connections with both previous and forthcoming material. This concern about connectivity of the subject in its teaching in view of students' praxeological development, could be addressed particularly by the approach presented in the next section.

## Problems with multiple solutions and interconnecting problems.

If what Chevallard (2012) called 'fake inquiry', defines the work that needs to be studied with fake or no motivation at all, 'real inquiry' naturally allows several passages to come to an answer. In this respect problems with multiple solution paths could be viewed as a proper didactic tool for training students. Indeed, any solution defines the work $O$ that students could meet during their activity of study and research with the view that there exist a broader work $O'$ for a given problem and thus work $O \subset O'$ has a potential to be extended. Observing learners involved in a finalized activity that allows multiple solution paths might give an insight of what potentially could happen in an open program of study and research.

The value of tasks that allow multiple solutions has been recognized in mathematics education (Leikin & Levav-Waynberg, 2008; Sun & Chan, 2009). Inspired by other works on multiple solution tasks and own practices, Kondratieva (2011) proposed to consider a special class of such tasks, namely, interconnecting problems. The latter are defined as problems that obeys the following conditions: (1) allow a simple formulation; (2) allow various solutions at both elementary and advanced levels; (3) may be solved by various mathematical tools from different mathematical branches, which leads to finding multiple solutions, and (4) can be used in different grades and courses and understood in various contexts.

Within *problems to solve* that call to find an answer, problems to prove the answer, play a special role because many aspects of proofs such as "explanation, exploration, justification of conjectures and definitions, empirical reasoning, diagrammatic reasoning, and heuristic devices" (Hanna & De Villiers, 2012, p. 3) are in the core of mathematical thinking. Proofs essentially constitute the logos that corresponds to mathematical practices. The ability of students to prove is a developing skill "beginning with the perceptions, actions and reflections", and building on "physical, spatial and symbolic aspects of mathematics" eventually enabling the learner to possess "more sophisticated thinkable concepts that have a rich knowledge structure" (Tall, Yevdokimov, Koichu, Whiteley, Kondratieva & Cheng, 2012, p. 45). Consequently, theoretical part of praxeologies develops along with students' engagement in more sophisticated practices from proper reflection on them and the desire to "know why", converging finally to logical necessity of certain conclusions.

Interconnecting proving problems may be used as means to identify and compare praxeologies specific to different grade level, including verbal, visual, empirical, generic, inductive, symbolic and deductive reasoning. Influenced by daily practice, primary and secondary teachers might have distinct views on the appropriate ways and means of mathematical argumentation (Lin, Yang, Lo, Tsamir, Tirosh, Stylianides, 2012). Teachers dealing with very young students use little symbolism and may be reluctant to accept other modes of argumentation (Simon & Blume, 1996). In contrast, secondary teachers often reject verbal and visual proofs as being invalid (Biza, Nardi & Zahariades, 2009) as they believe that all proofs must be formal algebraic (DREYFUS, 2000).

The goal of this paper is to discuss praxeologies available at different levels of schooling in view an interconnecting problem that was offered to K-12 teachers, and get some insight regarding their readiness to question the world.

## Data Collection and Findings

Since 2007 I had been teaching a graduate course at Memorial University (Canada), designed for in-service K-12 teachers and focusing on mathematical thinking (Mason, Burton & Stacey, 1982). One assignment is to discuss given interconnecting problem in groups. Participants are required to construct at least 3 different solutions at various levels of sophistication. The instructor produces a summary of all solutions and a whole class discussion follows aiming at re-connection of different areas of mathematics in terms of the problem in hands. Randomly formed groups combine primary and secondary teachers registered for the course. I hypothesized that the mixed group discussions would allow the participants to learn from and reflect on each other's praxeologies and beliefs (Kondratieva, 2013).

The following is an example of an interconnecting problem: Fred runs half the way and walks the other half. Frank runs for half the time and walks for the other half. They both run or walk at the same speed. Who finishes first? Explain your answer (Mason et al, 1982).

The data was collected in five different years. The class size varied from 12 to 18 students. Participants' on-line discussions, final solutions and individual journals have been analyzed. The types of solutions proposed by teachers varied from numerical examples and actual experimenting with walking and running to various graphical representations, algebraic and pure logical derivations. A possible scenario within one mixed group of top students consisted from the following stages: (1) answering the question using concrete numerical values; drawing corresponding graphs, reasoning with them and agreeing upon the answer; (2) expressing concerns about the validity of these methods to reach the conclusion and introducing a partly algebraic approach; (3) searching for algebraic expressions for total time in each case; comparing two expressions for time; (4) claiming that the algebraic approach is "the most certain but least human" and calling for logical/structural reasoning.

Some groups' study paths contained an evidence of connections between the secondary school approaches (algebraic, graphical, logical) and primary level activities (experimental and numerical), however, this required presence in the group of a good student familiar with secondary school practices and a good student with similar knowledge of the primary level. Some groups focused on concrete approaches (see Solution 1 below), treating examples with different sets of values for the length of race and speeds of running and walking as being different solutions. On the other hand, some groups of secondary teachers (10-12 grades) had hard time to go beyond an algebraic approach (Solution 2) and saw little value in looking at concrete numerical situations or their visual representations.

**Praxeologies available at different levels of schooling**

**Analysis of two typical solutions and related praxeologies**

Let us take a look at the two most typical solutions proposed by the teachers.

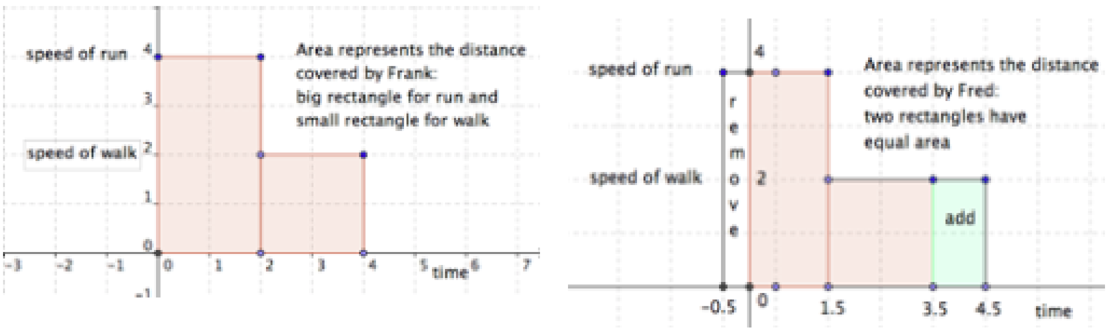
*Solution 1*. Consider the following example. Let the speed of the running be 4 miles per hour and the speed of the walking be 2 miles per hour. Let Frank run for 2 hours and walk for 2 hours. The total distance he covers in 4 hours is 4×2+2×2=12 miles. Then the time, that Fred needs to cover the same distance, is $\frac{6}{4}+\frac{6}{2}=4.5$ hours. Thus, Frank finishes first.

*Solution 2.* Let *d* be the length of the race and $v_1 \neq v_2$ be the speeds of running and of walking respectively. Since Fred moves with each of the speeds an equal amount of distance, $\frac{d}{2}$,, we have $v_1 t_1 = v_2 t_2 = \frac{d}{2}$, and so his total time of walking and running is $t_1 + t_2 = \frac{d}{2}\left(\frac{1}{v_1}+\frac{1}{v_2}\right) = \frac{d}{2}\left(\frac{v_1+v_2}{v_1 v_2}\right)$. Let Frank move with each of the different speeds an equal amount of time, $t$. Then we have $v_1 t + v_2 t = d$, so his total time is $2t = \frac{2d}{v_1+v_2}$. To compare the times we look at the ratio $\frac{t_1+t_2}{2t} = \frac{(v_1+v_2)^2}{4v_1v_2}$. One can observe that $(v_1+v_2)^2 - 4v_1v_2 = (v_1-v_2)^2 > 0$, and so $\frac{t_1+t_2}{2t} > 1$, that is the time of Frank is always less than the time of Fred.

Let us single out theoretical elements available at each level and related to the above solutions. At the Elementary (pre-algebraic) level, the theory required in this problem consists in defining (constant) speed as a ratio of the distance to the time spent to cover this distance (given as numerical values). This requires attention to the units of measurements: if du is a distance unit (e.g. km or mile) and *tu* is a time unit (e.g. hour, minute, second) then the speed will be measured in the compound unit *su=du/tu*. Rearranging numbers is the definition of speed, we see that the distance is a product of speed and time, and time is a ratio of distance and speed. Solution 1 gives an example of reasoning with concrete values. From the mathematical point of view, students perform arithmetic operations with numbers and justify their actions referring to facts such as “division is the inverse operation to multiplication”.

This concrete reasoning may be supplemented by a picture (see Figure 1), where speed of each sportsman is plotted as a function of time. Here we have another theoretical element at the pre-algebraic level: if one side of a rectangle is equal to a constant speed and another side is equal to the time traveled with such speed then the distance travelled is represented as the area of this rectangle. The length of a race travelled with two different speeds is represented by the area of two rectangles.

Figure 1
*Speed as a function of time for Frank’s and Fred’s courses.*




In our problem, rectangles representing Franks’ journey have equal width (along the time axis), while rectangles representing Fred’s journey have equal area. The following observation is an attempt to generalize the situation available at the elementary level: Assuming that the running speed is greater that the walking speed, the rectangle which represents running will always be taller than the rectangle that represents walking. Since in Frank’s case both rectangles have the same width, the taller rectangle will always be bigger (Fig 1, left). When we redistribute the area from the bigger rectangle to the smaller one (in Frank's case) to make the areas of the two rectangles equal (in Fred's case) the width of the part that we remove is always smaller than that of the part we add, so the time of Fred will always increase compare to the time of Frank (Fig 1, right).

Figure 2

*Distance as a function of time for Frank's and Fred's courses.*

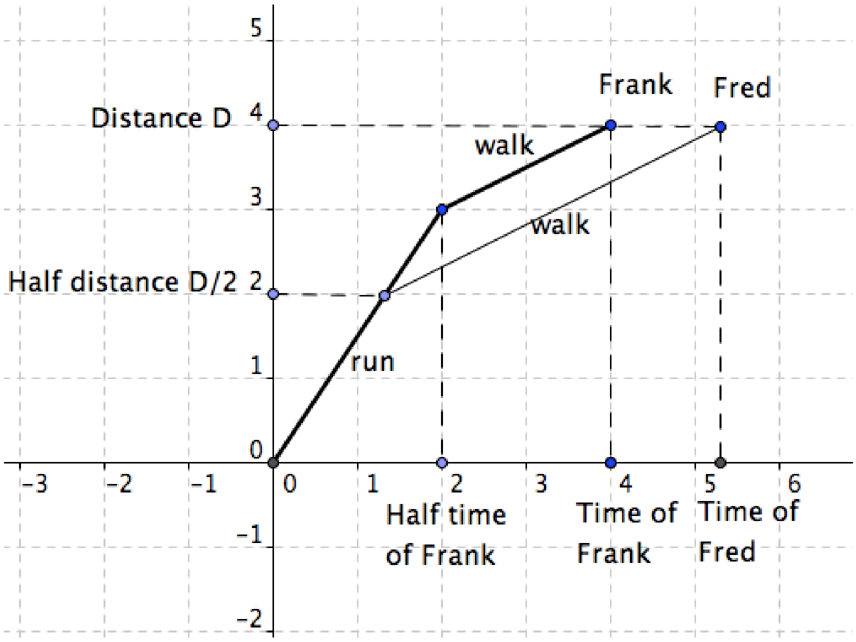



Figure 2 utilizes another theoretical component and its geometrical representation: For a constant speed, the graph of the distance as a function of time is a straight line with the slope equal to the speed. This pictorial representation again illustrates the fact that Frank finishes first. These graphical approaches along with logical attempts to generalize concrete numerical situations lead to the following structural insight: "Frank runs for *over half* of the course, while Fred only a half. Since Frank runs a further distance than Fred, he will finish the race first." We conclude that working with concrete numbers even at the elementary level may lead to solutions that fully justify the answer.

At the secondary (algebraic) level, working with concrete numbers no longer suffice for the theory, which now consists in algebraic manipulations of formulas. At this level working with concrete numbers rather belongs to students' praxis, while the algebraic approach is a way to build a theoretical explanation based on this praxis.

Solution 2 gives an example of an algebraic approach available at the secondary school level. One surprising discovery can be made by analysing this derivation: Regardless whether the running speed is less than or greater than the walking speed, the answer will be the same.

## The continuity of praxeological development

At the tertiary level, algebraic calculations become the field of praxis, while theory may include the reference to general inequalities such as AM > GM > HM, where AM, GM and HM stand respectively for arithmetic, geometric and harmonic means of a set of ($n \geq 2$ ) distinct numbers. In this case the problem situation itself could be generalized in the following way. Let $d$ be the total distance. Suppose Fred moves with each of n different speeds an equal amount of distance, $\frac{d}{n}$, while Frank moves with each of $n$ different speeds an equal amount of time, $t$. Then for Fred we $v_1 t_1 = v_2 t_2 = \cdots = v_n t_n = \frac{d}{n}$,, and so his total time is $t_1 + t_2 + \cdots + t_n = \frac{d}{n}\left(\frac{1}{v_1} + \frac{1}{v_2} + \cdots + \frac{1}{v_n}\right) = \frac{d}{HM(v_1, v_2, \ldots, v_n)}$. For Frank we have $v_1 t + v_2 t + \cdots + v_n t = d$, so his total time is $nt = \frac{nd}{v_1 + v_2 + \cdots + v_n} = \frac{d}{AM(v_1, v_2, \ldots, v_n)}$. Now, since for distinct numbers $v_1, v_2, \ldots, v_n$, the harmonic mean $HM(v_1, v_2, \ldots, v_n)$ is always less than arithmetic mean $AM(v_1, v_2, \ldots, v_n)$, we conclude that Frank is finishing first.

We summarize praxeologies available at each level in the following table:

Table 1:

*The continuity of praxeological development.*

| Level / Praxeologies | Primary/elementary (Pre-algebraic) | Secondary (Algebraic) | Tertiary (Analytic) |
|---|---|---|---|
| Praxis | Physical action; specific arithmetic and pictures. | Specific and generic arithmetic and pictures. | Symbolic calculations, use of algebraic rules, equations and inequalities. |
| Logos | Generic arithmetic and pictures. | Symbolic calculations, use | General analysis of equations and |

| | | | |
|---|---|---|---|
| | | of algebraic rules, equations and inequalities. | inequalities; axiomatic approach. |
| Development of | sense of generic structure. | symbolism. | formalism. |

Source: The author

The last line in Table 1 indicates the major direction of praxeological development. Observe that what may count as an explanation (theoretical element) at a lower level becomes part of students' praxis at the next level. Thus, at the elementary level, generic arithmetic and figures may "explain" the results obtained in any concrete case if the students start to see more general structure of many concrete examples. Note that some algebraic formulas (in our case, from basic kinematics, i.e. $d = vt, v = d/t, t = d/v$ may appear even at the elementary level, and students use them to make explicit calculations while sensing their more generic structure. Then at the secondary level, these generic arithmetic calculations become a part of the praxis that at its tern is "explained" by symbolic calculations and the use of algebraic rules. At the tertiary level, algebraic calculations become a part of practical tools, and then the theory involves elements of analysis, in particular, the study of more general inequalities. We suggest that the continuity of praxeological development described above, in addition to an advance from punctual to local, regional and global praxeologies (Chevallard, 1999), could be critical for students' grasp (with understanding!) of mathematics.

## Some observations with implications for teacher education

In order for teachers to support a continuing development of their students they need to be familiar with the range of praxeologies at least at the level adjacent to the level of their primary expertise (as a part of their horizon knowledge). Based on my observations, this was not always the case in the first place, and moreover, the teachers did not always possess a sufficient understanding of this need. However, when they collaborated on solving interconnecting problems in mixed groups, it was often evident that an exchange of different praxeologies occurred.

Note that pictorial and structural explanations discussed above are available at any of the three levels and may provide a common ground for the group discussion and lead to some interesting discoveries such as geometric interpretation of the AM-HM inequality (due to the fact that Figure 1 basically represents the algebraic Solution 2, which is a particular case of the method discussed for the tertiary level and involves this classical inequality). However, pictorial approach was not a popular solution, and if it occurred, many teachers stopped short of generalizing beyond concrete examples (e.g. of working with ‘generic’ units of measurement), confirming that people often tend to substitute empirical arguments for proofs (Lin et al, 2012).

Further, teachers were not always strong in generalizing their solutions, which perhaps was a consequence of some ‘defective’ praxeologies that focus mostly on the praxis component at their grade level, such as, working with numbers without seeing more general algebraic and/or structural pattern, or working with formulas without observing their relations to other representations.

My data illustrate that indeed a good question (such as a problem with multiple solution paths) in principle allows students to reason using tools of their own choice and employ mathematics in order to explain their answer. Being typical for a research setting, this behaviour is not common for school mathematics where the focus is on learning specific techniques to solve certain types of problems rather than on finding and justifying one’s own solutions, let alone comparing and generalizing them or arriving to new questions. An exposure to the inquiry paradigm triggered the following teacher’s revelation:

> I think that the beauty in problem solving is that there are so many ways a problem can be solved. It is evident in our different approaches that we may be teaching math at different levels which I feel will enhance our learning experiences. This is how problem solving should be happening in our classrooms: in collaboration not in isolation.

However, it remains to be seen what percentage of mathematics teachers really welcomes this opportunity to "question the world", and if their existing praxeologies are developed enough to meet this challenge. If teachers' justification deficiencies show up so clearly in this finalized version of inquiry, their success seems to be problematic in case of open programs of study and research.

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