EMP Educação Matemática Pesquisa
ISSN 1983-3156
A2
http://dx.doi.org/10.23925/1983-3156.2020v22i4p259-281

The Pósa method with ATD lenses: Praxeological analysis on math problems in Hungarian talent care education with 'recursion' in their logos blocks

La méthode Pósa avec les objectifs ATD: analyse praxéologique des problèmes mathématiques dans l'éducation hongroise aux soins des talents avec «récursivité» dans leurs blocs de logos

Dániel Katona[1]
Doctoral School of Mathematics, Eötvös Loránd University, Hungary
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
https://orcid.org/0000-0002-7040-4372

Abstract

The *praxeological analysis* of selected questions used in the Hungarian Pósa method is presented, focusing on a common element in their logos blocks, called *recursive thinking*. As part of a broader research with *reverse didactic engineering* methodology, aiming at theorizing the 'intuitively' developed Pósa method, the present findings are also compared to previous results and re-interpret the concepts of *kernel* and *web of problem thread*. Based on these results gained by using tools of the Anthropological Theory of the Didactic, the paper offers a partial description of the *didactic strategy of the Pósa method* for inquiry-based learning mathematics and raises questions for further research.

Keywords: Anthropological Theory of the Didactic, praxeological analysis, web of problem threads, kernel of problem threads, Pósa method

Résumé.

Nous présentons l'analyse praxéologique de certaines questions utilisées dans la méthode hongroise Pósa, en nous concentrant sur un élément commun à leurs logos blocs, appelé *pensée récursive*. Dans le cadre d'une recherche plus large qui met en place une méthodologie d'*ingénierie didactique inverse* visant à théoriser la méthode de Pósa développée "intuitivement", les résultats actuels réinterprètent les concepts de *noyau* et de *réseau de fils de*

[1] danikatona@gmail.com

problèmes. Sur la base des résultats obtenus en utilisant les outils de la théorie anthropologique du didactique, l'article offre une description partielle de la stratégie didactique de la méthode Pósa pour l'apprentissage des mathématiques basé sur l'enquête, et soulève des questions pour des recherches ultérieures.

Mots-clés : Théorie anthropologique du didactique, Analyse praxéologique, Réseau de fils problèmes, Noyau de fils problèmes, Méthode Pósa

Resumen

Se presenta el análisis praxeológico de algunas preguntas utilizadas en el método húngaro Pósa, centrado en un elemento común de su bloque del logos, llamado *pensamiento recursivo*. Como parte de una investigación más amplia basada en la metodología de la *ingeniería didáctica inversa*, cuyo objetivo es teorizar el método Pósa desarrollado “intuitivamente”, los presentes hallazgos también se comparan con resultados anteriores y reinterpretan el concepto de *núcleo* y *red de hilos de problemas*. Sobre la base de estos resultados obtenidos mediante el uso de las herramientas de la Teoría Antropológica de la Didáctica, se ofrece una descripción parcial de la estrategia didáctica del método Pósa para el aprendizaje de las matemáticas basado en la indagación, y se plantean cuestiones para futuras investigaciones.Research on the Pósa method.

Palabras clave: Teoría antropológica de la didáctica, análisis praxeológico, red de hilos problemáticos, núcleo de hilos problemáticos, método Pósa

The Pósa method with ATD lenses: Praxeological analysis on math problems in Hungarian talent care education with ‘recursion’ in their logos blocks

Reverse didactic engineering (RDE) on the Pósa method

Lajos Pósa has been organising weekend mathematics camps for 12-18 year-old gifted students in Hungary since 1988, using the now called ‘Pósa method’ (Győri & Juhász, 2018). He has developed an ‘informal’ and supplementary talent-care curriculum, together with a corresponding pedagogical practise. And he has done this ‘intuitively’, that is, without any research or a complete and explicit theoretical background, and without a complete written record of this nationally widely respected practise either.

In the frame of the current Content Pedagogy Research Program of the Hungarian Academy of Sciences (CPRP), as well of an on-going doctoral study at the Eötvös Loránd University, the author aims at the ‘a posteriori’ analysis, theorization and description of the method. Based on the result of these primary processes, the long-term goal is to modify the method for its use in public education and to contribute to potential national curricular changes in Hungary, that “may be carried out through the action of a charismatic leader” (Chevallard, 1992, p. 220), in our case, Lajos Pósa. The research also aims at contributing to the on-going conceptualization of inquiry-based mathematics education (Artigue & Blomhøj, 2013; Bosch & Winsløw, 2016).

This ‘a posteriori’ analysing and theorizing process is conducted through a research methodology called *reverse didactic engineering* (RDE). It is based on the conception of *didactic engineering research methodology* (Barquero & Bosch, 2015), in which didactic constructions are designed on the basis of a developed (though regularly modified) theory, and re-designed based on the re-formulated theory. In RDE, the implicit or even only partially existing theory shall be reconstructed through the analysis of the already developed didactic product, which may be re-designed at a later phase. During this process, already established intermediate-level theoretical frameworks are also used as research tools.

The Pósa math camps in Hungary: brief description of the context

Students start they 'Pósa camp life' usually in the beginning of grade 7, when around 30 of them are selected and grouped to meet during 2–3 whole weekends (from Friday to Sunday afternoons) a year, with homework between the sessions, together with a leading teacher and 2–7 assistant teachers (usually former camp students), until grade 12.

During the camps, as segments of a coherent 6–year–long 'building' procedure, the main sessions consist of students thinking on and discussing mathematical problems in groups of 2–4, in separate rooms. They approach 3–4 problems at a time, which offer appropriate challenge for them, with the aim that they autonomously solve problems and discover connections, with a complete freedom of making mistakes and the overall aim of the joy for thinking. During these special team work sessions (for rules see Győri & Juhász, 2018, p. 95) teachers monitor students' progress. Now and then, all the groups come together led by the leading teacher. They discuss students' solutions, implications, the relationships between the solved problems, and – as an important part – may pose new problems they are interested on the basis of the solved ones, some of which may be integrated to the course of the study.

However, perhaps the most decisive feature of the method is the structure of the problem set, the connection between the problems, which is to be partly revealed by the praxeological analysis that forms the essence of the present paper.

In September 2017, within the framework of CPRP, experimental (9^{th} grade) classes in two secondary schools have also been launched, with curricular constraints, but with basically the same methodology and to a considerable extent using the same problems that were developed for the math camps.

Web of problem threads (WPT)

WPT and the concept of kernel – previous research results

The problems posed in the Pósa camps and in the experimental classes are connected to each other by some common elements, which have been considered as mathematical methods, problem solving strategies or 'mathematical ways of thinking' to be used during working with and solving them. A set of problems, in only a partially fixed order that are connected to each other this way is called a *thread of problems* (Katona & Szűcs, 2017; Győri & Juhász, 2018, p.100).

As some different threads may have the same kind of connection, and also a thread may have a multiple way of connections, (Katona & Szűcs, 2017) introduced the concept of *kernel* to describe the connection, in order for differentiating it from the concept *thread*. Kernels are basically part of the a priori structure of the problems, which – as the whole WPT – are mainly pre-designed by the teacher. During this designing procedure, the problems are selected or constructed in order for giving birth to the kernels, and not (only) for their own sake. However, during particular implementations of the method, students' unanticipated ideas may establish new connections, even new kernels. Therefore, understanding the connections between the problems and the 'consideration of the whole' are essential parts of the students' work with the teachers' assistance.

ATD tools in the RDE research on the Pósa method

The present paper focuses on the second phase of the RDE process: As part of theorizing, the problems used in the Pósa camps and in the experimental classes are analysed with tools of the Anthropological Theory of the Didactic (ATD) (Bosch & Gascón, 2014; Bosch et al., 2019).

As a preliminary result, a praxeological analysis (Chevallard, 2006) on some selected Pósa problems is being presented in the present paper, concluding that one main characteristic of the Pósa method originates in a task- and curriculum-design that is mainly guided by special common elements of the logos blocks of the used praxeologies, which may correspond or at least strongly related to the concept of kernel of problem threads. In that respect, the concept of praxeology is used to understand the nature of earlier introduced concepts and may also help to understand what is learnt and how learning happens during the Pósa sessions.

The 'praxeological analysis' as theoretical background

Praxeology is a central concept and an important tool in the anthropological theory of the didactic (ATD). ATD, besides its other missions, aims at offering a theory of human actions, for instance actions of learning and teaching, and praxeology is the central concept to describe human actions. For the most recent and concise reference, see (Bosch, Chevallard, García & Monaghan, 2019).

A praxeology $\wp$ is an ordered pair (Π; Λ) in which Π is an ordered pair (T; τ) too, and Λ is also an ordered pair (θ; Θ); that is $\wp$ = ((T; τ); (θ; Θ)), though also commonly written as $\wp$ = (T; τ; θ; Θ), as in (Bosch et al., 2019). T, as the starting point of the whole concept of praxeology, is a set of (in some way connected) *tasks*, in which a task t ($t \in$ T) can be (almost) anything to be accomplished or realized by a human, though in our case it will be a mathematical task. In order for this accomplishment, the use of a *technique* is required, which is an element of τ, the set of some possible techniques, therefore τ is in relation to T. That is, different techniques can be used to solve a task in T, even if there are usually a small set of techniques that prevail in a given institution – sometimes only one – and appear to be the "normal way" of carrying out a type of task. The ordered pair (T; τ) is called the *praxis block* of the praxeology, that is (to be) done or conducted, also referred to as the 'know how' element of the praxeology.

If T is a single type of task, that, is for instance, T consists solely of such tasks that can be solved by a same technique, than $\wp$ is called a *point praxeology*.

In the other block, θ denotes the *technology*, which serves explanatory proposes, it is the part of the human action that justifies the use of the technique τ to accomplish tasks of T, in our case, it is the core element of the argumentation related to solving a mathematical task. θ depends on (T; τ), but for instance if $\wp$ is a point praxeology, than θ usually depends also on other types of tasks and techniques that are related to (T; τ). In fact, one of the roles of the technology is to create such connections among types of tasks and techniques, that is, to connect point praxeologies. If T consists of a set of different types of tasks that are "organized around a common technological discourse" (Barquero & Bosch, 2015 p.69), it is called a *local praxeology*.

However, the technology θ shall (or may) also be justified at a higher level, where it is usually connected to other technologies. This higher-level justification is Θ, and it is called the *theory* element of the praxeology. Θ is related to θ and also to other technologies. The ordered pair (θ; Θ) is called the *logos block* of the praxeology. $\wp$ = (T; τ; θ; Θ), that is basically at the level of argumentation and proving. If $\wp$ contains all point and local praxeologies 'of' a common theory, it is called a *regional praxeology*.

This 'definition' of a praxeology may suggest a static nature, which is typically not the case. "Human praxeologies are open to change, adaptation and improvement" (Chevallard, 2006, p. 23), they have an essentially dynamic nature.

Praxeological analysis (Bosch et al., 2019) is to study certain praxeologies, by unfolding, identifying and describing their praxis and logos blocks.

In the past years, research on ATD has been focusing on the institutional conditions enabling a paradigmatic change in teaching and learning mathematics, from the *paradigm of visiting works* (PVW) to the *paradigm of questioning the world* (PQW) (Chevallard, 2015), primarily through the development of a didactic design framework, called *study and research paths* (SRP) (Bosch & Winsløw, 2016; Bosch & Gascón, 2014; Chevallard, 2015; Barquero &

Bosch, 2015). However, the ATD theoretical framework and especially the tool praxeological analysis has been created to be applied to any kind of educational situation, belonging to any paradigm of teaching and learning. Whether to analyse elements of an SRP within PQW, where students are to inquire into questions for creating their own, not (necessary) pre-established answers, through studying available and relevant works and raising corresponding derived questions, or to analyse "pre-established praxeologies" which have already "turned into monuments" within PVW (Chevallard, 2006, p. 25), where the expected answers to the raised questions are basically pre-determined; the praxeological analysis may be in both cases appropriate to use in order for gaining a deeper insight into the situations and actions of students during the learning and teaching processes.

Selected problems from a selected Pósa thread – Object of study

a. The process for selecting the problems and the presented order of the problems

The selection was based on the praxeological analysis of a larger set of problems used in the Pósa camps or in the experimental classes. Problems having the same kind of element in their logos blocks, *recursive thinking,* were selected first. The problems to be presented in the next subsection for analysis are a subset of the set of these previously selected ones. On the one hand, they represent, even if not a wide range, but at least some different types of tasks. On the other hand some subtypes or variants of the same type of tasks are presented too.

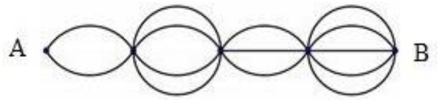
The problems are presented in the order they are usually posed to the 6^{th}-8^{th} grades students in the talent care math camps, and 9^{th} grade students in the experimental classes.

b. The problems to be analysed

Problem 1. In how many different ways can you get from A to B in the diagram below if you can only proceed from left to right?

Figure 1

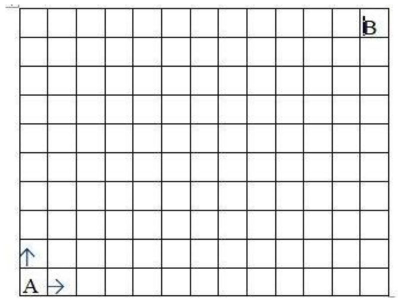
Paths in 'bubble figure' – Problem 1



Problem 2. In how many different ways can you get from the bottom left corner field to the top right one in a 10 x 13 rectangle grid, if in each step you can either move one (field) to the right or one upwards?

Figure 2

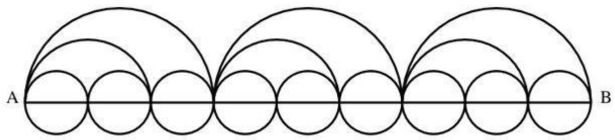
Paths in rectangular grid – Problem 2



Problem 3. Similar to Problem 1, with almost the same task, but with a different figure:

Figure 3

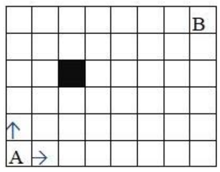
More complicated 'paths in bubble figure' – Problem 3



Problem 4. Similar to Problem 4, but you cannot step on the black square:

Figure 4

Paths in hollow grid – Problem 4



Problem 5. We notice that for the first some powers of 3, the decimal place is always occupied by an even digit. Is it true for every power of 3?

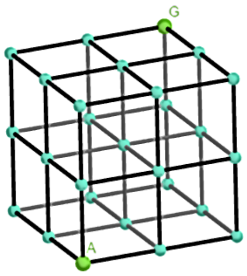
Problem 6. In how many ways can you (completely) cover a 2×12 sized rectangle (grid) with dominoes (1×2 sized rectangles, the spots on the dominoes are irrelevant)?

Problem 7.

> … an astronaut … lives in a space station that consists of 27 space modules… set at the vertices of the little unit cubes that make a 2×2×2 cube. There are passages between each neighbouring modules, represented by the edges of the unit cubes. Our astronaut can only use these passages to move between the modules. Our hero is now at the (green) module signed by vertex A, in Figure 6, and would like to go to the opposite vertex (G) of the two-unit cube. In how many different ways can they[2] do this, if they do not want to move away from their target? (Katona & Szűcs, 2017, p. 22-23)

Figure 5

The space station for the paths of the astronaut in Problem 7

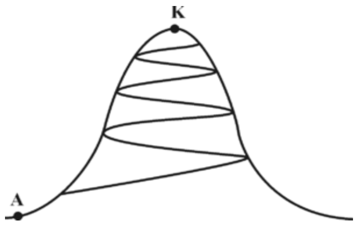


Problem 8. We go to a belvedere castle at the top of a hill (K in Figure 6). If we start below, from the village (A), how many different ways (paths) can we walk, if we always have to go upwards, and we can either use a segment of the serpentine, or of the steep paths (left of right), and we can independently decide at each crossing point whether we take the serpentine or the steep one?

[2] The word 'they' and its inflected forms are used as epicene (gender-neutral) singular pronouns.

Figure 6

Serpentine on the hill – Problem 8



Praxeological analysis on the selected thread of problems – description of the tasks, techniques and technologies

In this section, we conduct a praxeological analysis on the aforelisted problems. We describe the tasks, the techniques, and reveal the technology part of the logos blocks. We do not consider the theory parts in details, as we consider the decisive feature in focus to be part of the technology elements.

c. An interpretation of the elements of the problem-solving processes

In this section, we describe how elements of the problem-solving processes in the Pósa camps and experimental classes are interpreted, by using concepts, tools of ATD – in italics. The problems to be analysed are regarded as *questions*. The solutions, which are on the one hand teachers' anticipated solutions, but as well ones that have dominantly occurred as students' solutions, contain the *answers* to the questions. We also regard, for the praxeological analysis, the questions (the problems) as the *tasks*, the sequence of 'main' actions of the problem-solving process of elaborating the answers as the *techniques*, the explanation for the use of techniques for solving the particular tasks, or any supporting comments on these processes, that is the argumentation and proving elements as *technologies*, and the justification at a higher level as the *theory*.

Not all possible, not even all (in the camps and experimental classes) occurred *techniques* are presented, only the ones that are the most frequent, and more importantly the corresponding *logos blocks* of which contain the common element 'recursion' or 'recursive thinking', for the sake of the development of which these particular tasks were posed during the Pósa sessions.

Sometimes *technology* is described before the *technique*, reflecting a typical order of the ideas emerging during the problem-solving processes in the Pósa camps and the experimental classes, where the procedure of students working with the problems, that is, experimenting, conjecturing, testing, verifying, … is a whole, and no use of *technique* without revealed *technology* would make any sense or (real) value during these sessions. The *techniques* are sometimes not fully and explicitly written down when it seems trivial to figure them out for the expected reader.

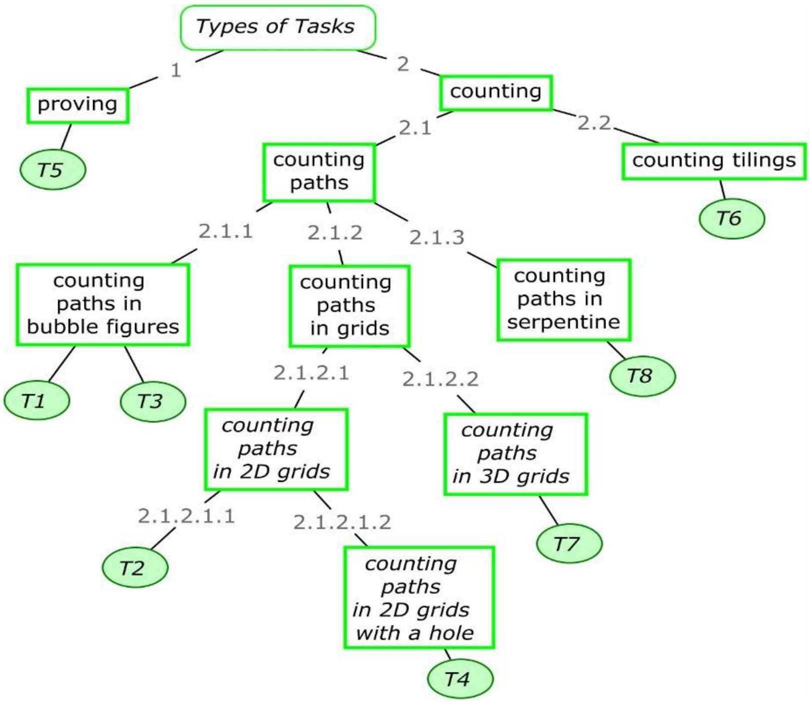
d. Types of tasks in the analysed sample

There are basically 2 main types of tasks: *counting* (different possibilities) and *proving*. The task embedded in Problem 5, denoted by Task 5 (T5) belongs to the proving type, all the other tasks belong to the counting type. However, in all presented cases, as in the case of each and every problem-solving process in the Pósa sessions, proving, arguing for the solution, searching for and formulating an explanation for the elaborated answer, or in other words, revealing, becoming aware of, expressing and discussing the corresponding technologies or even the theory parts are inseparable parts of the whole process.

The counting types may be separated into 2 subtypes: *counting paths*, in Task 1–4 (embedded in Problems 1–4) and Tasks 7–8 (Problems 7–8), and counting covering patterns (with dominoes), exemplified by Task 6 (Problem 6), which may also be called *counting tilings* task type. The counting paths type also contains 3 subtypes: counting paths in bubble figures (Tasks 1 and 3), in grids (Tasks 2 and 4 in 2 dimensions, and Task 7 in 3 dimensions) and the serpentine paths counting task type (Task 8). A graphical representation of this categorization is Figure 7 below.

Figure 7

Types of tasks to be analysed



e. Techniques and technologies for Problems 1 and 3

Technique 1A and 3A (techniques for Tasks 1 and 3, and Technique type A). Students may try counting the different paths one by one, with systematically registering the options (between intersection points) they had already taken. They may do it by drawing or colouring the different paths or only by imagining them. The operation they use here is *adding* (one by one).

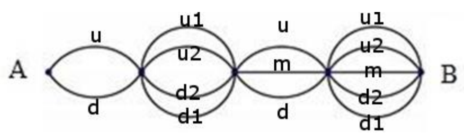
Technology 1A and 3A. By carefully determining cases and subcases, based on options between intersection points, they may 'quite surely' count all the different paths. They can use the operation adding (one by one), as those sets of possible paths the numbers of which they add do not have a (non empty) intersection, that is, they are disjoint. We call this the *additive method*. The technology they use is that *the number of elements of the union of disjoint finite sets is the sum of the numbers of elements of the sets*.

Students can also only start using this technique, and based on observations during the use of it, proceed on the following different techniques, Techniques of type A' or B.

Technique 1A' and 3A'. Similar to Techniques 1A and 3A, counting one by one, and using *addition*, but instead of drawing, colouring or imagining the different paths, students first code the paths e.g. by using sequences of letters, for e.g. *u* and *d* for up and down respectively and *m* for the middle way, where it is needed; see Figure 8. This technique requires the understanding of *symbolism*, in its technology part.

Figure 8

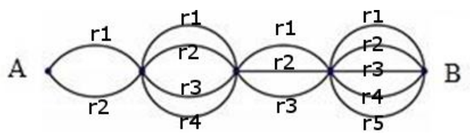
Symbolizing & addition' technique – version1 for Problem 1



The use of different symbols, denoting the possible options between neighbouring intersection points by *r1* for road 1, *r2* for road, etc., see Figure 9, may foster a shift to the use of Technique Bs.

Figure 9

'Symbolizing & addition' technique – version 2 for Problem 1



Technique and Technology 1B and 3B. Students realize that, e.g. in case of task 1, starting from point A, either you choose the higher or the lower path, and in both cases, you have the same number of choices later on (or, in the case of Problem 3, that you have 4 choices to arrive at B from the preceding point, and you have the same number of possibilities to arrive at this preceding point in all the 4 cases). This may result to noticing that if you count the number of paths between those pairs of points (edge points), which are connected by a path and for which any path going from the points between them (inner points) goes only to these inner or edge points (let us call these pairs of edge points together with their inner points *sections*), than you can multiply these numbers you have got for the sections to get the final correct answer. Counting the possibilities for the sections, and *multiplying* these numbers are the used techniques. The explanation behind multiplication is the so-called *product rule*, which is part of the technology and which relates to rules of counting the *number of elements of the Cartesian product of finite sets*.

f. Techniques and technologies for Problems 2, 4 and 7

Technique 2A, 4A and 2A', 4A'. Similar that of 1A, 3A and 1A', 3A', counting one by one, using the *additive method*.

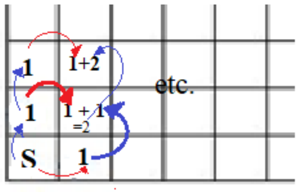
Techniques 2A' and 4A'is again based on *symbolizing* first, which again may lead to the use of techniques type B.

Technology 2B and 4B. For a given field X in the grid, you can either arrive from its left neighbour X_l or from its bottom neighbour X_b. On one hand, if you add the step X_b to X as a final step to any proper path from the bottom left corner S to X_b, you get a proper path from S to X; and using different S to X_b paths results in different S to X paths. The same holds for proper paths from S to X_l added the step X_l to X. Of course, starting with an S to X_b path results in a different S to X path than starting with an S to X_l one. On the other hand, all the different paths from S to X is either a path from S to X_l and then 1 step from X_l to X, or a path from S to X_b and then 1 step from X_b to X; and different S to X paths contain different S to X_l or S to X_b paths. This *bijection* is in the heart of the technology that gives account for the technique to solve the task. Here we use a kind of *addition method* again. For Task 4, the technology part is broader, corresponding to an additional technique needed, containing the *product rule*, explained in the followings.

Technique 2B and 4B. You can start counting the arriving possibilities for the particular fields (X) one by one, by starting from S, and in each step (for X) counting the sum of the arriving possibilities for X_l and for X_b. and then adding them. It continues until the arriving possibilities for all fields are counted, finally for the top right corner, which was asked. It is again a variation of the *additive method*, for a part of it, see Figure 10.

Figure 10

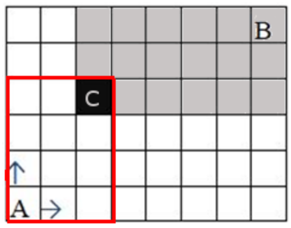
The 'additive method' technique for counting paths in a grid – Problem 2



For Task 4 (in very brief), you additionally need to count the number of paths in that you step on the black hole (B), using product rule as in Task 1 and 3, shown by Figure 11, and then to subtract it from the total number of paths.

Figure 11

Representing part of the technology behind multiplication – Problem 4



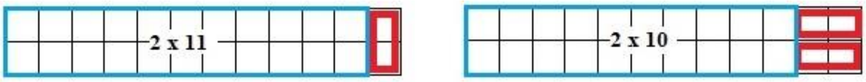
Technique and Technology 7. As a 3D variant of the techniques and technologies in Task 2, focusing on the B type of techniques and technologies, you can use the *additive method.*

g. Technique and technology for Problems 6 and 8

Technology 6 (only type B is discussed). We can place a domino in basically two ways: in a vertical or in a horizontal position. First, it is easy to see that if we place 2 dominoes horizontally in a way that the right side of one is just below the left side of the other (or the other way round) we will not be able to cover the whole rectangle, therefore if we place a domino horizontally we need to place another one just above or below it. That is, we have two building blocks, a 1×2 vertical one and a 2×2 one. Second, if we start with a vertical one, we have a 2×11 rectangle still needs to be covered, and the 1×2 vertical start plus any different 2×11 proper coverings results in different proper 2×12 coverings. Starting with a 2×2 building block, we have a 2×10 rectangle remained to be covered, and the 2×2 start plus any different 2×10 proper coverings results in different proper 2×12 coverings. Any covering with one type of start (building block) essentially differs from any other one with the other type of start. Third, any proper 2×12 covering is either a '1×2 start + proper 2×11 covering' type or '2×2 start + proper 2×10 covering' type, and different proper 1×12 coverings contain different proper 2×11 or 2×10 sub-coverings. Here, we also have a *bijection* (as in task 4) being in the heart of Technology 6. Of course, the same *correspondence* can be told to smaller rectangle segments, e.g. tracing the 2×11 coverings back to 2×10 and 2×9 coverings, etc.

Figure 12

Illustration of the main idea of the Technique for Task 6



Technique 6. Counting the possible numbers of coverings with each type of start (or ending) and adding these two numbers, you get the final result. We can now count the number of possible coverings for smaller rectangles, first for a 1×2 rectangle (: $=F_1$) = 1, for a 2×2 (: $=F_2$) = 2 (2 vertical blocks or 1 2×2 block (2 horizontals)), $F_3 = F_1 + F_2 = 3$, etc. you can calculate finally F_{12}.

We note that as part of the technology, students can also realize that these are the *Fibonacci numbers*, but the majority of them usually have not learnt about them before; it is discussed together with the teacher.

Technology and Technique 8 (only type B is discussed). If you look at the crossing points, which are all along the serpentine, numbering them according to their order in the serpentine line towards K, we have F1, F2, … F9 = K (we can have F0 = A). Let these symbols also mean the different arriving possibilities to these points. To K, you can arrive from either F8 or F9, and a similar *bijection* can be discovered to the one we found at Task 6, and similarly, you can discover the *Fibonacci sequence* and calculate K, by adding the numbers of arriving possibilities at the proper crossing points.

h. Technique and technology for Problem 5

Multiplying the possible last digits of powers of 3, which are 1, 3, 7 and 9 (known and proven before) by 3, we get 0, 0, 2 and 2 respectively as the digits taken to the decimal place. If there was an even number at the decimal place, say n, of a power of 3, then we get the last digit of $3\times n + 0$ or $3\times n + 2$ for the decimal place of the next power of 3, which digit is also even.

Pósa problems connected at the logos level, as a didactic strategy of the method – Preliminary result and conclusion

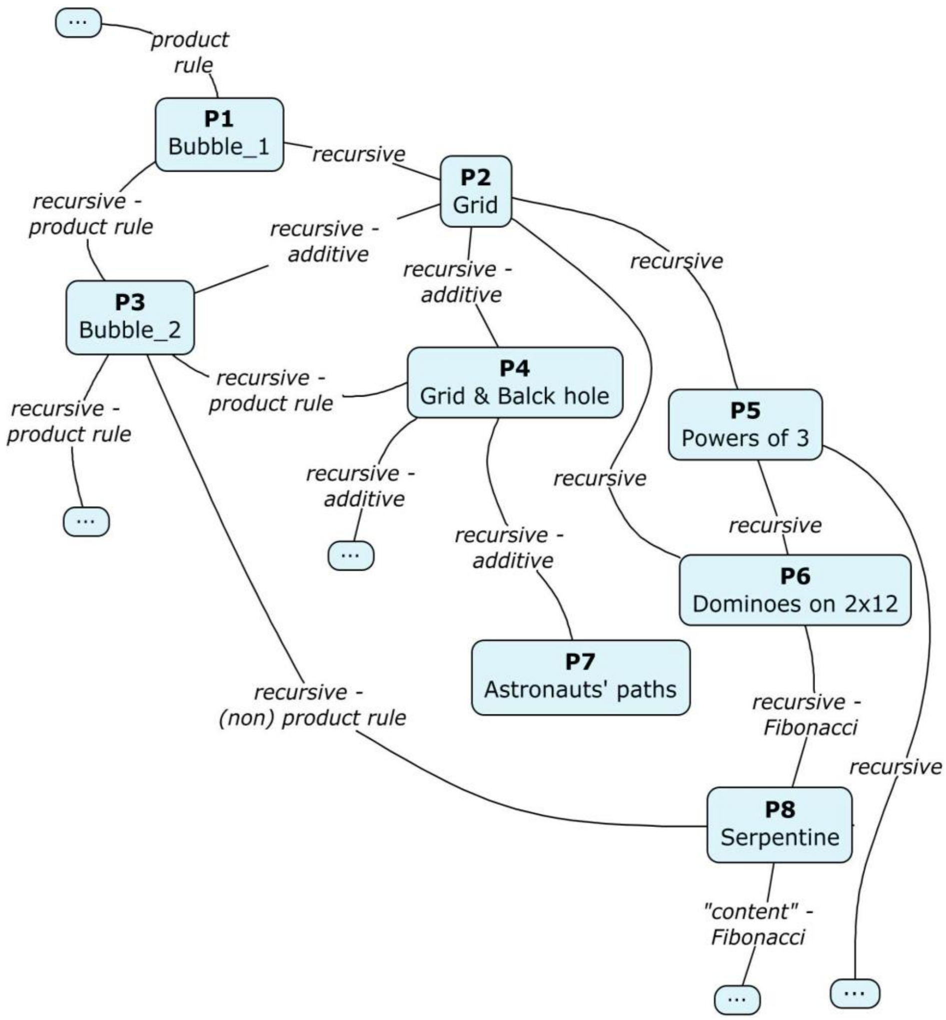
i. Common element of the technology parts – recursion

We can realize and lay focus on a common element of the *technology* parts of all the problems, in the cases of Problems 1-4 with respect to the B types of techniques and technologies, which is *recursion*. They all require the students to trace the original tasks back to ones that are 'structurally' exactly the same, and this process needs to be repeated for a (finite) number of times, until they arrive at an 'initial' task, which is easy to solve (or they can start with this easy one); that is, students need do, use and develop different kinds of *recursive thinking*.

In Technologies 1B and 3B, it is about tracing back to the so-called smaller 'sections'. This recursion leads to the use of the so-called product rule. In Technologies 2B and 4B, the recursive step is to trace the SX paths back to SX_b and SX_l paths. In technology 7, it's similar, but in 3 dimensions, so some fields have more neighbouring fields to take into account. In Technologies 6 and 8, it is the Fibonacci recursion. In all these technologies, in order for counting the different paths or coverings, students may count the same kinds of things in smaller segments of the figures, or in smaller rectangles for coverings. In Technology 5, the recursion step is between a power of three and the next power of 3[3], which step is in the heart of the proof presented.

Figure 13

The analysed problems connected by recursion in their logos blocks



The following graph, in Figure 13, represents the revealed structure of this 'technological' relationship, with the subtypes of recursion or with some other important elements in the technology part, such as product rule, showing the important revealed and

[3] The technique and technology for Task 5 is also an example of 'proof by induction'. Here, we do not aim at analysing the relationship between a 'recursive process' and 'proof by induction', although it seems to be an important research task during the continuation of the present research.

(during the sessions) discussed relationships of pairs of problems, and also indicating the connectedness to some other problems that could not be part of the presented analysis.

j. Relating presented ATD related research results to the previous WPT model and the concept of kernel

Recursive thinking, according to the WPT model (first, though not elaborately appeared in Katona & Szűcs, 2017) is (one of) the supposed *kernel*(s) of the threads formed by these (and other, partly similar) problems posed in the Pósa camps.

According to the results of the analysis of this present paper, we conclude that the kernel 'recursion' is a common element of the technology parts in the logos blocks of the problems that are part of the thread that is formed by this kernel. The kernel recursive thinking is an element of the intersection of the technology parts relating to certain techniques of the tasks that are embedded into the particular problems. In that respect, we have a set of *point praxeologies,* connected by their technology parts and thus forming a *local praxeology,* which appears as a decisive feature of the problem set used in the Pósa method.

In its current state, the Pósa WPT consists of hundreds of problems, which are connected to each other in multiple ways, through several kernels, such as *experimentation*, *yieldingness, invariant quantity*, *bounds (upper and lower)*, *pigeonhole principle*, *recursive thinking*, *induction (mathematical)*, *proof of existence*, *proof of impossibility*, *proof by contradiction and representational change*. They also belong to several mathematical content areas, such as geometry, functions and analysis, algebra, number theory, combinatorics, graph theory, probability and statistics. Finding further correspondence between the kernels and results of praxeological analyses, of which the present paper could only offer a sample, as well as conducting further praxeological analyses are needed and planned to be done in order for a deeper understanding on the structure and application mechanism of the Pósa WPT.

k. Discovering previously hidden local praxeologies through working with technologically connected point praxeologies – an element of the didactic strategy of the Pósa method

As concluded from the analysis of the present paper, considering also information on the Pósa camps collected so far and mostly presented in the 1st chapter, teacher(s) of the Pósa method present(s) (seemingly) isolated tasks of different types, appearing for the students during (the beginning of) problem-solving as a set of tasks from different point praxeologies. These tasks are related by a common technological element, which technological relation creates a local praxeology (about recursion) these point praxeologies belong to. The common technological element and the existence of the local praxeology is first hidden for the students, the local praxeology is for them to approach during problem-solving. The students develop techniques to solve the tasks; in the case of the analysed sample, for the students, there (probably) are 3 types of tasks: counting paths (P1-4 and P7-8), counting tilings (P6) and one with number regularity (P7). This appears at the level of praxis, when the students know the tasks and the techniques. Usually in parallel with this process, that is during, but sometimes before or after this praxis level procedure, students are required to create rationale for the techniques, usually at the level of proofs, which is also discussed together and becomes a commonality, so that the technology, and most importantly, the connectedness of the technology parts, the kernels are made explicit, and also to be used later. This is the revealment of the local praxeology they have approached, elaborated, discovered. Students understand that and why these problems are connected, why the tasks approaches are of the same type. In our case, they understand that recursion is a common generator of the techniques they used. This also allows students to elaborate new (related) techniques for solving new (related) tasks, due to the productiveness of this connecting technological element, the local praxeology, that it also has the capacity to generate new techniques to address new problems.

Brief list of issues **for further studies on the Pósa method in the ATD context**

– further praxeological analyses focusing on the logos blocks of the Pósa problems, and further support between the concept kernel and the intersection of logos blocks of particular sets of Pósa problems;

– *collecting (more) and categorizing kernels used in the Pósa method according to their functions in learning and doing mathematics;*

– *study dialectics in the application of the Pósa method, such as the dialectics of questions and answers and of media and milieu (Bosch et al., 2019) for revealing more of the structure of the problem set;*

– study the 'generating potential' of questions in the Pósa method, regarding them as *generating questions (Bosch et al., 2019)* and detecting types of generations used in the method;

– *study the relativeness of questions being generating, and what makes a question generating;*

– *study on the dynamicity of the WPT, comparing different implementations of the method with the same 'generating questions', whether different derived questions emerged;*

– *completing the description of the didactic strategy of the Pósa method;*

– *determining basic guidelines for the development of a new curriculum in Hungary based on the Pósa method, with the development of new (partly content-based) kernels corresponding to current curriculum requirements*

Acknowledgement

This study was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences.

The author owes utmost thanks to Marianna Bosch Casabò for her helping hand into the world of ATD research and for finalizing the present paper.

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