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Generating the *raison d'être* of logical concepts in mathematical activity at secondary school: Focusing on necessary/sufficient conditions

Générer la raison d'être des concepts logiques dans l'activité mathématique à l'école secondaire: se concentrer sur les conditions nécessaires / suffisantes

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Abstract

In this paper, we focus on the absence of the raison d'être for logical concepts, especially regarding necessary conditions and sufficient conditions, in mathematics at secondary schools. We investigated the fundamental role of these concepts in the mathematical organisation of mathematicians, which is related to their protomathematical and paramathematical values. Then we designed, implemented, and analysed a study and research activity with the aim to activate their functionality.

Keywords: Logical concepts, Anthropological theory of the didactic, Necessary/sufficient conditions.

Résumé

Cet article porte sur l'absence de la raison d'être des concepts logiques, en particulier des conditions nécessaires et des conditions suffisantes, dans les mathématiques secondaires au Japon. Nous étudions le rôle fondamental de ces concepts dans l'organisation mathématique de mathématiciens, qui est lié à leurs valeurs protomathématiques et paramathématiques. Ensuite, nous concevons, mettons en place et analysons une activité d'étude et de recherche ayant pour but d'activer leur fonctionnalité.

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Mots-clés: Concepts logiques, Théorie anthropologique de la didactique, Conditions nécessaires / suffisantes.

Generating the *raison d'être* of logical concepts in mathematical activity at

Though a number of studies about the teaching and learning of proofs have been carried out, mathematical proving seems to remain difficult to learn for a majority of students over the world. In Japan, mathematical proving is taught first in lower secondary schools, traditionally in connection with geometry, and then more developed proofs and their related concepts, including contraposition, reduction ad absurdum, and mathematical induction etc., are studied in upper secondary schools. However, these concepts seem far from well understood for Japanese students; in particular, most of them do not recognise the *raison d'être* of such concepts.

The research questions of this paper are as follows: what is the economy of the lack of the *raison d'être* of logical concepts on proofs in Japanese secondary schools? How can we generate the *raison d'être* of such objects? Against this backdrop, within the framework of the *anthropological theory of the didactic* (ATD) and the methodology of the *didactic engineering* within ATD, we designed, implemented and analysed a relatively short-scale inquiry-based teaching and learning activity with some predetermined target contents, that is, a *study and research activity* (SRA).

Preliminary analysis: a new school epistemology of logical concepts

Disappearance of the *raison d'être* of logical concepts at school

Within the *didactic transposition theory*, Yves Chevallard distinguished three types of notions which arise in mathematics teaching and learning: *protomathematical*, *paramathematical*, and *mathematical* (BROUSSEAU, 1997). Using these definitions, the actual mathematical notions correspond to mathematical objects to be studied, while concepts such as mathematical proofs at school are tools for studying mathematics rather than mathematical notions themselves to be studied; these are paramathematical notions. While both types of the above notions are recognised explicitly by a person or an institution in a given situation, some notions are not. The protomathematical notion is 'mobilized implicitly in uses and practices, its properties are used to solve certain problems, but it is not recognized, as a topic of study or even as a tool' (ibid., p. 153).

Now we will consider some logical concepts taught in secondary schools. Since the concept of mathematical proof is a paramathematical notion, its teaching is associated with other mathematical knowledge to be taught and, for example, produces new properties among known mathematical notions. Then, what do logical concepts work for? Some logical concepts, such as ad absurdum, necessary or sufficient condition, etc., have the effect of coordinating the consideration of proving, which we describe in detail in a later section. Some students may be able to use these effects unconsciously, without recognising such concepts, and successfully make a mathematical inquiry. Thus, we may say these concepts have the protomathematical nature at first; that is to say, logical concepts support implicitly mathematical proving. And then, they could shift to the state of notions similar to the mathematical proof: thus, logical concepts can become paramathematical objects in mathematical activities. However, this transition is not so much linear as repetitive: a logical concept goes back and forth between a state of being a tool and a state of being an implicit model in a given activity. Such a *dual functioning for directing proving activities* is, we regard, the *raison d'être* of logical concepts, that is, the fundamental functionality of them, which usually disappears at school, as we will describe below. This implies that these logical concepts could be learned in inquiry-based situations.

The Japanese educational guideline claims 'to understand the fundamental notions concerning sets and propositions, and to make use of these notions in considerations of phenomena' (MEXT, 2001, p. 20, our translation) as a content of mathematics to be taught in the first year of upper secondary school. And in fact, in Japanese school mathematics of this grade, logical concepts related to proving, such as necessary or sufficient condition, contraposition, and reduction ad absurdum, are studied in a small unit of 'sets' in rather decontextualised and theorised situations, that is, in connection with set theory, in spite of the claim from the guideline of proper use of these notions. Thus, when students learn logical concepts, these concepts are introduced as if they are mere mathematical notions, skipping both the paramathematical stages and the protomathematical stages. As a result, many students only memorise the facts such as 'when $P \Longrightarrow Q$, Q is the necessary condition for P', by phrasing it as 'the tip of an arrow is necessary' with a metaphor of hunters' material 'arrows'. They are then forced to tackle fill-in-the-blank tasks with the appropriate phrases, which merely checks their memorising:

– The condition x=6 is (   ) for x^2=36.

This is nothing but an extreme case of the phenomenon of *monumentalisation* of knowledge (cf. Chevallard, 2006). We think that this is the reasons for the poor role of such concepts and the absence of the *raison d'être* for them in many students' praxeological equipment. In short, *logical* concepts at school need relevant *mathematical* praxeologies wherein such concepts are available.

A reference model of necessary/sufficient condition

In the above general analysis of logical concepts at school, we argued that, at upper secondary schools, logical notions are parts of mathematical knowledge to be taught but lose their *raison d'être*, which is the twofold functioning with protomathematical value and paramathematical value in proving activities. From now, let me focus on the necessary/sufficient conditions. What exactly is their *raison d'être*? To answer this question, we refer not to the nature of the logical concepts in scholarly logical knowledge, but to the actual functions of them in the mathematician's inquiry. This means that we understand logic at school not as an independent disciplinary field but (mainly implicit) parts of mathematics, as we have already implied in the above.

In mathematicians' inquiry, the propositions or conjectures to be considered are, of course, not always biconditional. In many cases, a target condition in the inquiry, for example 'a given natural number *n* is a perfect number', exists, and more accessible conditions which have close relationships with the target are desired. Thus, for a target condition *P* and another condition *Q* being considered, it is necessary to describe the relationship between *P* and *Q*, and especially to distinguish '*Q* is necessary for *P*'and '*Q* is sufficient for *P*'. It is this distinction which enables the orientation of the consideration, that is, the clarification of what should be proved. Such a role of guiding proving activities appears not only in paramathematical form but also protomathematical form. In many cases, mathematicians operate this kind of reasoning naturally and unconsciously in their mathematical inquiries. We regard this proof-guiding functioning with protomathematical value and paramathematical value, rather than the nature as mere mathematical notions, as the *raison d'être* of necessary/sufficient condition at school, and as a principal ingredient for our *reference epistemological model* of them (BOSCH & GASCÓN, 2006).

The next step was to design an actual mathematical activity in the school institution which promotes the dissemination of the above role; in other words, we considered the possibility of the transposition of this knowledge into the school institution.

Design and *a priori* analysis: conditions for a quadrilateral to be a parallelogram

A priori analysis of the possibility of study and research activities as a generator of the *raison d'être* for logical concepts

We used the didactic organisation of *study and research path* (SRP) which starts from a *lively and generating initial question* (BARQUERO & BOSCH, 2015), for encouraging the emergence of the *raison d'être* for logical notions. We considered that SRP generally produces the *raison d'être* for such paramathematical notions similar to what happens with mathematical notions.

But on the other hand, our interest is related to the viability of the designed SRP in current educational systems. An SRP is usually designed and realised with a highly open inquiring trajectory and long-term period. These properties of SRP create a difficulty of designing an SRP focusing on a particular praxeology, and of the implementation of SRPs in ordinary secondary schools, at least in Japan, because many existing teaching methods and pedagogies postulate a closed trajectory and short-term period, following the monumentalism.

To manage these difficulties and constraints, we adopted a short-term SRP, or an SRA requiring the proper role of logical concepts, which are contents in the current Japanese curriculum. Our practice, which we describe below, consists of only two class periods of 45 minutes each. Such a practice using this content and at this scale can be readily implemented in Japanese secondary schools.

Mathematical design of initial question

In this section, we propose a mathematical activity for the upper secondary school level including inquiries, where what should be proved is ambiguous and needs to be clarified by the protomathematical nature of logical concepts including necessary and/or sufficient conditions.

Let us recall the characterisations of a parallelogram. What condition is sufficient for a quadrilateral to be a parallelogram? In Japanese lower secondary schools, they do not use the word 'sufficient', but take up the following five conditions in classes as 'conditions for a parallelogram':

– Two pairs of opposite sides are parallel. (Definition)
– Two pairs of opposite sides are equal in length.

– Two pairs of opposite angles are equal in measure.
– One pair of opposite sides are parallel and equal in length.
– The diagonals bisect each other.

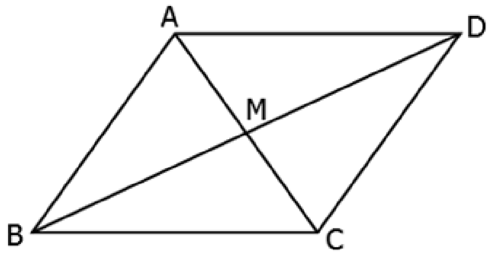
Each of these five conditions consists of two conditions out of the following:

– a pair of opposite sides are parallel;
– a pair of opposite sides are equal in length;
– a pair of opposite angles are equal in measure;
– and a diagonal bisect the other.

These are, namely for quadrilateral ABCD with its diagonals crossing at M, the following eight conditions : (a) AB=CD, (b) AD=BC, (c) ∠A=∠C, (d) ∠B=∠D, (e) AM=MC, (f) BM=MD, (g) AB∥CD, (h) AD∥BC. Using this we propose the following initial question.

Figure 1

Parallelogram



Q_0: In addition to the five conditions learnt in lower secondary school, are there any two conditions from (a) to (h) that make quadrilateral ABCD a parallelogram?

1. *A priori* analysis of possible mathematical inquiries

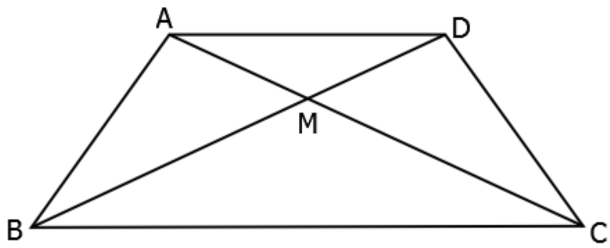
The 40 students in our study are in the first year of a typical upper secondary school in Japan (15-16 years old) and, in general, students in this school are not very competent in mathematics. The students were just about to study logical notions in the small unit of 'sets' and did not have any previous knowledge about these notions. Most of them were not even familiar with the notion of counterexample. In our design, we nonetheless prompted them to make their own inquiry against the initial question Q_0 in the first period, and the second period was devoted to the theorisation of the notions. In addition to the main teacher who is the usual teacher of these students, the two authors joined to lead the inquiry in the first class period.

Including the five conditions learnt in lower secondary school, there are 28 combinations and 16 elements of them are sufficient condition to make a quadrilateral a parallelogram, but 12 are not. In fact, various inquiries can be possible and some of them are rather accessible, even for beginners. The following three kinds of combinations are relatively easy to judge sufficiency:

– (a) AB=CD and (h) AD∥BC are not sufficient. A counterexample exists, of an isosceles trapezoid. The case of (b) and (g) is also the same.

Figure 2

Isosceles trapezoid as a counterexample.



– (c) ∠A=∠C and (g) AB∥CD are sufficient. Also the same for (c) and (h) etc. It can be easily proved by using the property of consecutive interior angles of parallel lines.

– With some insight, one can find a kite, which is a counterexample to the sufficiency of (c) ∠A=∠C and (e) AM=MC. The case of (d) and (f) is also the same.

All other cases are comparatively challenging. For instance, the counterexample for the combination of (a) and (c) might be found through the attempt to prove its sufficiency, while the consideration of the case using (c) and (f) may lead learners to the concept of reduction ad absurdum. The detail is mentioned by Hiroaki Hamanaka (2016).

Thus, this initial question has the possibility to induce various considerations involving the knowledge to be taught in existing educational systems.

Didactic design of teaching process

In the first period, the five characterising conditions of parallelograms were recalled, the initial question Q_0 was proposed, and then students were prompted to create their own inquiry in small groups, each consisting of approximately three students.

Our hypothetical trajectory of SRA starts from a type of tasks ‘to find sufficient conditions under which a quadrilateral becomes a parallelogram’ through the Q_0. The type of tasks includes twenty-eight tasks, each of which corresponds to two out of the eight conditions and belongs to a *genre of tasks* (cf. Miyakawa, 2012) or paramathematical type of tasks ‘to judge whether it is a sufficient condition under which an object becomes a specified object’, which we call the reference genre of tasks in this paper. The students might think this is a study about parallelograms, however, in this context, the inquiry about parallelograms is rather a possible condition for bringing about the praxeology at stake which involves this genre of tasks. This is the moment of the *first encounter* (cf. Barbé, Bosch, Espinoza & Gascón, 2005) with the mathematical praxeology at stake. This genre of tasks would produce the technique which consists of clarification of the proposition to consider, a search for a counterexample, and proving characterisation with the geometrical knowledge.

In the inquiry, when the students considered the combination of conditions (a) and (h), for instance, if they drew a figure of a parallelogram first, this figure would lead them to the consideration of a proposition different from ‘if (a) and (h) are true, then this is a parallelogram’, since this figure does not portray the assumptions, but instead, the conclusion to be proved. It would indicate the form and the direction of their considerations what figure they would draw first or what they would intend in drawing figures: this would indicate that they do realise, explicitly or implicitly, both what they assume and what they intend to conclude. Of course, they may face confusion in determining which is an assumption and which is a conclusion. However, this confusion could even help them to realise the difference between considerations of a proposition and its converse. In any case, for this task, they had to set up their own assumption and conclusion. This *moment is of the exploration of the type (or genre) of tasks*.

The important technique here is to clarify what proposition 'the combined conditions is the condition for a quadrilateral to be a parallelogram' is, that is, the *propositionalisation of the discussion* which identifies the assumption and the conclusion in discussion and formulates it as a proposition. In many cases, this technique would be performed implicitly, in other words, this technique has the protomathematical nature, while the consequent proof of the obtained proposition has the paramathematical nature. If one succeeds in the propositionalisation and obtains a result, the necessity of some concepts to explicitly express how the result involves the characteristics of parallelograms, that is, the *raison d'être* of the notions of necessary/sufficient conditions, becomes apparent. After undergoing these processes, protomathematical notions would change into paramathematical notions or mathematical notions in the second class period, where the inquiries performed in the first class would be reflected upon and reconsidered, and its techniques would be theorised. This phase is the moment of the constitution of *technological-theoretical environments* and *the institutionalisation*.

In addition, students might perform a transformation from the problem whether (a) and (h) are characterisations of a parallelogram or not, into whether there exists a quadrilateral which satisfies (a) and (h) but is not a parallelogram. This transformation would be done implicitly and it has a protomathematical nature first, but it can be developed into paramathematical notions of rebutting by a counterexample, and into a mathematical notion of counterexamples. Thus, the practical aspects of this notion, which promote the eligible consideration of judging a proposition, can be actualised in this inquiry. This phase is also the moment of the exploration, and the second class period is the technological–theoretical moment about the notion of counterexamples, where the negation of a given proposition is reviewed.

Our design has no moments of technical works and evaluation, because our experiment is strongly constrained by short didactic time periods. The outline of the model is shown in Table 1.

Table 1

Outline of our reference epistemological and didactic model of necessary/sufficient condition

	1st period	2nd period
Didactic moments	First encounter Exploration	Constitution of *logos* part Institutionalisation
States of logical notions	protomathematical	paramathematical

***In vivo* and *a posteriori* analysis**

When students created their inquiries in the first period, they started by drawing figures. At first, some of the students struggled with the question of how to draw the quadrilateral satisfying the selected conditions, which is not a task involving the mathematical praxeology at stake; that is to say, the students had not yet encountered the reference genre of tasks. However, in a short time they found themselves drawing a parallelogram and began to consider which figures should be drawn and what to consider with it, which involves the reference praxeology. We identify this moment as the first encounter with the genre of tasks at stake.

A considerable number of students realised, although implicitly, that what should be considered is not whether a parallelogram satisfies the condition, but, whether all the quadrilaterals satisfying the condition are parallelograms. This can be seen from their drawings of figures on the work sheets. Some of those students noticed that it is important to try to draw a figure which satisfies the conditions and is simultaneously as far from being a parallelogram as is possible. This is implied from the following students' conversations:

> – 'Well, since we have to consider whether this becomes a parallelogram or not, we'd better not model a parallelogram when we are drawing'.
> – 'I'll keep this unlike a parallelogram. What is important is to think outside the box'.
> – 'The result is yes for the combination of (b) and (g). Because it seems impossible to draw a figure which is not [a parallelogram]. Right?'

Thus we can recognise that the inquiry period, wherein students have to clarify the nature of the proposition to be considered for themselves could promote the distinction between a proposition and its converse, and the production of an implicit technique of transposition from the judgement of a proposition into the search for a *counterexample*. We can say this is the beginning of the moment of exploration.

In spite of the elaboration of the above techniques, students' final answer $A^{\heartsuit}$ was not fruitful as a response to Q_0. As a result, only a minority of students could find a counterexample on their own, and few students could complete the proof of the sufficiency of selected conditions. However, since we are focusing on the awareness of the role of logical concepts, what is important is not the actual mathematical answer $A^{\heartsuit}$, but the *protomathematical* process of the inquiry.

In fact, some students asserted that '[the quadrilateral with considered conditions] does not become [a parallelogram]', by indicating the possibility of a 'counterexample' which is constructed in the students' minds, although they did not know that word. We consider this kind of conjectures or statements to be a protomathematical occurrence of the notion of counterexample. Such a protomathematical occurrence is a crucial condition for the emergence of the *raison d'être* of any kind of logical concept within mathematical organisations.

We can also see the same phenomenon regarding the notions of necessary or sufficient conditions. In the first class, some students drew parallelograms and confirmed repeatedly that it satisfied the selected conditions. Although this implies that such students misunderstood what ‘the condition for a quadrilateral to be a parallelogram’ is, their struggles to prove the proposition they set indicate their recognition of the process: they did select an assumption and a conclusion on their own. Hence, it is possible even for such students to compare their reflections with others and realise the notions at stake in the second class.

In many groups, students summarised their affirmative or negative results using original Japanese expressions which mean ‘become’ or ‘not become’ for each pair of conditions which involved the notions of counterexample and necessary/sufficient conditions with protomathematical states. Then, in the second period, after student presentations of their results, the teacher asked and confirmed what ‘become’ means here. The teacher then wrote down the following and asked for the difference between them:

– ‘When (c) and (g) are true, a quadrilateral is a parallelogram’.
– ‘When a quadrilateral is a parallelogram, (c) and (g) are true’.

After some discussion, a student answered, ‘they are different. One means that the satisfaction of conditions (c) and (g) makes it a parallelogram, while the other means that there is a parallelogram first and then it satisfies the conditions’. Although the teacher shook students’ understanding by using the expression ‘when’ instead of ‘if’, this student pointed out the implementation in the sentences and explicitly distinguished one from the other. Thus, students understood the difference, and this distinction would be a technique in their praxeology equipments. It was at this point that the teacher introduced the notions of ‘sufficient condition’, ‘counterexample’, and ‘necessary condition’, and students, as expected, were able to learn these concepts as paramathematical notions which confirm their technique as a technology.

In conclusion, based on this experiment, we can conclude that the role of logical notions like sufficiency, necessity, and counterexamples, in other words the praxeology involving these notions, can live in the mathematical activities in secondary schools. Moreover, in some appropriate inquiries, the *raison d'être* of these notions arises rather naturally and regardless of the results of the inquiries; what is important in the genesis of these notions is its role of the coordination of paramathematical notions, not a direct coordination of mathematical notions. We consider that this lower susceptibility makes this proposed activity viable in secondary schools.

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