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Mathematics education in the context of certain classical debates in philosophy and mathematics

Educação matemática no contexto de alguns debates clássicos em filosofia e matemática

Educación matemática en el contexto de algunos debates clásicos en filosofía y matemática

L'enseignement des mathématiques dans le contexte de quelques débats classiques en philosophie et en mathématiques

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Abstract

The paper presents some illustrative turns in the history of the interactions between philosophy, logic, mathematics, and mathematical education since the 16th century. The underlying problem could be called the Aristotelian problem. Aristotle argued that any individual thing consists of a substantial form, which determines its general nature, and matter, which individuates the thing and makes it numerically distinct from other similar substances.

Keywords: Language, Logic, Philosophy, Mathematics, Different forms of complementarity.

Resumo

O artigo apresenta algumas voltas ilustrativas na história das interações entre Filosofia, Lógica, Matemática e Educação Matemática desde o século XVI. O problema subjacente poderia ser chamado Problema Aristotélico. Aristóteles argumentou que qualquer coisa individual consiste

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em uma forma substancial, que determina sua natureza geral, e matéria, que individua a coisa e a torna numericamente distinta de qualquer outra substância semelhante.

Palavras-chaves: Linguagem, Lógica, Filosofia, Matemática, Diferentes formas de complementaridade.

Resumen

El artículo presenta algunos giros ilustrativos en la historia de las interacciones entre la filosofía, la lógica, las matemáticas y la educación matemática desde el siglo XVI. El problema subyacente podría llamarse el problema aristotélico. Aristóteles había argumentado que cualquier cosa individual consta de una forma sustancial, que determina su naturaleza general, y materia, que individualiza la cosa y la hace numéricamente distinta de otras sustancias similares.

Palabras clave: Lenguaje, Lógica, Filosofía, Matemáticas, Distintas formas de complementariedad.

Résumé

L'article présente quelques tournants illustratifs de l'histoire des interactions entre la philosophie, la logique, les mathématiques et l'enseignement des mathématiques depuis le XVIe siècle. Le problème sous-jacent pourrait être appelé le problème aristotélicien. Aristote avait soutenu que toute chose individuelle est constituée d'une forme substantielle, qui détermine sa nature générale, et d'une matière, qui individualise la chose et la rend numériquement distincte des autres substances semblables.

Mots clés : Langue, Logique, Philosophie, Mathématiques, Différentes formes de complémentarité.

Mathematics Education in the Context of certain classical Debates in Philosophy and Mathematics

Leibniz and Spinoza

All philosophy is ultimately concerned with or even fundamentally shaped by the following question: Can we significantly influence our worldly destiny? Can one at least understand one's own life? Is there perhaps some hidden individual rational quietly ruling over each particular destiny? Or is life just a sequence of adjustments of the human subject to some universal law? In the latter case life would simply consist in adapting oneself to the universal law and in this case existence and thinking would appear as one. Spinoza believed the latter, while Leibniz invented various constructs to assure himself and others that God could only have created the best of all possible worlds (one among many) because for Leibniz God's will and his choice of a possible world are subordinated to God's reason or his logical reasoning.

Spinoza equates natural law and God, object and concept, or existence and thought. For Spinoza the existence of God and his essence, are one and the same (Ethics, Proposition I.20). Leibniz, on the contrary, puts God's spirit before nature and things. Spinoza writes:

> Our understanding would certainly be less perfect if the spirit were alone and knew nothing but itself." At the same time, what is most valuable is what is in "harmony with our nature. For when two individuals of exactly the same nature unite, they form an individual twice as powerful as either by itself (Spinoza, Ethics IV, Proposition 18).

Spinoza conceives the individual "spirit" as a universal individual, an idea that he saw accomplished in geometry:

> With regard to the statement that number is a negation and not anything positive, it is obvious that matter in its totality, considered without limitation, can have no numbers, and that number applies only to finite and determinate bodies. For he who says that he apprehends a figure, thereby means to indicate simply this, that he apprehends a determinate thing and the manner of its determination. This determination therefore does not pertain to the thing in regard to its being, on the contrary, it is its non-being. So, since number is nothing but determination, and determination is negation, number can be nothing other than negation: (...) *determinatio est negatio*. (Spinoza, Letter to a friend 1674)

Leibniz's philosophy and his view of the world was no less deterministic than that of Spinoza. Is was no less influenced by theological Ideas, but, at the same time, Leibniz's thinking was extremely individualistic and nominalist, resembling the views put forward by Protestants.

There are, in fact, two types of generality, *predicative* generality, on the one hand, and *continuity*, in the sense of Spinoza or Peirce, on the other (Peirce CP 5.102-103). An example may help clarifying this claim. The *Londoner* could be considered either as an *element* belonging to a given set, or as a concept encompassing a certain *type* of person. These two possibilities also exemplify the complementarity of object and concept, or of reference and sense.

Leibniz was a dualist in his logic and in his philosophy or metaphysics. Two fundamental principles, the *principle of the identity of indiscernibles*, on the one hand, and the *continuity principle*, on the other. These principles were both equally important to Leibniz, even though they seem to be incompatible with each other. Leibniz struggled to reconcile them.

Leibniz had inherited this problem from Aristotle. Aristotle had argued that any individual thing consists of a *substantial form*, which determines the general nature on the thing in case, and of a *matter*, which individuates the thing and makes it numerically distinct from other things belonging to the same substance. Aristotle is most often regarded as the great representative of a logic, that rests upon the assumption of the possibility of clear divisions and rigorous classification.

> But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language (…). (Lovejoy 1964, 58)

Aristotle thereby became responsible for the introduction of the principle of continuity into natural history.

And the very terms and illustrations used by a hundred later writers down to Locke and Leibniz and beyond, show that they were but repeating Aristotle's expressions of this idea (Lovejoy loc. cit.).

Leibniz believes that everything that happened could have happened differently and everything that exists could also have been different from what it is, had God willed it. But to accomplish his wishes God is constantly considering, as said, all possible worlds and he always chooses the best of all possible worlds, - namely that in which the uniformity of Nature representing the greatest variety of things, - is compatible with the determination by the laws of Nature. Against those who think that God might have made things better than he actually has, Leibniz replies that their opinions are

> in my judgment, based upon the too slight acquaintance which we have with the general harmony of the universe and with the hidden reasons for God's conduct. (Leibniz G. 1686, *III*).

Mark Stewart states the problem as follows:

> By raising God's choice to the level of possible worlds, Leibniz can have his *principle of sufficient reason* and eat it too, in a sense: that is, he can grant that all things within our world are linked together in a necessary way, while still maintaining that the world as a whole does not necessarily have to be the way it is. The reasons for the world, he says, lie in something extramundane (Stewart 2006, 238).

There is necessity in the actual world, but this necessity is just one of many. Things, therefore, are necessarily what they are in the actual world, but this necessity is a contingent one because had God chosen a different world, things could have been different. But still a strictest logic and necessity would reign in any of those worlds.

The difference between Spinoza and Leibniz described above is the expression of a deep rift between meanings (ideas) and things. For example, Leibniz proposed to develop a *characteristica geometrica* to reflect geometrical insight. On the other hand, Leibniz's ambition also included the creation of a *calculus ratiocinator,* which was conceived of by him as a method of symbolic calculation that would mirror the processes of human knowledge and reasoning. These goals are incompatible (Otte 1989, 21ff). "If controversies were to arise" says Leibniz, "it would suffice to sit down at the abacus and say to each other: *Let us calculate*!" (Leibniz 1684). But in consequence of his fundamental *principle of the identity of indiscernibles* Leibniz at the same time developed an intensional logic that has strongly influenced Bolzano and Frege. In fact,

> when Leibniz's project began to be realized in the nineteenth century, its two logical components were taken up by different research traditions. The "algebraic" school represented by Boole, Peirce, and Schröder sought to develop in the spirit of Leibniz's *calculus ratiocinator* mathematical techniques by means of which different kinds of human reasoning could be mastered. In contrast, Frege's objective was, as he himself noted in his *Begriffsschrift*, to put forward a *characteristica universalis* in Leibniz's sense, a *Formelsprache des reinen Denkens* (Hintikka 1997, Introduction, I.).

The split between conceptual logic and the algebraic calculus, that was caused by Leibniz's double orientation is also noticeable psychologically. For example, the well-known Gestalt psychologist Max Wertheimer (1880 - 1943) comments on the mathematical solution of Zeno's paradoxes by means of a geometric series, resp. the convergence of that series. Fundamental for this solution is the following calculation. One multiplying the series

$S = (1 + a + a^2 + a^3 + a^4 + ...)$ by a and subtracts the outcome from the initial series S, which immediately leads to $S - aS = 1$. Wertheimer writes:

> It is correctly derived, proved, and elegant in its brevity. A way to get real insight into the matter, sensibly to derive the formula is not nearly so easy; it involves difficult steps and many more. While compelled to agree to the correctness of the above proceeding, there are many who feel dissatisfied, tricked. The multiplication of *S* by *a* together with the subtraction of one series from the other, gives the result; it does not give understanding of how the continuing series approaches this *value* in its growth. Real understanding proceeds by considering what happens in the growth of the series and derives the law of this growth, leading to the limit. Many do not bother really to understand. They are satisfied to have the result (Wertheimer 1945).

In an appendix to his book Wertheimer presents an alternative approach to this problem. The essential characteristic of it consists in relying on the *meaning* of just a few relevant concepts (fraction etc.). "If I want to understand", he says, "I must realize from the beginning what the first term $1/a$ means as a part of its whole" (Wertheimer 1945, 218). There is nothing wrong with that. But people like Leibniz or Wertheimer demand too much: formal aesthetics and personal intimacy at the same time.

Euclid and Descartes

Euclid's *Elements* are considered to provide the first axiomatic presentation of geometry and of mathematics in general. But Euclid's geometry is concerned with the possibilities of constructing real geometrical figures and is therefore synthetic. In fact, Euclid systematized handling real geometrical objects (Fowler 1987) and introduced the geometrical postulates and theorems on the basis of an experience-based certainty.

For example, in the very short argument of §35 (theorem 25) of book I of Euclid's *Elements* the word "equal" occurs more than 10 times, with three different meanings: *congruence, equality of area,* and *numerical identity*. Among the "moderns", Leibniz and Bolzano selected *congruence* as the essential equality relation, Grassmann took *equality of area,* and Descartes or Cantor and the rest of modern mathematics understood equality as *numerical identity.*

Things, if considered strictly from a historical point of view, could become still somewhat more complicated, because the choice also depended on the kind of logic employed. For example, set-theory was initially essentially a logical conception (Ferreirós 1999).

The points in the plane or in the space of Euclid or Descartes or Newton, as well as, the Points of today's school geometry are all isolated, each marking a special place. As a consequence numerical identity becomes the only equality relation. Leibniz saw things differently. He understood Euclid in the traditional way, conceiving the congruence relation as the fundamental geometric equality. It yet turned out that congruence would not allow geometry to be algorithmized, because congruent parts can be put together in different ways, so that different "sums" arise. Grassmann drew attention to this in his *Geometric Analysis* (Otte 1989, 21ff).

This failure of Leibniz's construction of a geometric calculus was a direct result of his attempt to combine the Platonic conceptualization of geometry with the algebraic version of Aristotelian logic developed by Petrus Ramus. The works of Aristotle, rediscovered in high medieval Spain and translated into Latin, had caused a tremendous intellectual revolution in European universities, "their turbulent *Aristotelianized* schools of art and theology became the seedbeds of scientific thought" (Rubenstein 2003, 7).

Many historians credited Petrus Ramus (1505-1572) with a particularly important role in the academic debates about Aristotle's logic and rhetoric, while mathematicians and scientists regarded Ramus as mathematically and scientifically illiterate. But Ramus' efforts to convert the Aristotelian syllogistic logic, which seemed quite useless in mathematics, into an algebraic calculus proved essential. He seems to have been among the first to suggest that algebra deserved greater importance in logic. And he advocated a different, arithmetized representation in geometry. Ramus insisted that

> geometry, music and astrology cannot be sustained without numbers: These arts must therefore explain numbers and subordinate themselves to their service (Rossi 2000, 99).

Ramus influenced Isaac Beeckman (1588-1637), a very close friend of Descartes in Holland. Descartes's contact with Ramist methods was fostered by his acquaintance with Beeckman. This connection led to the idea of creating analytic geometry. In 1619, Descartes outlined some new ideas of establishing an analogy between arithmetic and geometry in an important letter to Beeckman of March 26 (Shea 1991, 44). Twenty years later Descartes published his *Geometry*. It begins, with the following programmatic statement:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the length of certain straight lines is sufficient for its construction.

Descartes program is based on connecting arithmetic and geometry by a diagrammatic model of arithmetic operations in the plane. Descartes determines the geometric quantities sought by solving specific equations instead of operating with geometric constructions. In this way, Descartes generalizes the Euclidean concept of construction. The simple “machines” of Euclid’s geometry, compass and ruler, are generalized in Descartes’ geometry becoming “machines” that allow to construct curves or functions, instead of just points.

Scientifically, the progress from Euclid to Descartes is reflected by the different problem-solving capacities of both systems. For example, the problem of doubling the square in the plane can be solved using Euclid’s methods (cf. Plato’s dialogue *Meno*), but the Delian problem of doubling the cube cannot be so solved. Descartes solves it by introducing additional objects (points, variables).

The strength of algebra lies in the possibility of turning even unknown objects into objects of mathematical operation and calculating with them. The algebraic solution to the Delian problem looks as follows.

> Let x be the edge of the cube sought.
> We then get the equation $x^3 = 2$.
> This leads to $x^4 = 2x$.
> Now substituting $y = x^2$.
> yields: $y^2 = 2x$.
> This means that x is constructed as the intersection point of the parabolas $y=x^2$ and $y^2=2x$.

The new variable y gives a new reference point and this leads to new relations. That’s the trick of mathematics: its indexical signs. Algebra is based, in fact, on a *logic of relations*, rather than on a logic of subject-predicate sentences (object-property logic). An example by the psychologist *D. Kahneman* is useful in further explaining this. Kahneman writes:

> Do not try to solve it but listen to your intuition:
> *A bat and ball cost $ 1.10. The bat costs one dollar more than the ball.*
> *How much does the ball cost?*
>
> A number came to your mind. The number, of course, is 10: 10 ¢. The distinctive mark of this easy puzzle is that it evokes an answer that is intuitive, appealing, and wrong.

> Do the math, and you will see. If the ball costs 10 ¢, then the total cost will be \$ 1.20 (10 ¢ for the ball and \$ 1.10 for the bat), not \$ 1.10. The correct answer is 5 ¢ (Kahneman 2010, 44).

In any case, psychology (Kahneman) identifies two methods and two forms of thinking. One depends on the meaning of the signs and words and dominates philosophy and the humanities. The other establishes a logic of relations between things, the known and the unknown, and is characteristic for mathematics.

Mathematical existence

Historically the issue of solving algebraic equations forced mathematicians to consider introducing new "imaginary" numbers. These proved helpful in factoring polynomials of 3rd and 4th degree which then made it possible to find general solutions to certain types of equations. In the beginning these new objects were perceived as mere *operational* symbols, rather than as true numbers.

A similar problem arose already at a more elementary level, in connection with the issue of factoring quadratic polynomials. It turned out that even simple quadratic equations could not always be solved by rational numbers. This of course, had been already known to ancient Greek mathematics. To cope with the algebraic irrationals like the square root of 2, Euler, in his *Complete Guide to Algebra*, invented a mixed notation: $a + b\,(2)^{1/2}$ or if we abbreviate $(2)^{1/2}$ by t: $a1 + bt$ for the enlarged set of the algebraic numbers, as he did not see how he could deal with the irrational numbers otherwise. The semiotic approach presented the solution to an ontological problem, creating a new representation.

To anybody familiar with the notion of *algebraic structure*, Euler's approach may suggest an analogy with the *imaginary* numbers, because 1 and t are linearly independent vectors over the rationals, in exactly the same way, as 1 and i are linearly independent vectors over the reals. However, the axiomatic idea of a vector-space did not yet exist during the 18th century and therefore nobody saw and explored that analogy before the 19th century. New mathematical disciplines, like algebraic number theory or vector calculus and linear algebra had to wait.

It seems like a sleight of hand to create new mathematical objects - such as complex numbers - by expanding the application of given structures. Many mathematicians only felt reassured when Gauss proposed a geometric interpretation of the imaginary or complex numbers. But from the point of view of mathematical justification, geometry may be superfluous here, because theories and concepts are simply instruments of thought, which are primarily intended to fulfill specific functions.

Here is an example of an analogous problem that dates back to the times of Nicolaus Copernicus, who not only recalculated the orbits of the stars, thereby causing a major revolution in respect to the understanding of the place of mankind in the universe, but who also carefully reflected upon the economics of money and its devaluation. Until the 1970s, the objective value of banknotes worldwide was secured by government promises that anyone could exchange their paper money for gold bars. In addition to the payment function, money therefore also had a substantial value. In the US, however, President Nixon withdrew that promise in 1971. The separation of *function* and *substantial value* had begun earlier.

As a result of the Great Depression after World War I, attempts were made to understand money as a *pure function of economic activity*, just as the mathematician understands numbers as purely functional instruments for specific purposes (even Gauss realized that numbers were the products of our own making, as he wrote in a letter to Bessel in 1830). This shift is well illustrated by film with the title *Das Wunder von Wörgl* (*The Miracle of Wörgl*) that tells the following true story. The global economic crisis after World War I is in full swing. The Tyrolean community of Wörgl - a small town near Salzburg in Austria - is facing bankruptcy. The Mayor rescues the community bringing back employment and prosperity through the introduction of a circulation-secured, purely functionally designed, local money (you cannot deal with this money you can only spend it and if you do not spend it loses its value). The idea was based on the economics developed by *Silvio Gesell* (1862-1932). The famous British economist *John Maynard Keynes* comments:

> Gesell's chief-work is written in cool and scientific terms, although it is run through by a more passionate and charged devotion to social justice than many think fit for a scholar. I believe that the future will learn more from Gesell's than from Marx's spirit.

Structuralist trends appeared since the advent of "pure mathematics" at about the turn to the 19th century and structural analogy became a powerful research instrument. Hilbert's logical mentor *Paul Bernays* explains:

> In the philosophy of mathematics it is a common thesis, by which one tends to characterize what is specific to mathematics, that existence in the mathematical sense means nothing but freedom from contradiction. (…) This thesis is neither as simple nor as self-evident as it might seem (Bernays 1976, 93).

The thesis implies that in mathematics the existence of so-called "ideal objects" means nothing other than the existence of certain functional legal relationships. In order to interpret a formal-axiomatic theory as knowledge, certain intended applications are required, because the theory itself is nothing but a formal instrument. These applications can be very diverse. So-called *fictionalism* – one more type of modern nominalism - has developed a related conception of mathematical objectivity. Hartry Field, a very prominent fictionalist, explains:

> Nominalim is the doctrine that there are no abstract entities. The term 'abstract entity' may not be entirely clear, but one thing that does seem clear is that such alleged entities as numbers, functions and sets are abstract - that is, they would be abstract if they existed. In defending nominalism therefore, I am denying that numbers, functions, sets or any similar entities exist (Field 1980, 1).

H. Field, M. Balaguer or L. Tharp saw mathematics as a kind of *fictionalism*. Tharp writes, for instance, that mathematical statements involve claims about relationships between very specific *concepts*. He explains his ideas by the following very brief story. "*The only people in our story are Gertrude and Hamlet. Gertrude is a queen. Hamlet is a prince, and Gertrude is Hamlets mother"* (Tharp 1989, 168). Tharp then continues:

> Given these two stipulations which constitute our story, various consequences follow from the meanings of the concepts ›prince‹, ›queen‹ and ›mother‹, and are evidently true-in-the-story: for example, no princes are queens; Gertrude and Hamlet are distinct; Hamlet is not Gertrude's mother. None of these conclusions follow logically from the given story, however (Tharp 1989, 168 f.).

And the difference between "8 + 5 = 13" and "8 + 5 = 15" is simply that the first equation is true in the "grand tale" of ordinary arithmetic and the second is false. In analogy to the fact that the sentence "Gertrude is Hamlet's son" would be wrong in Tharp's story. Fictionalism in its many forms may be interesting, but it has gained little prominence within mathematics because it fails to accommodate the fact that a mathematical theory cannot or should not be definitively coupled to any particular meaning.

Education and mathematics as a language

Mathematics is often referred to as a language. Up to about 1800 mathematics consisted mainly in problem solving and the search for more powerful methods and new applications ruled the scene. Since then mathematics became most valued as a language. This development was a result of the strong educational role attributed to mathematics and to arithmetics in particular in Humboldt's educational reform, and the associated idea of a unity of research and teaching. The conception still exists today.

The intuitions of the classical spirit, split between rationalism (Descartes) and empiricism (Locke), were replaced by debates over the nature of language and communication. This shift went hand in hand with the search for a perfect match between sign and meaning. In this sense Condillac (1714-1780), Humboldt (1767-1835) and Bolzano (1781-1858) could be considered the ancestors of linguistic philosophy which made its way into the debates abut the function of language in education. Descartes saw in society only a source of superstition and error. Condillac - originally a student of Locke - thought differently. In a letter to a friend Condillac writes:

> Before social life, natural signs are properly speaking not signs, but only cries that accompany sentiments of pain, joy, etc., which people utter by instinct and by the mere form of their organs. They must live together to have occasion to attach ideas to these cries and to employ them as signs. Then these cries blend with the arbitrary signs. (Condillac 2010, XXVII)
> That's what my entire system comes down to in this matter. Social intercourse gives occasion (1) to change the natural cries into signs; (2) to invent other signs that we call arbitrary; and these signs (the natural as well as the arbitrary) are the first principles of the development and progress of the operations of the mind. I admit that on all this my work is not clear enough. I hope I'll do better another time (Condillac 2010, 845).

Since then mathematics became most valued as a language. Effros writes:

> The formulation and solution of problems has provided what is regarded by many as the most characteristic feature of the subject. It must be stressed, however, that the primary purpose of problem solving is to facilitate the discovery of new mathematical concepts and to gauge the success of these methods. (…) Our premise is that mathematics is a language, since it provides both conveyance for and a substantiation of our thoughts. It is that aspect of mathematics that explains the key role it plays in modern science (Effros 1998, 132).

The core of the conception of mathematics as language consists in the axiomatic method and in deductive reasoning. This approach came about as soon as it was realized that one was free to create all kinds of algebraic systems in which the variables stand for completely unspecified objects. Mathematics became Meta-Mathematics (a system of axioms in Hilbert's sense presents a theory after all!)

Especially algebra became meta-algebra, became transformed from an analytical language (Condillac) into a theory of structures. The works of Evariste Galois (1811-1832), Hermann Grassmann (1809-1878), A. DeMorgan (1806-1878), George Boole (1815-1864) or Benjamin Peirce (1809-1880) in the United States, to name just a few, provide a clear idea of this new algebraic spirit. We have gained a glimpse on these developments when discussing the problem of the imaginary numbers in section III.

The dynamic heart of this approach consists in a complementarity of idea and application, of concept and object. In his criticism of Edmund Husserl's emphasis on conceptual thinking, Frege emphasized that mathematicians define neither concepts, nor their contents, but rather their extensions:

> For the mathematician, it is no more correct and no more incorrect to define a conic section as the circumference of the intersection of a plane and the surface of a right circular cone than as a plane curve whose equation with respect to rectangular co-ordinates is of degree 2. Which of these two definitions he chooses, or whether he chooses another again, is guided solely by grounds of convenience, although these expressions neither have the same sense nor evoke the same ideas. (Frege, quoted after Dummett 1991, 32)

In school, however, teachers and students need to put far greater effort into communicating mathematical ideas. And with respect to mathematical thinking, it seems very relevant indeed which definition is chosen, which perspective is taken, or how a problem situation is represented, exactly because the various "expressions neither have the same sense nor evoke the same ideas". In mathematics education this is often discussed as the distinction between *basic (computational) skills* and *conceptual understanding* (compare, for instance, Wu 1999; Radu 2002). Conceptual thinking plays a far greater role in school, than in theoretical mathematics.

From an educational perspective two concepts A and B are not the same, even if contingently or necessarily all A's are B's and vice versa, because different concepts help to establish different kinds of relationships and thus influence problem solving activity development in different ways. Two concepts can be extensionally equivalent and yet function differently depending upon the cognitive context they are embedded in.

Effros construes "understanding" as the ability of forming concepts and links this to his conception of a linguistic view of mathematics:

> As in the other disciplines the most valuable product of our enterprise has been discovering concepts. These concepts in turn represent extensions of our ability to use language, the mediator of human understanding. (Effros 1998, 139).

Teachers often even believe, that the main objective of school-mathematics resembles that of philosophy (creating concepts), or more narrowly that of logic (learning to use language correctly). Given this state of affairs, a process is important which Thurston describes under the label of mental compression:

> Mathematics is amazingly compressible: you may struggle a long time, step by step to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as whole, there is often a tremendous mental compression (Thurston 1990, 847).

Mathematics, then emerges, on the one hand, as the creation or production of meaning and depends in its development on hypostatic abstractions, that is, on abstractions from actions, rather than from ready-made objects. Abstraction must, on the other hand, be accompanied by the inverse process of interiorization of the explicit; a process which transforms the explicit symbolic representations into internal means, helping the learner to build new intuitions and thus expand its range of feeling, thinking, and acting.

Mathematics in society

Nowadays one often hears the complaint, that science and technology have become so abstract, so technically and mathematically demanding, and that, as a result, it has become increasingly difficult to properly incorporate them into general culture. This has been sometimes referred to as a new barbarianism:

> Pour la première fois dans l'histoire de l'humanité, savoir et culture divergent. (…) Les déterminations géométriques auxquelles la science galiléenne tente réduire l'être des choses sont des idéalités. Celles-ci loin de pouvoir rendre compte di monde sensible, subjectif et relatif dans lequel se déroule notre activité quotidienne se réfèrent nécessairement à ce monde de la vie, c'est seulement par rapport à lui qu'elles ont un sens, c'est sur le sol incontournable de ce monde qu'elles sont construites. De ce point de vue si l'on considère la terre non pas comme une planète qui tourne autour du soleil (…) mais comme ce sol de toute expérience auquel les idéalisations scientifiques renvoient inévitablement, il faut reprendre la folle sentence de Husserl et dire avec lui: l'arche-originaire terre ne se meut pas (Henry 1987, 18).

This gap between intuitive grasp and theoretical concepts is not limited to technology and mathematized science. It also has nothing to do with the fact that not everyone can be taught the latest scientific techniques and theories. Rather, it has to do with the inner and outer worlds falling apart. Communication cannot be reduced to language, but requires the existence of a common world and shared customs or world views. In this sense, mathematics educators and philosophers had always understood Euclid and Euclid's elements as valuable educational content.

Euclid was not importance for Galileo, Leibniz, Bourbaki, etc. but Euclid became important for the (higher) educational institutions. From an ideological and philosophical point of view Euclid's Geometry did not lose its relevance due to the advancements in pure mathematics towards analytic or projective geometry (Richards 1988).

However, the comprehensive reform of mathematics education after the *Sputnik shock* of 1957, which came to be known under the name "New Math", and which tried to narrow the gap between mathematical research and the schools was essentially based on the opposite philosophy. In 1959, Jean Dieudonne (1906-1992), spokesman for the Bourbaki group, gained a lot of attention by proclaiming: "Down with Euclid! Death to Triangles!". He argued that the teaching of geometry at school should be replaced by linear algebra.

In learning mathematics, we become aware of and experience structures. That is what mathematical experience is. Finite sets are not enough, so axioms are used to define the structures. Hilbert and his followers used axioms, for example in geometry, to express structures. In the 50ies structures entered the teaching of mathematics at the universities. With respect to the schools, however, language and logic ruled the day, not mathematical structuralism. This also has become the dominant position in analytical philosophy.

John Rawls (1921–2002), America's eminent political philosopher, recounts in his memoirs (*Future Pasts*, 2003) a constant dispute of this kind between Burton Dreben, a logician considered the highest authority on the recent history of philosophy, and the Harvard mathematician Gerald Sacks:

> Sacks then distinguished between syntax and structure, saying that what is important for logic is structure, not syntax. What we do in studying mathematics is learn about various structures. Sacks granted that syntax is simpler and that it therefore makes sense that we begin teaching logic with syntax. But, he argued, syntax is not what is fundamental. Sacks argued that Dreben thinks that syntax is fundamental because it is simple, and that this is simply wrong.

The primary function of language seems to have been originally, “to describe the spatio-temporal processes which surround us, and whose topology is transparent in the syntax of the sentences describing them” (Thom 1971, 698).

In order to alleviate the difficulties of teaching mathematics, it has therefore been proposed that classical Euclidean geometry and its language be given a greater role in mathematical teaching because, from various linguistic points of view, it could form a bridge between the mother tongue that we unconsciously acquire and the formal algebraic language of school mathematics. The eminent mathematician, R. Thom said in his address to the 2nd International Congress for Mathematics Education in Exeter in 1972, among other things:

> The language of elementary geometry offers a solution to the following problem: to express in a one-dimensional combination – that of language – a morphology, a multi-dimensional structure. Now this problem recurs in a form ›everywhere dense‹ in mathematics, where the mathematician has to communicate his intuitions to others. In this sense, the spirit of geometry circulates almost everywhere in the immense body of mathematics, and it is a major pedagogical error to seek to eliminate it (Thom 1972, 206 f.).

The syntax of our language is poor and simple, but the sense and meaning of what is said is usually intuitively immediately apparent. We saw an illustration of this earlier on in our presentation of Tharp’s fictionalism. Geometry has a richer structure, consisting of all possible constructions and movements in space. However, sense and meaning are still more or less intuitively present. At the same time, the meanings are more clearly identifiable through the definition of geometric equality (congruence) than is the case with the linguistic meanings. But the operational options of geometry are, as point out above, very limited. So the analytical element in geometry remains tied to algebra (This reminds one of Leibniz’ problems with his project of a geometrical characteristics).

Algebra is structurally much more complex than language or geometry because it is not constrained by any boundaries other than the requirement of logical consistency But it is vague or abstract in its meanings. The sense or the meaning of an algebraic equation lies in the syntax, it just consists in calculating.

Psychologically, our language and the social context carry the greatest weight. Psychologists have used this fact to test people's social empathy. Elsa Ermer and Kent Kiehl of the University of New Mexico, Albuquerque, decided to examine prison inmates' moral sensitivities using the *Wason Card Test* (Ermer & Kiehl 1992).

In the first presentation of the problem, four cards are placed on the table, each having a number on one side and a color on the other. The cards laid out show something like: 3, 8, brown, and red. The rule is: "If a card shows an even number, then it is red on the other side". Question: "Which cards do you have to turn over to decide if the rule has been broken?" Now consider the following problem. The rule to be tested is: "If you rent a car, you should refill the tank with gasoline afterwards". The following cases are these "1) Dave didn't borrow the car; 2) Helen borrowed the car; 3) Brianne filled the tank with petrol; 4) Kirk didn't fill the tank." As before, you are required to decide which cards to turn over to see if the rule has been broken. In terms of formal logic, the problems are the same. However, as Ermer and Kiehl found out in their large-scale test, in some of the groups tested, most people found the second problem easier to solve, than the first.

The same phenomenon appeared with the following problem. There are rabbits and chickens on a farm. Together they have 9 heads and 24 legs. How many rabbits are in the yard? You just have to allocate the corresponding number of feet to each head. This is not easy for some children to grasp because of the "cruelties against the animals involved. For many children it was easier to understand the problem when it was dressed in the form of 2 and 4 bed hotel rooms. Distributing beds among rooms seemed less "cruel" than distributing feet to heads.

Conclusion

Ultimately, whether in science or in education, we seem unable to escape the debates caused by the opposition of *syntax* and *semantics*, of the *functional* and the *substantial* approach described above. Heijenoort aptly coined this opposition as a distinction between two types of logic:

> The universality of logic expresses itself in an important feature of Frege's system. In that system the quantifiers binding individual variables range over all objects. As is well known, according to Frege, the ontological furniture of the universe divides into objects and functions. Boole has his universal class, and De Morgan his universe of discourse denoted by ›1‹. But these have hardly any ontological import. They can be changed at will. The universe of discourse comprehends only what we agree to consider at a certain time, in a certain context. For Frege it cannot be a question of changing universes. (Heijenoort 1967)

From our point of view, however, it is important to highlight the fact that mathematicians work under the paradigm characteristic for the algebraic school initiated by De Morgan, George Boole, and George Peacock in England, Robert Grassmann and Ernst Schroeder in Germany, by Charles S. Peirce in the US, by Giuseppe Peano in Italy etc. As pointed out by Heijenoort, under this paradigm, *sign* and *object* retain a relative independence from each other. This complementarity of reference proves helpful in advancing mathematical knowledge, whether in research or in school.

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