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Teachers' Reflections on Fraction-by-Fraction Division: Understandings and Philosophies

Reflexões de professores sobre divisão de fração por fração: compreensões e filosofias

Reflexiones de profesores sobre división de fracción por fracción: comprensiones y filosofías

Réflexions des enseignants sur la division fraction par fraction : compréhensions et philosophies

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Abstract

In this text we present the results of a collaborative qualitative investigation, carried out in 2020 and 2021, with a group of teachers on the knowledge needed by the teacher who will teach fraction-by-fraction division in elementary school. The methodology used was the reflection on the practice of six teachers regarding their understanding and knowledge about the subject in weekly meetings of the group. It brings a dialogue between teaching and learning as it seeks to understand the concept and its didactic transposition, based on the group's reflections and what think authors of mathematics education and in the light of philosophy. Alternative algorithms for accessing the formal concept and its relevance are discussed when looking for an instrumental and relational understanding of a task. One of the authors had previously used such task on an assessment instrument with pedagogy students in 2021. The study suggests that it is possible to teach this topic with understanding, as long as the teacher has clear objectives, conceptual and pedagogical knowledge for teaching fractions and operations with fractions.

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Furthermore, it recommends that the teacher consider the limits of contextualization, but with the awareness that mathematics is not produced in a vacuum. Finally, it shows how a group of six teachers reflected on their own knowledge, when they highlighted strengths and weaknesses in the teaching of such a topic.

***Keywords**:* Division, Fraction, Teaching, Learning, Reflections.

Resumo

Neste texto, apresentam-se resultados de uma investigação qualitativa colaborativa, realizada em 2020 e 2021, em um grupo de professores sobre conhecimentos necessários ao docente que vai ensinar divisão de fração por fração no ensino fundamental. A metodologia utilizada foi a reflexão sobre a prática de seis professoras acerca de seus entendimentos e conhecimentos sobre o tema em encontros semanais do grupo. Traz um diálogo entre ensino e aprendizagem, ao buscar a compreensão do conceito e sua transposição didática, com base nas reflexões do grupo e no que pensam autores de educação matemática e à luz da filosofia. Discutem-se alternativas de algoritmos para o acesso ao conceito formal e a relevância deste, ao procurar o entendimento instrumental e relacional de uma tarefa. Uma das autoras tinha usado anteriormente com estudantes de pedagogia tal tarefa avaliativa em 2021. O estudo sugere ser possível ensinar este tema com compreensão, desde que o professor tenha objetivos claros, conhecimento conceitual e pedagógico de como ensinar fração e operações com frações. Ademais, recomenda que o professor considere os limites da contextualização, mas com a consciência de que a matemática não se produz em um vácuo. Enfim, mostra como um grupo de seis professoras refletiu sobre os próprios conhecimentos, quando evidenciaram potencialidades e fragilidades no ensino de tal tema.

Palavras-chave: Divisão, Fração, Ensino, Aprendizagem, Reflexões.

Resumen

En este texto se presentan los resultados de una investigación cualitativa y colaborativa, realizada en 2020 y 2021 por un grupo de profesores, relativa a los conocimientos que necesitará un profesor que tenga que enseñar la división de una fracción por otra fracción en la escuela primaria. La metodología que se utilizó fue la reflexión sobre la práctica de seis profesoras y su comprensión y su conocimiento sobre el tema en encuentros semanales del grupo. Después de un diálogo entre la enseñanza y el aprendizaje, para buscar comprender el concepto y su transposición didáctica, basado en las reflexiones del grupo y en lo que piensan los autores de la educación matemática y a la luz de la filosofía. Se discuten algoritmos alternativos para acceder al concepto formal y a su relevancia, cuando se busca una comprensión instrumental y relacional de una tarea. Una de las autoras ya había utilizado previamente la tarea en 2021 en una tarea de evaluación de este tipo con estudiantes de pedagogía. El estudio sugiere que es posible enseñar este tema con comprensión, siempre que el profesor tenga objetivos claros, conocimiento conceptual y pedagógico de como enseñar las fracciones y operaciones con fracciones. Además, se recomienda que el profesor considere los límites de la contextualización, con la conciencia de que las matemáticas no deriven en algo insustancial. Finalmente, se muestra cómo un grupo de seis profesoras reflexionó sobre sus propios conocimientos, mostrando las fortalezas y las debilidades en la enseñanza de este tema.

Palabras clave: División, Fracción, Enseñanza, Aprendizaje, Reflexiones.

Résumé

Dans ce texte, nous présentons les résultats d'une recherche qualitative collaborative, réalisée en 2020 et 2021, dans un groupe d'enseignants sur les connaissances nécessaires à l'enseignant qui enseignera la division fraction par fraction au primaire. La méthodologie utilisée était une réflexion sur la pratique de six enseignants sur leur compréhension et leurs connaissances sur le sujet lors de réunions de groupe hebdomadaires. Le texte fait dialoguer enseignement et apprentissage, cherchant à comprendre le concept et sa transposition didactique, à partir des réflexions du groupe et de ce que pensent les auteurs de l'enseignement des mathématiques et à la lumière de la philosophie. Des alternatives algorithmiques sont discutées pour accéder au concept formel et sa pertinence dans la recherche d'une compréhension instrumentale et relationnelle d'une tâche. Un des auteurs avait déjà utilisé une telle tâche évaluative avec des étudiants en pédagogie de la même année. L'étude suggère qu'il est possible d'enseigner cette matière avec compréhension, tant que l'enseignant a des objectifs clairs, une connaissance conceptuelle et pédagogique des fractions et des opérations avec des fractions. De plus, il recommande à l'enseignant de considérer les limites de la contextualisation, mais avec la conscience que les mathématiques ne se produisent pas dans le vide. Enfin, il montre comment un groupe de six enseignants a réfléchi sur ses propres connaissances, lorsqu'ils ont montré des forces et des faiblesses dans l'enseignement d'une telle matière.

Mots-clés : Division, Fraction, Enseignement, Apprentissage, Réflexions.

Teachers' Reflections on Fraction-by-Fraction Division: Understandings and Philosophies

In this article, we present a selection of studies carried out by teachers on fractions in 2020 and 2021. We present alternatives for teaching and learning issues involving the concept of dividing fractions, as well as considerations on the relevance of this content in basic school. We dialogue with the Philosophy of Mathematics Education that underlies these actions and knowledge that go beyond instrumental understandings of how to perform mathematical calculations (Skemp, 1976, 1987). These reflections were made possible in weekly virtual meetings by the Study Group on Mathematics Education of Espírito Santo [Grupo de Estudo em Educação Matemática do Espírito Santo - GEEM-ES][4], in which our knowledge and practices are the objects of study and research. The group is part of the extension project at the Federal University of Espírito Santo, which aims at the initial and continuing education of teachers who teach mathematics and is coordinated by professors Brum and Santos-Wagner.

Preparing teachers to teach mathematics has been a major challenge recently, especially in the early years. With that in mind, Brum carried out activities in 2021 with her pedagogy classes at Ufes to challenge prospective teachers to find creative ways of teaching mathematics that favor understanding. The teacher who understands the concept knows where to start, where to arrive, and how to explain it to their students in various ways when proposing mathematical activities (Skemp, 1976, 1987; Schulman, 1986, 1987, 2014; Ball, Thames, & Phelps, 2008). That is, the teacher has a relational understanding of the mathematical concepts (Skemp, 1976, 1987), has knowledge of the mathematical content, and has pedagogical knowledge of this content (Shulman, 1986, 1987, 2014) and thus, has specialized knowledge of mathematics teaching (Ball, Thames, & Phelps, 2008). More than that, they know how to choose paths that allow for didactic transposition, understood here as an attitude in which.

> the teacher must construct problem situations in which the mathematical knowledge pointed out is recontextualized and repersonalized to become the student's knowledge, i.e., a more natural response to the indispensable conditions for this knowledge to have meaning (Almouloud, 2011, p. 156).

When Almouloud (2011) deals with the recontextualization and repersonalization of mathematical knowledge for the construction of a new concept, we understand that the dialogical situations created between teacher and student in the classroom allow the latter to create, risk, and validate solutions that can be generalized through the teacher's mediation. In

[4] The GEEM-ES meetings, before the pandemic, were held weekly, for two hours, in person.

this sense, the teacher needs to have consolidated the mathematical knowledge they intend to address, an essential condition for them to know how to assess what the student knows and does not know about the concept, which requires reflection on one's content knowledge and pedagogical content knowledge (Shulman, 1986, 1987; Santos, 1993). In the sense of Ball, Thames, and Phelps (2008) and Rojas, Flores, and Carrillo (2015), pedagogical content knowledge is called specialized teaching knowledge. Thus, the text we present tries to answer the following questions: 1) Why teach fraction division to students in the early years? 2) What knowledge is needed by the teacher who intends to teach fraction-by-fraction division to students in the first segment of elementary school? 3) How can group studies help teachers in continuing education understand why, for what, and how to teach fractions in the early years?

Thus, to answer the questions, we report dialogues and reflections of teachers-members of the GEEM-ES when solving one of the questions of the written evaluation involving fraction-by-fraction division. This task and five other tasks had been applied to the pedagogy students at Ufes by professor Brum, and later, they were brought to our group. This way, the text will show how we recontextualize and repersonalize what we knew and did not know about this content, highlighting strengths and weaknesses. That is, we broaden our understanding and knowledge of division, causing changes in our strategies on how to teach the subject. This made us ponder and dialogue, during 2020 and 2021, about understandings shaped in the group about fractions (Santos & Rezende, 1996), and deepened in 2021 and 2022 in studies present in the literature on teaching and learning of fractions (Vaz, 2016; Guerra & Santos da Silva, 2008; Junior & Wielewski, 2021). At the same time, we reflected on the practice itself in the light of philosophy, as every practice hides thinking about what, how, and why to do something (Skemp, 1976, 1987; Ernest, Skovsmose, Bendegem, Bicudo, Miarka, Kvasz, & Moeller, 2016; Shulman & Shulman, 2004). Therefore, we agree with Bicudo, Monteiro, and Baier (2019) when they state that

> Philosophizing opens horizons, favors criticism; it goes beyond what to do and how to do something towards why to do it and how it is done. Therefore, researching in, with, and in the Philosophy of Mathematics Education is to remain in the movement of thinking about the activities that are developed when Mathematics Education is carried out, whether they are about research, teaching or learning; as well as those that occur in daily life or that are related to Education public policies. (p. 2)

These authors' provocations align with our concerns as educators of teachers who work in the classroom. Therefore, we read and studied several authors to maintain this movement of reflecting on our pedagogical actions and how we prepare and encourage pre-service and in-service teachers who teach mathematics in elementary school. We sought literature on didactic transposition, scientific knowledge, and taught knowledge; content knowledge, pedagogical content knowledge/teaching expertise; and knowledge of the purposes and values of education and its philosophical and historical base. After that, we brought some theoretical and methodological notes about teaching, learning, and assessments of fractions in basic school and what some authors think about the subject. During the study, we tried to understand this teaching and learning in the light of the Philosophy of Mathematics Education, presenting our way of understanding and doing it, enriched by the group's discussions. Above all, being conscious that it is not enough for the teacher to know the content and know how to teach it. It takes motivation and thoughtfulness to realize the beauty of mathematics that exists in everything, from the simplest things in everyday life. About this, Mathias states:

> [...] I see the essence of the concept (...) in the urgency of a poet's words after he has written them intending to disturb his/her readers. I see it in newspaper reports, in the variation of time and things, in sensations. I see it to the point of no longer knowing when it ceased to be what it is in the calculus book or the poet's head. (Mathias, 2015, p. 3)

Thus, our challenge is to show the beauty of learning and teaching fraction-by-fraction division based on our understandings, reflections, and dialogues in GEEM-ES. Finally, we end the text by discussing and reflecting on some answers to our research questions.

Theoretical notes

Harouani (2015), when discussing the reason for teaching mathematics in schools, states that, often, neither students nor teachers can say why we study the mathematics offered there. He says this is not due to a lack of efforts by higher education, which propose new pedagogies, adequate methodologies, and visions of citizenship education. Still, "perhaps school math has nothing to do with usefulness; perhaps it is primarily a product of an education system whose main purpose is not learning, but socializing and certifying students" (Harouani, 2015, p. 10). We experience situations in schools where this statement makes sense (Hoffman, 2012). Often, teachers were concerned about completing the program, forgetting students' learning. In turn, when asked why they studied mathematics, students always answered: "To be someone in life" or "To get a good job". It is as if teachers and students were only fulfilling a societal ritual by ignoring street mathematics, in which stallholders use reasoning and calculations far from what we experience at school (Nunes, Carraher, & Schliemann, 2011).

Researcher Harouani (2015) mentions the texts of the school problems that may not make sense to students, corroborating what Lopes (2008) questions about the study of fractions in elementary school, "Should we really be teaching fractions to our kids?" to which Lopes (2008, p. 2) adds: "Should we teach fractions the way we always teach?" The answer would certainly be "No". Commenting on Peter Hilton (1980)[5], who calls "misleading applications" the problem situations that try to create a context for using fractions, Lopes (2008) qualifies them as "pseudo-practices" as they are far from the children's world and absolutely meaningless. For example, he mentions a situation found in teaching material from 2007: "João ate 3/17ths of a cake, his brother ate 5/9 of what was left. How much is left for his sister?" (Lopes, 2008, p. 4). Today, we still find similar situations, invented by teachers or in printed and digital didactic material to justify the use of operations with fractions.

Thus, it constitutes a challenge for pedagogy teachers to form prospective teachers who know the mathematical content and master the pedagogical knowledge of this content/specialized teaching knowledge. Especially in times of non-contact classes, such as we experienced during the pandemic in 2020 and 2021, when teachers had to face six major challenges: a) learning to use platforms; b) preparing different classes; c) creating ways of interacting with students; d) looking for ways to make connections with other knowledge; e) seeking to contextualize and validate teaching practices; and f) motivating students. Those new demands required –and still require– effort and creativity from Brazilian and foreign teachers, which agrees with Harouani (2015), Lopes (2008), because using fractions is challenging as most of the time they deal with content applicable to the world of adults. However, as researchers of our practice, we concluded that this does not mean that the content cannot come near the child's world, thinking about living situations and suggesting research with family members on the application of fractional numbers. Mediated by the teacher, the child will find applications in car odometers that measure fuel, in homemade recipes, and measures such as meters, liters, and kilograms. The latter allows the systematic and simultaneous exploration of fractions and decimal numbers. When the teacher has clear objectives and understands and knows how to teach fractions, they will be able to lead a debate for the understanding of fractions of discrete numbers in social situations, as explained by Lopes:

> Fractions of a discrete collection, such as 1/2, 2/3, 3/4, and 3/5, appear in chapters of the Federal Constitution or the bylaws of state or municipal city councils as references for passing laws or changing the constitution. There is a context where the calculation of 3/5 of 513 or 2/3 of 81 is not artificial; with two thirds of the votes of federal deputies, a process of impeachment of the President of the Republic can be initiated; 1/3 of the

[5] Lecture given by Peter Hilton at ICME IV, in 1980, in Berkeley, USA, under the title "Do we still need fractions in the elementary curriculum?".

> ministers of the court of accounts are chosen by the President of the Republic, 2/3 by the National Congress (Lopes, 2008, p. 6).

Let us think of the rich debates in which today, in 2022, in the country's current political situation, examples such as those cited above would provide an understanding of fractions of quantity and their social implications within the vision of humanist mathematics[6]. Let us imagine the effects of this discussion when politicians want to approve a Proposed Constitutional Amendment (Proposta de Emenda Constitucional - PEC) that will alter the lives of families under the most varied pretexts. After all, teaching mathematics is knowing that it does not happen in a vacuum. About this, Ernest and colleagues say:

> [...] the aims, goals, purposes, rationales, etc., for teaching mathematics do not exist in a vacuum, belonging to people, whether individuals or social groups (Ernest 1991). Since the teaching of mathematics is a widespread and highly organized social activity, its aims, goals, purposes, rationales, and so on, need to be related to social groups and society in general, while acknowledging that there are multiple and divergent aims and goals among different persons and groups (Ernest, 1991, apud Ernest et al., 2016, p. 3).

After explaining to the child, the meaning of the Constitution and the number of deputies needed to approve the amendment, let us imagine the child calculating and discussing the matter with the family. This type of mathematics is the one that can leave the academy and gain the school in a transdisciplinary way because being "a teacher is much more than teaching contents; it is educating through them" (Mathias, 2015, p. 3). An engaged teacher understands their students intellectually, socially, culturally, and personally from a perspective of global and historical development (Shulman & Shulman, 2004). The debates in our study group went beyond the assessment process involving fractions with an understanding of the concept, it incited us to think about the applicability and necessary prerequisites for its approach based on the four levels of knowledge commented by Shulman (2014). When the teacher knows the content, they conduct the classroom like an orchestra, aware of the steps to be taken, to come and go working in any field of knowledge. Recontextualizing what the author says about English language teaching, we think about how it would be in solving a problem: 1) denotation – the decoding of the utterance; 2) connotation – understanding the utterance; 3) interpretation – the relational understanding of the statement in the construction of the concept; and 4) application of the assessment – the generalization and application of the concept in other

[6]Philosophy that understands mathematics as a human creation, made by man and for man (D'Ambrósio, 2006).

situations, perceiving connections and reading the world through them (Shulman, 2014). To achieve skills at such levels, we analyze our knowledge as teachers and teacher educators, reflecting on three categories of the knowledge base according to Shulman (2014):

> • content knowledge; [...]
> • pedagogical content knowledge, that special amalgamation of content and pedagogy that is the exclusive territory of teachers, their special means of professional understanding; [...]
> • knowledge of the ends, aims, and values of education and its historical and philosophical basis (Shulman, 2014, p. 206).

Wielewski's (2008) work on instrumental and relational thinking in dialogue with Skemp (1989)[7] clarifies Shulman's (2014) categories of knowledge to us. Paraphrasing Wielewski (2008) when commenting on Skemp's (1989) ideas, if we seek a teaching that aims at understanding, we will say that this will only be possible when promoted in a way that does not only allow learning by habit. That is, it needs to be intelligent learning in which the learner establishes relationships between knowledge structures acquired through experiences that they perceive personally. Then, it is up to the teacher to create opportunities for the student to make connections that access those structures, as good teaching projects the learner forward from their mental stage (Vygotsky, 1993, 2007). Thus, the teacher must reflect on their mathematical knowledge and evaluate it. In addition, it needs to analyze its ability to provide teaching and learning situations capable of incentivizing the formation of students' mental structures to help them create their understanding.

In Wielewski (2008), we find a scheme that shows ways of constructing learning based on Skemp (1989). It summarises the path of construction of relational thinking, not instrumental thinking by habit, repetitive, and circumstantial.

[7] Skemp, R. (1989). Mathematics in the primary school. London: Routledge.

BUILDING		TESTING
From our own encounters with the physical world: **Experience**	Mode 1	The again expectation of a physical event: **Experiment, prediction**
From other people's schemes: **Communication**	Mode 2	Comparison between other people's schemes: **Discussion**
From the inside, by formation of higher order concepts: through extrapolation, imagination, insight: **Creativity**	Mode 3	Comparison between one's personal knowledge and beliefs: **Internal consistency**

Figure 1.

Adapted from Skemp's Building and Testing Scheme, apud Wielewski (2008, p. 66)

The scheme above summarizes the ways of structuring thoughts that must be present in an interconnected way in the education of both the teacher and the student when they want to understand concepts. For example, when building understandings of fraction division, I need to ask myself, according to the scheme in Figure 1: (i) What do I already know about this subject based on my experience? A: I can experiment according to my expectations. (ii) What do others say about it? A: I can search the literature or interact with teachers for new information and discuss points of view. (iii) Based on experiences, studies and dialogues with other teachers, what can I do? A: From then on, I can draw new conclusions, extrapolate the content, and create new strategies, comparing and validating them. Thus, I will think about what I know and what I do not know, in a process of self-assessment and reflection on my thinking before and after the study (Santos, 1993, 1997). In other words, developing metacognitive practices in which I think and reflect on the knowledge I do or do not have (Santos, 1993, 1997).

From those reflective studies, the teacher will choose a way to make content learnable through dialogue, instigating the student, or visual representation. We often use the latter in this work as a way of "seeing" what happens in a fraction-by-fraction division. This happens because the geometric representation induces the learner to use "creativity, driven by the visualization process", which "is the development of a thoughtfulness that can expand the modes of knowledge and ways of approaching the unknown" (Cifuentes & Santos, 2019, p. 19).

Methodological notes

The methodology used was collaborative qualitative research on practice, made possible by the study group in a movement in which action and reflection walk hand in hand (Fiorentini & Lorenzato, 2007). Next, we show question IV, proposed by Brum, the object of this study. First, we present and discuss resolutions using conventional algorithms and then report alternative ways to construct the concept of fraction-by-fraction division. Afterwards, we discuss the relevance of the content, possible connections between knowledge that the text of the problem addressed, and reflections made possible by our practice, dialoguing directly with the authors already mentioned.

The strategy for collecting and producing data was to distribute six assessing questions among ten teachers in a virtual environment. First, each member showed the solution to the question numerically and suggested other algorithms. Afterwards, each participant made their presentation, explaining to the others their ways of thinking and the concepts involved in the issue. In a third moment, each group member was challenged to research and report how they would teach their students one of the questions, seeking, in the available literature, new understandings about the content addressed. Question IV triggered this study, in which we analyze the information produced:

> 4) Linda had 4 2/3 meters of fabric. She is making baby clothes for the bazaar. Each dress pattern needs 1 1/6 meters of fabric. How many dresses can she make with the fabric she has? Solve the problem and then reflect: How would you explain through drawing to a child who had not understood it? [...] (Adaptation of the collection by Brum – Pedagogy – UFES, 2021).

Professor Brum challenges us to explain through drawing, which refers to the work of Cifuentes and Santos (2019). They call this type of representation geometric visualization. The authors defend the method for making visible what happens in structuring a concept. We agree with this thought because it bridges the visible and the abstract. It is one of the forms of materialization in mathematics that we have used most in our experience because, in fact, it is historical knowledge used in any situation. We represent something to view, review, remember, or redeem at later times. Therefore, it is one of the best tools to help each person's mind to remember, internalize ideas and understandings of mathematical content (Santos, 1997), and, thus, serves as a didactic transposition for any content (Almoloud, 2011).

Due to some members' absences, we had six teachers participating and producing the data for this question, five on the day of the meeting in which this issue was discussed and one by phone after the meeting. Three participants have a degree in Mathematics and postgraduation in Mathematics Education, and three have other degrees (including two master's degree students in Mathematics Education and one completed the master's degree). We will try to answer the following questions: 1) Why teach fraction division to early-grade students? 2) What knowledge is needed by the teacher who intends to teach fraction-by-fraction division to students in the first segment of elementary school? 3) How can group studies help continuing education teachers understand why, for what, and how to teach fractions to students in the early years?

To answer the questions, we dialogued with the authors mentioned above, reflecting on our mathematical knowledge and mathematical pedagogical knowledge/specialized teaching knowledge to teach fraction-by-fraction division. We analyzed the images of tasks performed and dialogues constructed with the teachers, which we identified with the initials of their names. And for the reader to understand better, we highlight some evidence of answers in italics.

Resolutions and discussions

We started the discussion with my –Hoffman– solution, bringing three more resolutions from other participants afterwards to illustrate some indications of answers to the questions of the study. To solve question IV, thinking about how I learned to solve problems involving fractions and mixed numbers through modules of the *Madureza Ginasial* (youth and adult education for the last years of elementary school) course at the Instituto Universal Brasileiro (the early 1970s), I numerically applied a division operation: $4\,\frac{2}{3} \div 1\frac{1}{6}$. Translating the mathematical thought expressed in the operation, *I asked, because I had already understood the ideas of division: How many times does 1* $\frac{1}{6}$ *fit in 4*$\frac{2}{3}$*?* Then, I applied the memorized algorithm in which I transformed the mixed numbers into improper fractions, then inverted the second term, which represents the divisor, transforming the fraction into its inverse and multiplied: $4\,\frac{2}{3} \div 1\,\frac{1}{6} = \frac{14}{3} \div \frac{7}{6} = \frac{14}{3} \times \frac{6}{7} = \frac{54}{21} = 4$. Thus, I got four dresses as an answer. As we can see, until now, I have employed an algorithm with no concern about explaining what I understood of the concept. *It is a practice that evidences instrumental understanding, as it identifies integers and fractions greater than the integer, and the calculus itself is just knowledge by habit* which, according to Skemp (1976, 1987), can be used automatically and sometimes forgotten. However, when I explained the conceptual idea of division, thinking about how many times the divisor fits in the dividend, my arguments already show the pedagogical knowledge/specialized teaching knowledge about division that I am deepening and rebuilding in GEEM-ES (Bazet & Silva, 2015; Hoffman, Oliveira, & Souza, 2015).

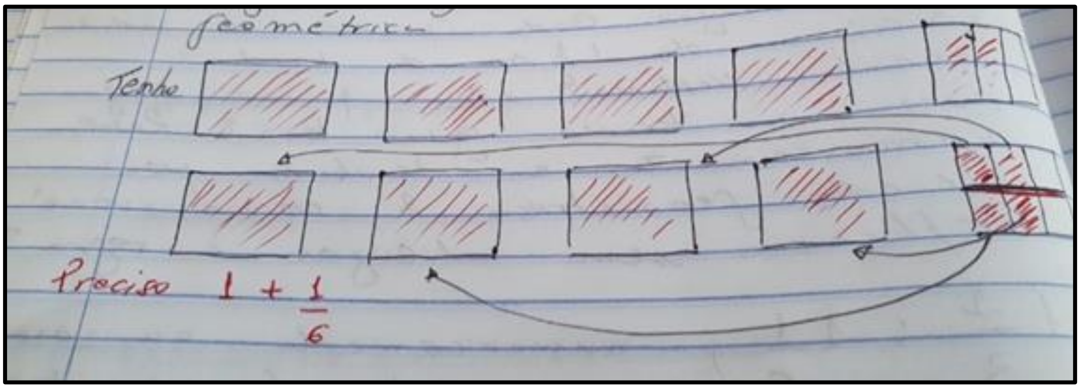

Figure 2.

Hoffman's geometric representation

However, Professor Brum challenged us to show geometrically what happens in this division with the question: How would you explain through drawing to a child who had not understood it? This question is in line with the thinking of Cifuentes and Santos (2019). If Linda has a full four meters plus two-thirds and needs a full meter and a sixth to make the baby dresses, we can make the representation we see in Figure 2. The representation of what you have and what you need for each dress shows that, initially, each dress fits a whole meter, but it remains to think about the distribution of $\frac{2}{3}$. *By redividing them so that they become equivalent to* $\frac{4}{6}$*, I visualized that it could be possible to redistribute them, making them correspond more* $\frac{1}{6}$ *to every integer, and I will have* $4 \times 1\frac{1}{6}$; so, Linda can make four dresses. It is a very simple reasoning and accessible to children when they have already understood the equivalence of fractions. Thus, some kind of relational understanding of equivalent fractions of the same numerical value or measurement is needed. Then we can show that, for each dress, we think about the exact measure of what I need $1\frac{1}{6}$, in relation to the whole fabric, which measures $4\frac{2}{3}$, that is, we thought of visually representing $1\frac{1}{6}$ of $4\frac{2}{3}$. But, for this to be possible, it is necessary to transform thirds into sixths, using the idea of equivalence of fractions. *In my geometric resolution, I visually used the idea of distributing an integer, initially, and then, as I already used the idea of equivalence, I distributed* $\frac{4}{6}$ *intuitively. This demonstrates relational understanding* (Skemp, 1976, 1987, Santos-Wagner, 2008), *because I was able to simplify the content to be taught through visualization*, as Cifuentes and Santos (2019) state.

AD applied even faster reasoning and performed the numerical calculation through simplification, as seen in Figure 3.

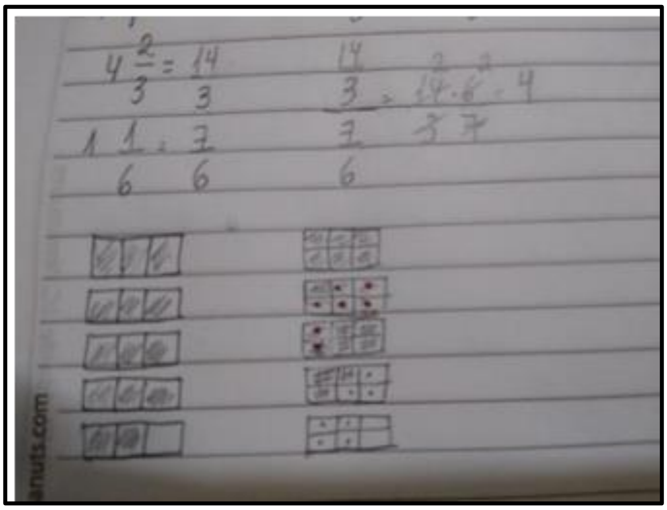

Figure 3.

Possible AD's resolution

Transcribing the operation $\frac{14}{3} \times \frac{6}{7} = \frac{2}{1} \times \frac{2}{1} = 4$. AD used the algorithm as I did. *She transformed the mixed numbers into improper fractions, inverted the second term, transforming it into its inverse fraction, simplified the fractions, and multiplied, as taught by the practical rule, already internalized by habit* (Skemp, 1976, 1987; Santos-Wagner, 2008). When we asked her why she solved it as she did, she said that she had learned to solve it through the quick method and preferred teaching through the numerical algorithms. *It seems that she has an instrumental understanding of the content, but we cannot say whether she has pedagogical knowledge/specialized teaching knowledge (*Shulman, 1987, 2014; Ball, Thames, & Phelps, 2008) *and relational understanding of the content to explain the reasons for this practice* (Skemp, 1976; 1987). She identifies an idea of division in the problem, sets up the operation and solves it using the procedures she had learned. Her geometric representation, however, is more didactic than that shown in Figure 2, as it transforms thirds into sixths by the principle of equivalence: $4\,\frac{2}{3}$ turns into $\frac{28}{6}$. Then, she uses different colours and icons to show that they fit 4 times $\frac{7}{6}$ in $\frac{28}{6}$. Thus, four dresses. *She explores and demonstrates the idea of measurement or quotas, showing pedagogical knowledge of divisional content/expert teaching knowledge* (Shulman, 1987, 2014; Ball, Thames, & Phelps, 2008). She shows to have grown professionally since her participation in GEEM-ES about twelve years ago (Hoffman, Oliveira, Souza, 2015).

When Brum asked us to solve geometrically and explain to each other in various ways, she forced us to review what we had understood and what knowledge we had or did not have about fractions, mixed numbers, the equivalence of fractions, and division (Santos, 1993, 1997). At that moment, two participants did not present a solution that could be visualized geometrically. They admitted that they were unfamiliar with the content and had always had difficulties approaching it. *One of the colleagues who* before exhibitions *said she had not understood the question and that after viewing the resolution with drawings made by the group participants, she understood better*. This is directly linked to what Cifuentes and Santos (2019, p. 2) state: “the notorious immobilization of mathematics teaching may be related to a teaching conception of mathematics as a rigid, totally logical, and algorithmic science, whose main purpose is the application”, which corroborates the texts by Skemp (1976, 1987), Shulman (1986, 1987, 2014), (Ball, Thames, & Phelps, 2008) and Lopes (2008) as previously commented.

In our experience, we have seen in schools the adoption of algorithmic calculation without any concern with visualization, exploring creativity or understanding the context and the concept involved. Usually, teachers repeat the methods through which they themselves learned, without questioning or reflecting on the understanding of the concept. We believe this fact is directly linked to the difficulties that students face in subsequent studies and how they emotionally relate to the subject. When we teach the algorithm without relational understanding, just by knowing the habit, as Wielewski (2008) states, commenting on Skemp (1989), we are developing learning that the student possibly does not retain. *The teachers' awareness of our lack of knowledge about fraction division was one of the learnings of the group up to that moment* (Santos, 1993, 1997).

Other resolutions with a geometric demonstration: learning in the group

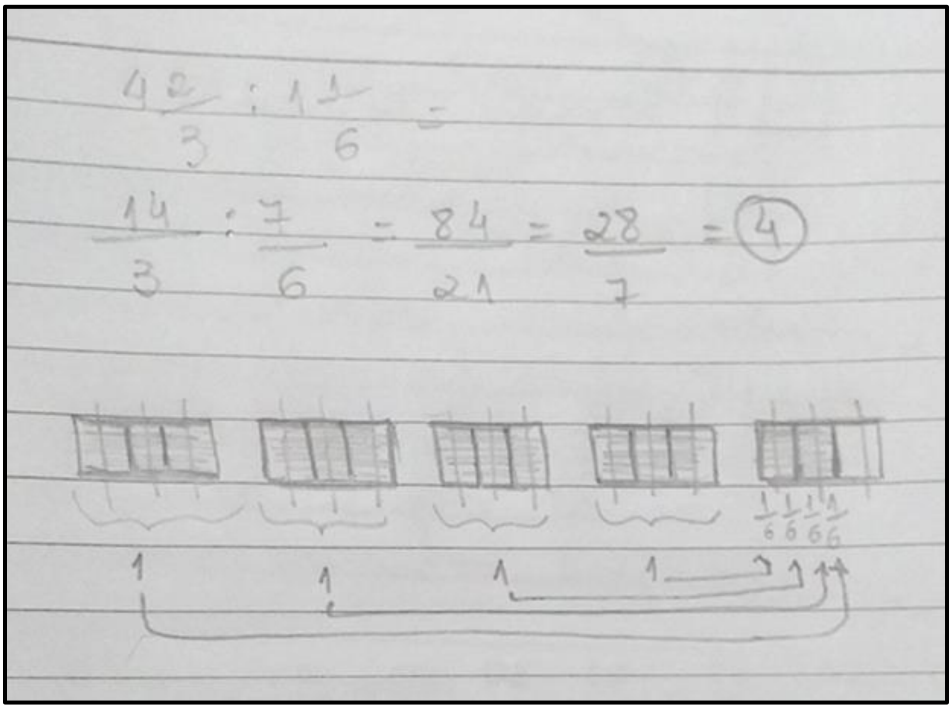

Figure 4.

HI's resolution

In the resolution of Figure 4, it is clear that teacher HI, another member of the group, uses the formal algorithm in the simplest possible way, obtaining by simplification the result of $\frac{28}{7} = 4$. *We found that she is familiar with division, as she moves skilfully in the multiplicative field*. Right away, *she performs the geometric demonstration, transforming thirds into sixths by the equivalence principle and makes the distribution of* $\frac{7}{6}$ *for each dress*. This teacher, a member of the GEEM-ES since 2019, gave evidence of deepening her understanding of multiplication, division, and fractions in the study meetings on the book of *Números* [Numbers] (Santos & Rezende, 1996). In 2021, she confirmed, through her resolutions and explanations to the group, how she is expanding her pedagogical knowledge/specialized knowledge of mathematics teaching (Shulman, 1986, 1987, 2014; Ball, Thames, & Phelps, 2008).

However, in later discussions, teacher HI *concluded that, for a 5th-grade student, numerical resolution carried out in this way, alongside the geometric one, may be an additional obstacle, as the student certainly will not understand where the sevenths obtained by simplification came from* (Almouloud, 2011; Shulman & Shulman, 2004). *The teacher reflects on her knowledge and repersonalizes it, as she thinks about what can happen in the 5th-grade classroom in light of her professional experience and her knowledge of the students*. Visualization will only help with formalization if the student finds meaning in it. Therefore, the group’s studies concluded that the representation and exploration of creativity in solving mathematical questions must always precede formalization and students’ dialog with each other and between teacher and students when they comment and explain different task resolutions.

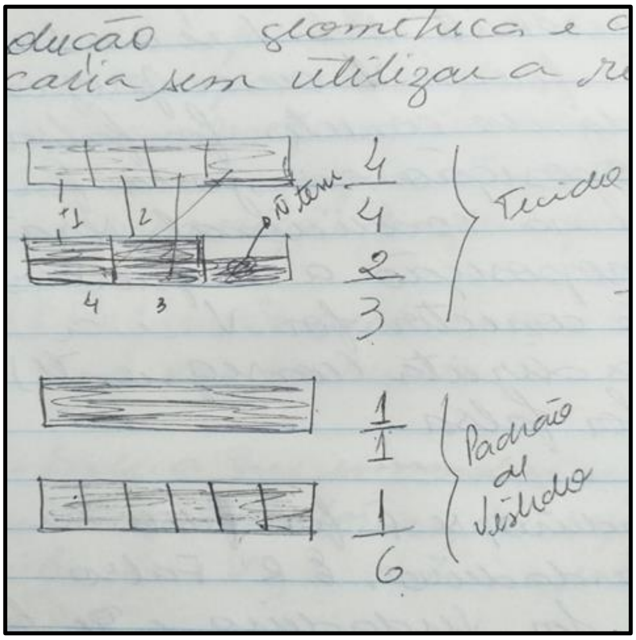

Figure 5.

SI’s first resolution.

SI’s resolution shown in Figure 5 is interesting because she clearly separates and represents the pattern of each dress in her mind. Visually, this helps the student understand the idea of measurement, of “how many fit” or of dimensions, but the representation of the fabric available ($4\frac{2}{3}$) is wrong. When the teacher represents $\frac{4}{4}$, she has only one integer. The student does not visualize the fabric to be divided. It seems that SI lacks clarity in the representation of what the $4\frac{2}{3}$, but she explained that she has little familiarity with geometric proof and got it wrong before the second reading. It was necessary to clarify the content knowledge and then think about how to make it learnable (Shulman & Shulman, 2004). *This teacher is very afraid to solve mathematical tasks, but when she manages to relate them to practical everyday things and her experiences, she feels more comfortable in carrying them out and demonstrates some intuitive knowledge* (Gómez-Chacón, 2003). In many moments, she repeated that she had forgotten the procedures and needed to revise formulas, so several exercises were necessary for the group, in which some formulated simpler oral problems for others, such as: how many integers do I have in $\frac{16}{3}$? If I have $3\frac{4}{7}$, how many sevenths do I have in total? Through these dialogues and others between the teachers and SI, she gained more confidence in herself, and she seems to have acquired some knowledge of this content of fractions (Shulman, 1986, 1987, 2014; Shulman & Shulman, 2004; Ball, Thames, & Phelps, 2008).

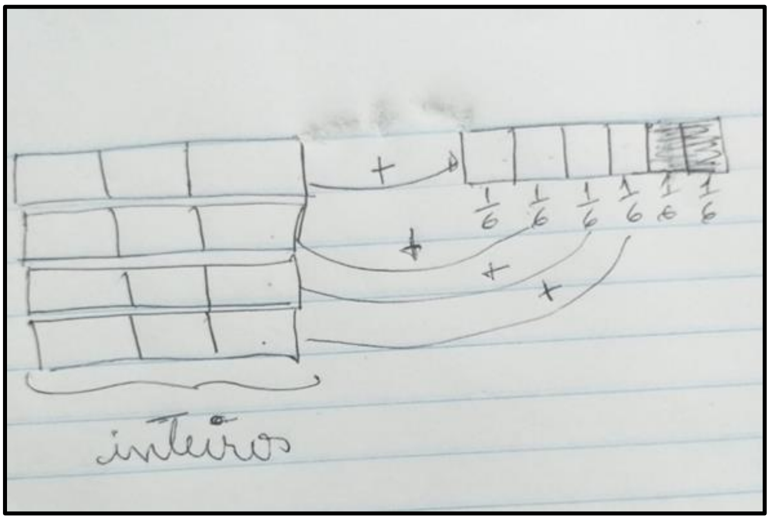

Figure 6.

SI's second geometric resolution

In the resolution of Figure 6, teacher SI redid her reasoning with a very simple and clear distribution. She used the additive principle and distributed the parts. From the experience we have with 5th-grade students, we believe that this would be how they would explain if they had prior knowledge about the equivalence of fractions. The resolution is similar to that of Figure 4 and is easier to see and understand. *It shows how exchanges of ideas influenced SI to rethink her understanding of the representation of integers in fractions* (Skemp, 1976, 1987). Thus, we were looking for the skills highlighted by Shulman and Shulman (2004), when they state that good teachers act from the perspective of forming learning communities, learning from their own experiences and those of their peers. For this, the authors emphasize that teachers must be motivated, reflective, and seek knowledge with the courage to examine their knowledge and their "know-how", also thinking about the consequences of their actions.

The know-how was the point of discussion in these representations (Figures 4, 5, and 6). An effective didactic transposition with visualization must help clarify concepts, not confuse them. The teacher must be conscientious not to make mistaken demonstrations that would cause more doubts to students because this would not contribute to their learning (Cifuentes & Santos, 2019). The resolutions so far demonstrate an understanding of the two ideas of division: division as distribution and division by quotas or measure (Bazet & Silva, 2015). But would geometric representations help explain the rule of thumb for dividing fractions by fractions? How to go from visualization to formalization? Visualization with drawings, when done well, helps the student understand what happens when dividing fraction by fraction, *but the rule of thumb still felt like magic. Until that moment of the study, four colleagues still had doubts*

(Skemp, 1976, 1987; Shulman, 1986, 1987, 2014)

Necessary knowledge for the teacher: explain the rule of thumb

What if Brum asked us how to explain to the students the numerical algorithm? In other words, why, in the fraction-by-fraction division, do we multiply the dividend by the inverse fraction of the divisor? If the student asks that question, how do we answer it? Before we work with students on division and multiplication with fractions, the equivalence of fractions and understanding of multiples and divisors must be already clear to them. Based on this principle, they may understand the need to transform the terms of operations with fractions into the equivalent fractions whose denominators are the same. Once this concept is consolidated, one will understand that dividing $4\,\frac{2}{6} \div 1\,\frac{1}{6}$ is the same thing as dividing $\frac{28}{6} \div \frac{7}{6}$. Then, it is up to the teacher to mediate the understanding that, in this operation, the student will calculate $\frac{7}{6}$ in $\frac{28}{6}$. The idea of division as a measure or quota is another indispensable prerequisite at this point, as the student is measuring how many times $\frac{7}{6}$ fits in $\frac{28}{6}$.

AD's representation in Figure 3 could be transcribed as in Figure 7, using colours that facilitate visualization. Each colour represents a measure in which it is easily understood that it fits four times; thus, there will be four dresses. And how to explain that it is actually multiplying $\frac{28}{6} \times \frac{6}{7}$? We often hear that dividing and multiplying fraction by fraction is very easy, as these operations involve simple calculations. The problem is when we challenge the student and the teacher to explain the reason for those procedures. We also heard that it is not unnecessary to teach multiplication and division of fractions by fractions in 5th-grade classes, as the concepts involved in this content would only be consolidated in the 7th grade. Are these assertions true? We defend the development of activities that prepare children's mental structures from an early age so that the construction of the concept of fractions, their operations and applications are understood and consolidated in the future. Perhaps here, we can remember Lopes (2008) when he calls the teaching of fractions at school didactic aberrations in misleading applications, whose meaning a child hardly finds. Nevertheless, Shulman and Shulman (2004) encourage us to think that almost everything is possible as long as the teacher knows what he/she is doing.

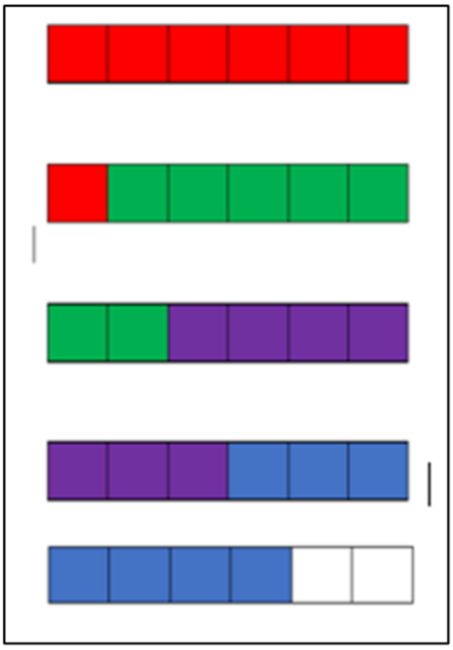

Figure 7.

Geometric representation with the idea of measurement transcribed by Professor Hoffman

Does the problem of baby dresses make no sense to the student? What kind of activities could arise from a problem-solving text like this from a more discursive perspective? We thought of some possibilities that might help: a) taking baby clothes to the living room; b) observing the meter and understanding its decimal divisions into multiples and submultiples; c) talking to sewists or visiting baby clothes factories; d) researching costs and visiting fabric stores and ready-to-wear stores; e) interviewing sewists who make mass production and compare their earnings with that of the store that sells the final product, and f) talking about the social groups that buy these clothes. These activities point to what Ernest and colleagues (2016) call mathematics linked to what we do in society. Based on activities like the above, we think it might make perfect sense to approach fraction division. We understand that not everything is possible, but it is always possible to do some example activities or others to recontextualize and repersonalize the content in question (Almouloud, 2011). After all, one does not buy whole pieces of fabric, and sewists certainly use a standard measurement for each dress. The group maturely reflected on "their own visions of the desirable and the possible [...]" (Shulman & Shulman, 2004, p. 261).

Content knowledge and pedagogical content knowledge: the rule of thumb

Santos and Rezende (1996) state that it is up to the teacher to decide when to introduce fractions-related concepts and how to do this in a way that does not create obstacles to understanding. In this sense, we agree with Brum, whose commented activity is instigating, as she suggests that it is the teacher who needs to be clear. The teacher must know the content; only then will he/she be able to master the pedagogical knowledge of that content/specialized teaching knowledge and decide the best time to teach it, making it learnable. The fraction-by-fraction division was a valuable inducement because it gave the group the opportunity to look at what it knew and what it did not know about this content (Santos, 1993, 1997). During the studies, we noticed that four members of the participating group showed weaknesses in understanding the concept, so, they preferred to apply the algorithm and memorization of the calculus because they felt unsure about how to explain to the student the rule of thumb beyond visualization (Shulman, 1986, 1987, 2014; Ball, Thames, & Phelps, 2008).

Santos and Rezende (1996) propose working with simple and routine problems, activating previous knowledge, as suggested by the old master Polya (1945/1978). They suggest starting from knowledge of operations with natural numbers to build the basis for understanding division by fractions. Example: “A school received 60 bags of powdered milk to be consumed evenly over three weeks. How many bags of milk will be used each week?” (Santos & Rezende, 1996, p. 175).

It is a simple problem with the idea of distribution but when solving it, the student realizes that by calculating the 20 bags per week, he found the third part of the total 60. That is, $60 \div 3 = 20 = 60/3 = 1/3$ of 60, so $1/3 \times 60 = 20$. However, for the student to understand why we multiply $1/3 \times 60$, we need to get them to understand that, when multiplying by 1/3, they are calculating the third part “of” 60. Cristiano Muniz (2009, p. 104) states that “associating the preposition OF to the multiplicative context is a very fertile pedagogical and epistemological tool, i.e., $2 \times 7 = 2$ groups OF 7 [...]”, similarly, 1/ 3 out of 60 = 20. It is a concept that needs to be clarified for the student, who begins the study with operations with fractions, as they were used to operating on the set of natural numbers in which the result of a multiplication of two numbers always results in a larger number. The reasoning for those who initiate the concept of multiplication by rational numbers is counterintuitive. Therefore, we agree with Santos (1997): it is necessary to explore further the notion of inverse fraction and its effect in calculations so that there is an understanding of the multiplicative inverse. What is the multiplicative inverse of 3? And why? Every natural number can be written as a fraction. Thus, 3 equals 3/1, which also represents “a thing” that I can divide into three parts, and to arrive at it, I again need to think of three times these three parts ($3 \times 1/3$), that is, perform the inverse operation, so I can multiply any fraction by its inverse and get the number 1 as a result. By analogy, I might think that the reciprocal of 3 is 1/3, the reciprocal of 3/2 is 2/3, and so on. It is a very simple logic. Indeed, any number divided by itself will give the unit, which is the neutral element of the multiplication operation. This knowledge is essential for students to understand the rule of thumb of division by fractions.

Santos and Rezende (1996, p. 176) follow the explanation with another conjecture: “if another school receives 60 bags of milk to be consumed in half a week, what should be done to know how many bags of milk the school will use in one week?” Students will certainly notice that consumption will double. In the previous examples, we know that 60 bags of milk were consumed in two weeks and to know the milk consumption in a week, we divided by 2 or multiplied by its multiplicative inverse, which is 1/2: 60 ÷ 2 = 60 /2 = 1/2 of 60 = 1/2 × 60 = 30. So, if we look at the problem by Santos and Rezende (1996), in which we know the consumption of milk in half a week, we will do 60 ÷ 1/2 or 60 × 2/1 = 120, using the multiplicative inverse principle or reasoning that the consumption for the whole week is twice as much as consumed in half a week.

Of course, we will have to do several exercises with the children so that they understand the concept of inverse fractions. We think it would not be possible to explain to a child and expect them to understand the rule of thumb of a problem like the one proposed by Brum without taking many other examples like those of Santos and Rezende (1996) first. By that time, the four teachers in our group had already come to understand better the content involved.

New understandings of the conventional algorithm for the proposed problem

After working with the students on simpler situations through the examples mentioned above, we resume the problem proposed by the teacher, who asked how many baby dresses I can make with $4\,\frac{2}{3}$ m, whose measure is $1\frac{1}{6}$ m for each dress. Maybe the students will already begin to understand the reason for inverting the fraction that represents the divisor in the division operation. An easier explanation is to show the student that, when multiplying any fraction by its multiplicative inverse, we will transform it into 1, a neutral element in the multiplication operation. This knowledge explored alongside one of the properties of division, in which we can multiply both terms of a division by the same number and thus obtain the same result finally clarifies the rule of thumb. Eg: 25 ÷ 5 = (25 × 2) ÷ (5×2). Analogously, in the proposed problem, after we transform the mixed numbers into improper fractions, we would have $\frac{14}{3} \div \frac{7}{6}$ and, applying the multiplicative inverse property of the divisor, we have $(\frac{14}{3} \times \frac{6}{7}) \div (\frac{7}{6} \times \frac{6}{7}) = \frac{84}{21} \div \frac{42}{42}$ $= \frac{84}{21} \div 1 = \frac{84}{21} = 4$. As we can see, we transform the divisor into the neutral element 1, leaving, in practice, only the first multiplication.

The teacher must go a long way with the student so that they understand the rule of thumb of the fraction-by-fraction division. Students must have clear what a fraction is and the ideas involved; move with dexterity in the multiplicative field, understanding that multiplication and division are reciprocal operations in which one does while the other undoes; understand equivalence and comparison; in other words, have built up a sense of numbers that allows them to easily make comparisons, measurements, equitable distributions, or measurements. Then, they will also risk other less conventional solutions. And this was also the achievement of our group.

All teachers involved understood the ideas involved in fraction-by-fraction division and the conventional algorithmic calculation process after this study. Even so, two teachers admitted that they were not confident enough to carry out with the student all the steps shown in this work. Therefore, they admitted they needed to prepare better and reflect on how to develop a class to explore fraction-by-fraction divisions (Santos, 1993, 1997; Santos-Wagner, 2008). Therefore, adequately prepared, they would not risk avoiding the content, as they did not know how to explain to the student the reason for the inversion of the term that represents the divisor in a fraction. In Skemp (1976, 1987), we find that understanding means knowing how to do and why to do something, as both understandings merge, complement each other, and interrelate.

Teachers do not always accept well that it is not enough to teach formulas and rules for instrumental and relational understanding to meet and become effective. Sometimes they prefer to teach as they learned. This conception became clear in our group when teacher AD constantly said: “I teach the fastest way”, “At school I was praised for the quick way I used to solve problems”, or “The student wants quick answers and paths to resolution”. This is true in many schools and corroborates what Skemp says:

> By many, probably a majority, his attempts to convince them that being able to use the rule is not enough will not be well received. ‘Well is the enemy of better,’ and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this (Skemp, 1976, p. 5).

However, we insist because, as the author above and others commented, we believe that teaching with understanding means knowing how to do something and why to do it; only then will the teacher know how to explain in different ways any subject that he/she proposes to teach. And, at the end of the study, AD also agreed with us: *the best path does not always represent the “good”, that which will provide us with knowledge that can be applied in other situations or related to other concepts*.

Group learnings: other alternative algorithms

✓ **Paper Folding**

Paper folding strips is also very useful for better understanding fraction-by-fraction division operations. For our example, we can use four strips of paper and one is folded into thirds and show them to the students while asking them to help us write what they represent: 4 2/3. Then, we take another whole strip and another one folded into three parts and folded in half, transforming it into six, which will be our measurement, again asking them to help us write it, systematizing the operation on the side strips, as shown in Figure 8.

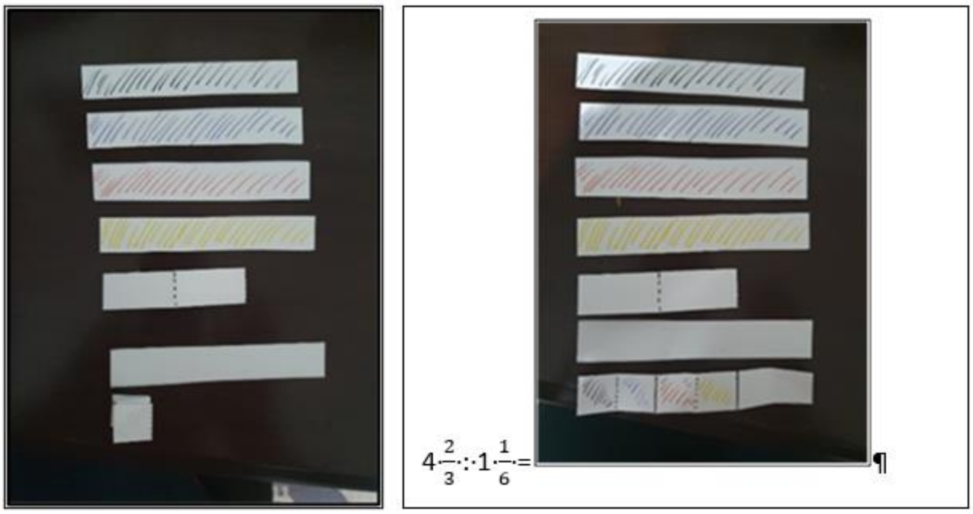

Figure 8.

Resolutions' stage of division with paper folding

In the first image, we see the first five strips representing the fabric Linda has, $4\,\frac{2}{3}$; and, in the last two, we see the measurement of each dress, $1\,\frac{1}{6}$. How many times is it possible to superimpose this measurement of a dress on all the fabric I own? The student will have no difficulty measuring the whole. Their query will be consider $\frac{1}{6}$ more. With some mediation from the basic education teacher, they will see they can redivide $\frac{2}{3}$ and turn them into $\frac{4}{6}$. Opening the folding and placing the strips side by side, as shown in Figure 9, they will notice that the measurement fits in four times. We consider this one of the most didactic ways for the student to understand what happens when dividing fractions by fractions. Hence, they will also easily infer that each whole meter now corresponds to $\frac{6}{6}$ and Linda's whole fabric corresponds to $\frac{28}{6}$.

Working with paper folding is playful, the student can use colours to highlight the folding, systematizing the learning built through collage in the notebook and text production by noting the conclusions. Therefore, our group understood the possibility of going beyond "arithmetization", in which algorithms are privileged, since the demonstration is almost always possible, and the student needs this power of seeing, as stated by Lorenzato (2006). The proof is that, in our group, teachers with long classroom experience needed demonstrations. This idea is corroborated by Cifuentes and Santos (2019) when they state that to understand is to make evident what is not. Thus, there is a need to see to understand. We then conceive visualization as a resource to materialize concepts, abstract or not, giving them shape, movement, and bringing them to the world of our insights, in order to be "seen" by it.

However, we are not naive. We know, as Mathias (2017) very well reminds us, that it is not enough to know the content and know how to teach it in the way we deem appropriate. We must consider the reality in which our student is inserted, to know the political pedagogical project of the school, and consider these socio-historical dimensions. They will be able to indicate the best form of didactic transposition and the language to be used to motivate and encourage in-service and pre-service teachers to seek creative ways of teaching mathematics that can delight the student. The form of didactic transposition that may work for one class may not work for another.

✓ **Numerical calculations**

We know there are clashes in which teachers defend that teaching mathematics equips students to speed up calculations. When we advocate concrete demonstration, we are not saying that this should not be one of the goals, but we believe its achievement comes from visualization, so both traditional and creative methods must be enabled. Therefore, our group went further and, still regarding the division in question, suggested that, in later stages, after the student gets familiar with the resolutions using creativity, we could ask: And, in the formal algorithm, would we have other ways of solving this operation? Rafael Vaz (2016) shows the fraction-by-fraction division using the division itself. In this algorithm, the numerator and denominator are divided by the divisor, i.e., numerator by numerator and denominator by denominator. Let us see what that would look like: $4\frac{2}{3} \div 1\frac{1}{6} = \frac{14}{3} \div \frac{7}{6}$. We realized that we could not divide 3 by 6, so we looked for a fraction equivalent to $\frac{14}{3}$ that will enable us to do so; hence, $\frac{14 \times 2}{3 \times 2} = \frac{28}{6}$. So, we do $\frac{28}{6} \div \frac{7}{6} = \frac{28 \div 7}{6 \div 6} = \frac{4}{1} = 4$.

A student who receives a similar operation for simple procedural training, without going through the steps described above can try to solve intuitively the division operation like the one shown here. Most likely, the teacher will say that he chose the wrong path. *Honestly, before this study, certainly four participants in our group would act like this because they were unaware of the algorithm presented by Vaz (2016).*

When we provide the student with various activities in which they need to think about numbers and their regularities, they will move easily in the multiplicative field and perceive equivalences. They will be able to choose the algorithm that best suits them. *That is what happened to our group. Everyone today has a new look at dividing fractions by fractions*.

What is the relevance of teaching fraction-by-fraction division to early years students?

Regarding the relevance of fraction-by-fraction content, the group concluded that all activities aimed at the flexibility of mathematical thinking must be present in our classrooms but in such a way that the students experience, create, explore, play, visualize, question, estimate, and approximate without falling into what Cifuentes and Santos (2019) call "arithmetization". Logic and applicability will be anchored in intuition and perception, so any content can be explored in the classroom as long as we know how to do it, so that the student feels attracted to it. Thus, mastering mathematical content will open up possibilities of thinking that may be liberating, as we seek meaning for it. And, after our study, all our colleagues agreed that it is possible to teach fraction-by-fraction division with comprehension. It is possible that students still repeat, as Lorenzato (2006, p. 91) points out: "I know how to do it, but because that is how I should do it, I do not know it", because it is a path to be built. What cannot happen is that the teacher does not know it. It is up to the teacher to emphasize the whys and encourage the essentially human curiosity.

When we return to the question about why to teach fraction division, we could say that this choice should be made by the teacher, considering his/her objectives and interests and students' previous knowledge. This decision should never be based on limiting the teacher's content knowledge and pedagogical knowledge/specialized teaching knowledge. Nor be mandatory through curriculum prescription, guidelines, or teaching manuals. Nor disregard the school's pedagogical political plan. The teacher must have, above all, the courage to reflect on their knowledge and seek to excel it in the exchange of experiences with their peers. In Shulman and Shulman (2004), we find:

> The learning proceeds most effectively if it is accompanied by metacognitive awareness and analysis of one's own learning processes, and is supported by membership in a learning community. Indeed, this model may well apply to the learning processes of students as well as it does to the learning processes of teachers. (Shulman & Shulman, 2004, p. 267).

Therefore, teaching or not teaching the content discussed above should be based mainly on reflecting: Do I know how to teach division of fractions? What do I know and what do I not know about the topic? What are the actual demands of the class? How will this content contribute to developing students' sense of numbers? Is the difficulty level involved adequate, or would other simpler activities be recommended? How can this content contribute, so my student becomes more able to learn other topics, better preparing them to play their role in society? What do local and national guidelines say about this content? Our group courageously sought to answer these questions and matured in this small learning community.

Conclusion

The test question proposed by Professor Brum made us think about several issues involving the teaching and learning of fractions, the teaching of mathematics in general, and about problem solving (Santos-Wagner, 2008). The professor was happy with the proposition because it did not fall into what Lopes (2008) calls misleading applications or dishonesty of presentations, as the problem about babies' dresses and their measurements is a possible context and was easy to demonstrate. Another high point of this evaluation was that it instigated the pre-service teachers and us to question our knowledge of this content. Before the study, four teachers in our group had doubts or a lack of relational and conceptual understanding. However, they all agreed that after this study, we all feel more confident in fraction-by-fraction divisions and how to teach them. Many of us, teachers who teach mathematics in elementary school, learn to apply algorithms and memorize them without knowing the reason for their function, much less their historical construction. And so, we do not always know how to make specific contents more accessible to our students, nor do we even ask ourselves about their actual relevance. Many times, we prefer to leave the content for the second segment of elementary school, claiming that the child is not prepared to understand it when, in fact, we are the ones who are not prepared to teach it.

The text by Paul Ernest and colleagues (2016) reinforces our conviction that doing mathematics is much more than simply bringing specific content to the classroom to comply with a programme. Doing mathematics is also seeking subsidies in philosophy and looking at school mathematics with a new attitude, daring and believing that any content can be taught if we know how and when to do it as philosophy

> [...] provides thinking tools for questioning the status quo, for seeing that 'what is' is not what has to be'; to see that the boundaries between the possible and impossible are not always where we are told they are. It enables commonly accepted notions to be probed, questioned and implicit assumptions, ideological distortions or unintended prejudices to be revealed and challenged. It also, most importantly, enables us to imagine alternatives. Just as literature can allow us to stand in other people's shoes and

> see the world through their eyes and imaginations, so too philosophy and theory can give people new 'pairs of glasses' through which to see the world and its institutional practices anew, including the practices of teaching and learning mathematics, as well as those of research in mathematics education. (Ernest et al., 2016, p. 4)

Bringing this evaluation applied to pedagogy students to be discussed in the group of professors in GEEM-ES and the UFES extension project allowed us to exercise these different perspectives, re-discuss practices, create strategies, and exchange them with colleagues, and learning from each other. The perception of some weaknesses led us back to study in search of new understandings. Furthermore, group discussions validated and clarified both conventional and unconventional algorithms. In this way, we could all perceive that the use of creativity makes mathematics teaching enjoyable because it can awaken the pleasure of discovery and rediscovery.

We found the need to further discuss the relevance of fraction-by-fraction division content in the early years, not to remove it from the program nor to defend its presence in basic school just by complying with guidelines, but to think about how this content can be better taught and why. We think that mastery of fraction-by-fraction division will contribute to other learning, as it requires the learner to master equivalences and increase the sense of numbers, a skill that provides the flexibility of thought. We would say this is an important and liberating step towards understanding rational numbers in their entirety in subsequent studies. We believe that mathematical knowledge should be appreciated for its beauty, social applicability, ability to read the world, and the hope of empowering people to make it a better place. In this sense, it is always possible to find reasons for learning any content and to teach and learn with creativity and understanding.

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