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**Problem solving and teaching equations of 2nd degree: meta-analysis of two professional master's research studies**

**Resolución de problemas y enseñanza de ecuaciones de 2º grado: metaanálisis de dos estudios de maestría profesional**

**Résolution de problèmes et enseignement des équations du 2e degré : méta-analyse de deux recherches de master professionnel**

**Resolução de problemas e o ensino de equações do 2º grau: metanálise de duas pesquisas de mestrado profissional**

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**Abstract**

This paper aimed to analyze proposals of teaching 2nd degree Equations developed in research's postgraduate with a focus on problem solving and the conceptions of Algebra. We developed bibliographic research in which we obtained two dissertations that underwent a meta-analysis. The results show that both are interested in teaching approaches to 2nd degree Equations related to Problem Solving. However, misunderstandings between theory and practice are evident regard to limitations and potential of each approach to Problem Solving, in addition to the recurrence of exercises related to the systematization of techniques through repetition, reinforcing a fundamentalist-analytical view of Algebra. It is necessary to consider the importance of expanding studies that present sequential proposals for teaching 2nd degree Equations during Problem Solving, consistently employing the different approaches (about, for and via), and the conceptions of Algebra that are linked to this teaching.

**Keywords:** Algebra; Mathematical education; 2nd degree equations; Problem solving.

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## Resumen

Este artículo tuvo como objetivo analizar propuestas para la enseñanza de Ecuaciones de Segundo Grado en investigaciones de posgrado con enfoque en la resolución de problemas y las concepciones del Álgebra. Desarrollamos una búsqueda bibliográfica en la que obtuvimos dos disertaciones que se sometieron a un metaanálisis. Los resultados muestran que ambos están interesados en la enseñanza de ecuaciones de segundo grado relacionadas con la resolución de problemas. Sin embargo, se evidencian malentendidos entre teoría y práctica ante las limitaciones y potencialidades de cada enfoque de Resolución de Problemas, además de la recurrencia para los ejercicios relacionados con la sistematización de técnicas a través de la repetición, reforzando una visión fundamentalista-analítica del Álgebra. Es requerido considerar la importancia de ampliar estudios que presenten propuestas secuenciales para la enseñanza de Ecuaciones de Segundo Grado en medio de la Resolución de Problemas, empleando consistentemente los diferentes enfoques (para, sobre y via), y las concepciones de Álgebra que se vinculan a esta enseñanza

**Palabras clave:** Álgebra, Educación matemática, Ecuaciones de segundo grado, Resolución de problemas.

## Résumé

Cet article visait à analyser les propositions d'enseignement des équations du 2e degré dans la recherche postdoctorale en mettant l'accent sur la résolution de problèmes et la conception de l'algèbre. Nous avons développé une recherche documentaire dans laquelle nous avons obtenu deux mémoires qui ont fait l'objet d'une méta-analyse. Les résultats montrent que les deux sont intéressés par l'enseignement des approches aux équations du 2e degré liées à la résolution de problèmes. Cependant, les incompréhensions entre théorie et pratique sont évidentes face aux limites et aux potentialités de chaque approche de Résolution de Problèmes, en plus de la récurrence d'exercices liés à la systématisation des techniques par la répétition, renforçant une vision fondamentaliste-analytique de l'Algèbre. Il est important de considérer l'importance d'élargir les études qui présentent des propositions séquentielles pour l'enseignement des équations du 2e degré au milieu de la résolution de problèmes, en utilisant systématiquement les différentes approches (sur, pour et via), et les concepts d'algèbre qui sont liés à cet enseignement.

**Mots-clés :** Algèbre, Enseignement des mathématiques, Équations du 2e degré, Résolution de problèmes.

## Resumo

O presente artigo objetivou analisar propostas de ensino de Equações de 2º grau em pesquisas de pós-graduação com foco na resolução de problemas e na concepção de Álgebra. Desenvolvemos uma pesquisa bibliográfica em que obtivemos duas dissertações que passaram por uma metanálise. Os resultados mostram que ambas apresentam interesse por abordagens de ensino das Equações de 2º grau relacionadas à Resolução de Problemas. Entretanto, evidencia-se incompreensões entre a teoria e a prática frente as limitações e potencialidades de cada abordagem da Resolução de Problemas, além da recorrência a exercícios relacionados à sistematização de técnicas por meio da repetição, reforçando uma visão fundamentalista-analítica da Álgebra. Cabe considerar a relevância da ampliação de estudos que apresentem propostas sequenciais de ensino de Equações de 2º grau em meio à Resolução de Problemas, empregando com coerência as distintas abordagens (sobre, para e via), e às concepções da Álgebra que estão atreladas à esse ensino.

**Palavras-chave:** Álgebra, Educação matemática, Equações de 2º grau, Resolução de problemas.

## **Problem Solving and Teaching Equations of 2nd Degree: meta-analysis of two professional master's research studies**

In what concerns school algebra and the manner of conceiving it, there is consonance between what was exposed by Fiorentini, Miorin and Miguel (1993), The Parâmetros Curriculares Nacionais (PCN) (Brasil, 1998) and the Base Nacional Comum Curricular (BNCC) (Brasil, 2018). For those, the objective of Algebra must be based on the development of the algebraic thought, which must be explored during the schooling from actions which: search for regularities; present variations between magnitudes; and make it possible to generalize. However, the studies by Lins and Gimenez (1997) and Kieran (2004) affirmed, at the time, the didactic books - main allies of the teachers in the teaching process - presented a traditional approach to Algebra, based on rules to be followed by manipulating symbolic expressions and equations, characterizes as theory/exercise. Recently, regarding the content of 2nd degree Equation, Pereira, Doneze and Proença (2023) found that textbooks maintain the organization of the teaching-learning proposal for such content considerably similar to the works used during the Modern Mathematics Movement (MMM).

In contrast to the traditional approach to teaching, it is suggested approaching problem solving in teaching and learning of Mathematics (Brasil, 1998, 2018). Authors as Schroeder and Lester Jr. (1989), Onuchic and Allevato (2011) and Proença (2018, 2021) affirm the usage of mathematical situations, which configure possible problems for the students, must be employed as starting point for the teaching of mathematical contents, being this known as the approach via/through problem solving.

Schoen (1994) already proposed a series of recommendations for an efficient teaching of Algebra with focus on problem solving. Between the recommendations, it is highlighted: basing the learning of new subjects in knowledge and the understanding the students already have; and introducing algebraic topics with applications by means of real problems/situations. In the same direction, Schoenfeld (2020) sustains that the most advantageous possibility of teaching Algebra is that in which the students must be engaged in activities as arithmetic generalization, the relation of interdependence between magnitudes or as a mean of searching the solution for situation in which the main information is unknown and thus it is resorted to the equations.

In front of this, it is taken as reference to this study the subject of Equations of 2nd degree, resorting to the problem solving as a possibility of composing teaching strategies, in a manner we asked the following question: how the Problem Solving has been investigated in the classroom focused on the teaching and learning of Equations of 2nd degree? Thus, we focused

on the objective of this article which consisted in: analyzing the proposals of teaching Equations of 2nd degree in postgraduate research focused on problem solving and in the conception of Algebra.

For this, we structured the present article in the following sections: The teaching-learning of school Algebra in the scope of Equations of 2nd degree and the problem solving; the methodological aspects, in a manner we did a gathering of Brazilian thesis and dissertations attached to the Banco de Teses e Dissertações da Capes and the Biblioteca Digital Brasileira de Teses e Dissertações (BDTD) so we could do the meta-analysis, utilized to outline the results, from the assumptions by Severino (2007), Lovatto et al. (2007), Gil (2008), Bicudo (2014) and Oliveira and Borges (2021); and, lastly, about the final considerations of this investigation. Given these and other skills forwarded by the BNCC (Brazil, 2018), Proença, Campelo and Santos (2022) highlighted that problem solving in such an official document indicates the application of mathematics studied in problems.

### **The teaching-learning of School Algebra: The case of Equations of 2nd Degree**

In Brazil, according to Miorim, Miguel and Fiorentini (1993), the teaching of Algebra - for a determined period - was marked by the reformist aspiration to the Modern Mathematics Movement (MMM). This period was characterized by the exacerbated rigor to the usage of mathematical terms and expressions, which reduced the teaching to the structuralist study of definitions before they could be built from the exploration of problems. The Equations of 2nd degree by settling in the block of Algebra contents also suffered the implications of the movement. In the works elaborated by Osvaldo Sangiorgi, precursor of MMM in Brazil, it becomes evident how the equations were approached: About the definition of Equation of 2nd degree, Sangiorgi (1967) elaborates and answers a possible question which could appear during the teaching of the subject: Why does the coefficient of  $x^2$  must be necessarily different of 0?

Well, if  $a = 0$ , then you would have:  $0x^2 + bx + c = 0$  or  $bx^1 + c = 0$  which is an equation of first degree in the variable  $x$ , if  $b \neq 0$  (remember the biggest exponent of  $x$  now is 1). You already know how to solve equations of first degree since the 2nd year of junior high, therefore you know that  $\forall b, c$  and  $b \neq 0$   $bx + c = 0 \Leftrightarrow bx = -c \Leftrightarrow x = -c/(b)$  where  $V = \{-c/(b)\}$  and, therefore, the root of the equation is the number  $-c/(b)$  (Sangiorgi, 1967).

From a conception called fundamentalist-structural, where the introduction of structural properties of the operations, which justified the passing of the change from a content of Algebra to another, would be sufficient to make the students capable to identify and apply these same structures in different contexts which succeeded them. The teaching of Equations of 2nd degree

passed to be seen as manipulation of letters and procedures linked to Arithmetics. Works such as this of Sangiorgi (1961, 1997), turned to the 4th grade of junior high, presented such teaching based in the technical sequence (algorithm)/practice (exercises). This processological practice is translated into: initially, it is done an instructionist explanation and self-explaining, based on the techniques and methods prescription; lately, there is a group of exercises called application or attention tests, where the instructions presented in the previous stage of the sequence will require the process of replication (Miorim, Miguel & Fiorentini, 1993, Lins & Gimenez, 1997, Kieran, 2004).

In the official documents, considering by chronological order the Parâmetros Curriculares Nacionais (PCN) (Brasil, 1998), to the teaching of Equations of 2nd degree it was recommended a deep study of the techniques of equation solving in situations-problem with the objective of involving the variation of magnitudes and making it possible the understanding of the letter as an unknown. Without pointing exactly in which year, the PCN (Brasil, 1998, p. 88) wove succinctly recommendation for the study of the content, suggesting approaching the “solving of situations-problems which can be solved by an equation of second degree which roots are obtained by factoring, discussing the meaning of these roots in confrontation to the proposed situation”. Recently, the Base Nacional Comum Curricular (BNCC) (Brasil, 2018) recommended the 8th grade as the initial contact with the content, suggesting an education which favors the students “solving and elaborating, with and without the usage of technologies, problems which might be represented by polynomial equations of 2nd degree of the kind  $ax^2 = b$ ” (Brasil, 2018, p. 313). On the other hand, in the 9th grade, the study must favor the action of “understanding the factoring processes of algebraic expressions, based on their relations with the notable problems, to solve and elaborate problems which might be represented by polynomial equations of 2nd degree” (Brasil, 2018, p. 317).

Although in the last years the documents cited above excelled the solving and elaboration of problems or mathematical situations which made it possible the confrontation with the content, Lins and Gimenez (1997) already commented on research which employed the usage of real or realistic situations, connected to authentic activities or situations from daily life, as an investigative possibility of glimpsing the content as a tool capable of organizing or guiding the process of solution, not primarily as the direct object of the study. Currently, research as that of Gonçalves and Proença (2020, p. 211), supported by the BNCC recommendations for Equations of 2nd degree, wove the following question: If the students of the ninth grade should learn this aspects, then is it possible the conceptual and procedural knowledge of High School students on these mathematical contents are presented as developed?

In a general manner, the performance of the students, both in exercises of application and in contextualized situations, revealed itself in deficit in the conceptual and procedural scopes by recognizing examples of Equations of 2nd degree only in the reduced form, they did not identify the coefficient relation with the procedure in the solution formula and did not correlate the content to the available situations. The researchers question the teaching based on the presentation of Equations of 2nd degree only in their reduced form and in the memorization of formulas and recommend a teaching based on solution of contextualized situations which make it possible identifying the characteristics of the content, besides exposing the students to different procedures of solution attributing meaning to the recurrence of algorithms (Gonçalves & Proença, 2020).

The context mentioned above matches the analysis realized by Pereira, Doneze and Proença (2023), by analyzing didactic books selected by the PNLD to be implemented in the quadrennium 2020/2021/2022/2023, found the processes of teaching and learning of Equations of 2nd degree start by discussing situations from extra-mathematical context, associated to the laws of Physics and to mathematical linked to concepts of Geometry, as are of rectangular figures. However, during its course the researchers denounced there is a great structural similarity to works of the period of the Modern Mathematics Movement, strongly rooted in the structural-fundamentalism based on the propositions by Miorim, Miguel and Fiorentini (1993), Lins and Gimenez (1997) and Kieran (2004).

Lastly, Pereira, Doneze and Proença (2023) found that in different stages of the teaching and learning proposal from the books, there is an exacerbated usage of exercises of exhaustive repetition and explicit requisition of the usage of solution methods. This way, they indicate the organization of conceptual teaching and learning of Mathematics, consequently of Algebra, from the Problem Solving as a starting point and/or as possibility of application and investigation of the learned knowledge by ending the process of teaching according to what is proposed by Proença (2021).

### **The Teaching-Learning of School Algebra: The Case of Problem Solving**

In a conditional manner, maybe it is only a sufficient condition enough to recurring to Problem Solving so the teaching and learning of Algebra and in special Equations of 2nd degree occur as the aforementioned researchers recommend. However, for each conception of Algebra there is an approach of Problem Solving and Problem Elaboration to be employed in a manner to potentialize the processes of teaching and learning. In this sense, Pereira, Doneze and Proença (2022) demonstrated that in the didactic books approved in the PNLD are found five

kinds of activities for the students to elaborate problems. Meanwhile, the discussion if the activities proposed by the teachers and developed by the students are problem solving or of applications in exercises have been constituting fields of investigation. In the view of Echeverría (1998), there are different denominations and meanings for it.

According to Echeverría (1998), exercises are not always related to the repetition of mathematical operations, but also to activities which do not require the students to make decisions about what procedures and strategies are going to be useful during the process of solution. There are those who allow being solved by the repetition of a determined technique previously taught by the teacher as it was discussed by Miorim, Miguel and Fiorentini (1993) and by Lins and Gimenez (1997); the second, not only require the systematization of evident techniques, but require some procedures are inserted in the solution, as the translation from the natural language to the mathematical language which for Miorim, Miguel and Fiorentini (1993), concerning algebraic contents such as Equations of 2nd degree, it would be characterized as a linguistic-stylistic conception, in which Algebra is seen as a specific language with the finality of expressing and leading to procedure which are also specific.

On the other hand, Echeverría (1998) presents that problems are situations which propose the challenge of being solved without a known algorithm which leads directly to the solution, independently of the fact that in a determined task a student knows or not this algorithm previously (Echeverría, 1998).

In front of this definition, the author characterizes the school problems as being: quantitative problems; qualitative problems. The school problems of the qualitative kind are those which aim to require the students to observe, judge and manipulate mathematical objects without in fact achieving a mathematical solution or not. Some skills are required, as an example; classifying, serializing, ordaining, analyzing and evaluating. While those of the quantitative kind would require data manipulation to reach a numerical solution or not, which aim to “translating daily experiences for a mathematical language, establishing conjectures and hypotheses or exploring and modeling the strategies of problem solving acquired in informal contexts” (Echeverría, 1998, p. 50).

At this point it is evidenced by the recurrence to mathematical situations which may be faced as problems with the end of objectifying the teaching and learning of Equations of 2nd degree which is meaningful for the students, in other words, which make possible the construction of new knowledge in detriment of their previous knowledge. To do it, it requires knowing different approaches of Problem Solving, distinct in themselves according to the teaching objectives of the teacher.



This recommendation is close to the recommendation woven by the National Council of Teachers of Mathematics (NCTM, 1980) and by researchers as Lester Jr and Mau (1993), Masingila and Lester Jr (2001) and Proença (2018), which affirmed the importance of an environment driven to the investigation of new knowledge in relation to those the students already have, transforming the conception that teaching is transmitting information in an act of helping the students to build understanding on the mathematical concepts and processes. In order to do it, they commune with the fact that Problem Solving should become the focus of teaching mathematical contents in the classroom. Facing this fact, researchers Hatfield (1978) and Schroeder and Lester Jr (1989) presented three approaches of Problem Solving to teaching: teaching about the Problem Solving; teaching for the Problem Solving; teaching via Problem Solving.

According to Schroeder and Lester Jr (1989), teaching about Problem Solving makes reference to the model proposed by Polya in the early decades of the 20th century based on 4 stages: understanding the problem; elaborating a plan; executing the plan; review or retrospect. In this approach, the students take themselves as problem solvers, and acquire and use strategies as recurring simpler problems or for patterns in similar solutions. In the teaching of Equations of 2nd degree, an example would be characterized as investigating in the solution of notable products the procedures or strategies which allowed solving an equation, even if it is not acknowledged or given such importance to the concepts which might appear to be employed seeing that solving the Equation of 2nd degree would be the object of study itself.

When teaching for Problem Solving, Schroeder and Lester Jr (1989) affirm it is about glimpsing Mathematics as a possibility of practical application, in which the main objective of learning it is being able to use it. The teacher focuses on which manner the Mathematics is being taught might be applied in the problem solving of routine situations or not. By choosing this approach, the teacher focuses on the students' skills in applying the acquired knowledge from one problem to others. In the teaching of Equations of 2nd degree it would characterize a similar situation to that found in the didactic books since Sangiorgi (1961, 1697) and that during the last years has remained according to what is pointed by Miorim, Miguel and Fiorentini (1993) and Lins and Gimenez (1997). It would be treated after the teaching of every concept connected to Equations of 2nd degree and the teacher presenting a group of situations which would make possible the application of this concept, based on theory/practice.

When teaching via/through Problem Solving, for Schroeder and Lester Jr. (1989) the problems are seen as the initial means to teach and learn Mathematics - in this sense- the reaching of a mathematical topic, its concepts and techniques are developed as answers during

the Problem Solving. The problems are seen as vehicles which will work to drive the teaching and learning of the content. In the teaching of Equations of 2nd degree it is characterized as how, in a few didactic books, contextualized situations are presented in the beginning of each content as a possibility to, by mean of investigation and group work, make it possible for the students to use previous knowledge connected to Algebra as Equations of 1st degree, notable products and factoring or even the Arithmetics to reach a solution which lately will serve the teacher as possibility of articulation for the objective of teaching.

In Brazil there are two great branches connected to the teaching via/through Problem Solving. The first is about the Grupo de Trabalho e Estudos em Resolução de Problemas (GTERP), which proposes a script of stages to aid the teachers in the conduction of Problem Solving according to Onuchic and Allevato (2011), being: Preparation of the problem.; Individual reading; Group reading; Problem solving; Observing and encouraging; Register of the solutions on the board; Plenary; Search for consensus; Formalization of the content; and proposition of new problems. The second it the Grupo de Estudos em Resolução de Problemas na Educação Matemática (GERPEM), which in the most recent proposal, Proença (2021), consists in an organization of the Mathematics teaching based on four stages, being them: Usage of the problem as starting point; Formation of the concept; Definition of the consent; Application in new problems. The first of the stages is found based in five actions to aid the teachers' work in the classroom by incorporating the Teaching-Learning of Mathematics via Problem Solving (EAMvRP) presented in Proença (2018), to be known: Choice of the problem; Introduction of the problem; Aid to the students during the solution; Discussion of strategies; Articulation of the strategies of the students to the content.

Both corroborate with Schroeder and Lester Jr. (1989) making it possible to affirm that if the objective is teaching Mathematics the best approach is teaching via/through Problem Solving. Following such affirmation, the researchers Pereira and Proença (2023), structured a Hypothetical Trajectory of Teaching-Learning of Equations of 2nd Degree via Problem Solving (THEAE2QvRP) and found the hypothetical gathering of questions, doubts and difficulties of the students intertwined to the problem solving and possible solutions, when based on the assumptions of the EAMvRP, make it possible guiding the teachers to a new and efficient process of teaching-learning.

In view of such discussions, Maia-Afonso and Proença (2020), when proposing to investigate Problem Solving approaches in professional master's theses, highlighted that the three approaches were covered, but in the majority that reported using teaching via Problem Solving, there were cases in which that the focus was directed to teaching for Problem Solving.

However, in this investigation, the objective does not reside in positioning or discussion which are the potentialities and limitations of each approach. The intention is revealing and confronting the distinct approaches and articulations of Problem Solving with the Equations of 2nd degree which might be found in the scope of the researches developed in the Brazilian scientific scene.

### **Methodological Procedures**

The present investigation has as objective analyzing teaching proposals of Equations of 2nd degree in postgraduate research focused on problem solving and the conception of Algebra. In face of its qualitative and bibliographical nature (Gil, 2008) were employed meta-analysis elements as method of investigation. As pointed out by Bicudo (2014, p. 9), it is an “investigation supported on comparisons and analysis of primary data of the researches, taken as significant in relation to the subjects put under focus”. For Lovatto et al. (2007) there are some main justifications to employ the meta-analysis, between them: obtaining new results; synthesis of contradictory results; increase of analytical precision. According to Santos, Oliveira and Borges (2021), structurally, some of the main stages to be followed in investigations which recurred to meta-analysis are: defining the study problem; identifying primary relevant studies; selection (inclusion/exclusion); data extraction; synthesis and analysis of the data with evaluation of the quality of the evidence; and discussion of the results.

Having the objective of presenting new theoretical knowledge on the interpretations of other research, for the selection, extraction and synthesis of the data it was searched support in assumptions by Severino (2007) and Gil (2008) about guidelines for reading, analysis and text interpretation; in general, consist in: textual, thematic and interpretative analysis, problematization and personal synthesis.

In this manner, by defining the problem of the investigation, the search for primary studies was initiated from the Biblioteca Digital Brasileira de Teses e Dissertações (BDTD), where in the field of ‘advanced search’, having as knowledge the different names about the theme, two researches were done. In the first, there was an insertion of keywords: Equation of second degree and Problem Solving, finding correspondence in 12 dissertations and 3 thesis; quadratic equation and Problem Solving were found 15 dissertations and 1 thesis; polynomial equation of second degree and Problem Solving was found 1 dissertation. The second search counted on the addition of the teaching terms and learning to the keywords previously defined, being then: teaching and learning of Equation of second degree and problem solving with correspondence in 5 dissertations and 1 thesis; teaching and learning of quadratic equation and

Problem Solving finding 2 dissertations; teaching and learning of polynomial equation of second degree and Problem Solving without finding any results.

The second bibliographic source was the Catálogo de Teses e Dissertações da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), in which were done three insertions: in the first, ‘Equation of second degree AND Problem Solving AND teaching AND learning’; the second, ‘polynomial equation of second degree AND Problem Solving AND teaching AND learning’; in the third, ‘quadratic equation AND Problem Solving AND teaching AND learning’. The expressions AND between the keywords are boolean connectives which aid in the correspondence between words. In the first insertion, as results 114 investigations. In the second insertion, as result were obtained exactly the same 114 investigations of the previous insertion. In the third insertion, 10 investigations were obtained.

The refining of the results obtained in both databases occurred from the reading of the title, abstract, summary, keywords and parts of the text which referred particularly to the subjects: Equations of 2nd degree, Problem Solving and teaching-learning. Chart 1 evidences the research by Coutinho (2016) and Kuroiwa (2016) as results of the refining which constituted the corpus of analysis.

Chart 1.

*Characterization of the researches which constituted the corpus*

<b>Research</b>	<b>Title</b>	<b>Degree/Kind</b>	<b>Key words</b>	<b>Databank</b>
Coutinho (2016)	An application of the Problem Solving in the teaching of Equations of 2nd degree	Dissertation/Professional Master’s	<ul style="list-style-type: none"> <li>• <i>Equação de segundo grau AND Resolução de Problemas AND ensino AND aprendizagem</i></li> </ul>	<ul style="list-style-type: none"> <li>• CAPES<sup>2</sup></li> </ul>
Kuroiwa (2016)	A peculiar approach of the Equation of Second Degree in Elementary School and High School	Dissertation/Professional Master’s	<ul style="list-style-type: none"> <li>• <i>Equação de segundo grau e Resolução de Problemas;</i></li> <li>• <i>Equação de segundo grau AND Resolução de Problemas AND</i></li> </ul>	<ul style="list-style-type: none"> <li>• BDTD<sup>3</sup></li> <li>• CAPES</li> </ul>

<sup>2</sup> <https://catalogodeteses.capes.gov.br/catalogo-teses/#/>

<sup>3</sup> <http://bdtd.ibict.br/vufind/>

			<i>ensino AND aprendizagem</i>	
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With the researches selected, we aimed to outline the reading units which, for Severino (2007), are thematic sectors of the researches which form meaning to be analyzed in face of the objective of the investigation. Chart 2 presents the reading units to be considered in the analysis of the research.

Chart 2.

*Reading units to be analyzed in the selected researches*

<b>Reading units</b>	Objective(s)
	Teaching Proposal

The reading units served as aid for the realization of a selective and analytic reading (Gil, 2008) of such parts of the research, making it possible to extract and organize key fragments of both proposals. In this manner, the analysis was given in terms of three categories: a) About the voice of the problem and its introduction – referring to the possible problems chosen and how its teaching was given; b) About the problems and their characteristics in face of Algebra – referring to the conception of Algebra involved in the usage of the possible problems; c) About the approach of teaching of Problem Solving – referring to the teaching approach, if the teaching about, for or via/through problem solving. Thus, it was searched to gather problematization which sustained the elaboration of a reflexive synthesis after each organization as a new source of theoretical contributions about the investigated subject (Severino, 2007; Gil, 2008).

**Analysis and Data Discussion**

From the reading units, the research by Coutinho (2016) was described and analyzed based on fragments presented in Chart 3.

Chart 3.

*Description proposed by Coutinho (2016)*

<b>Coutinho (2016)</b>
<b>Objective:</b>
“[...] debating about the subject equations of 2nd degree, but utilizing unusual methods of solution of the equations, such as the method of completing squares and the geometrical method by Al-Khwarizmi” (p. 13).
<b>Teaching Proposal:</b>

The proposal was developed in a private school from the city of Niterói, Rio de Janeiro, with 27 students from the 9<sup>th</sup> grade. The didactic book adopted by the school is the *Araribá Plus Matemática 9º Ano*, and also the activities notebook from the same collection. “The present topics in the unit of the didactic book which approached equations of 2nd degree were introduced by a motivational problem, that is, for each subject presented in the unit, a situation was elaborated in a manner that their mathematical modeling could satisfy such context. Furthermore, three activities were applied in this class, to be known, two workshops, one about the method of completing squares and the other about the geometrical method by Al-Khwarizmi, and a list of problems” (p. 14).

1. Below it is found the first example presented to the students, referring to the definition of Equation of 2nd degree.

Juliana made a rug to decorate her room, observe it:



Figure 1.

*Problem utilized to start the study of equations of 2nd degree extracted from Coutinho (2016)*

To make this rug, Juliana knitted, one onto the other, square-shaped retils of fabric, all with the same dimensions. Knowing the rug ended with 4.050 cm<sup>2</sup>, how can we calculate the measurements of each side of the retail squares?

2. This situation was proposed and, without more difficulties the students understood that each side of each square would be the unknown  $x$ . Given the length of the rug has 10 squares and the width has 5 squares. the measurements of the sides of the rug will have, respectively,  $10x$  and  $5x$ . Since the rug has rectangular shape, its area will be given by the rectangle area, that is,  $S = 10x \cdot 5x$ , but in the enunciation, it was said this area had a value of 4.050cm<sup>2</sup>. Therefore, we can elaborate the following situation (p. 56):

$$10x \cdot 5x = 4050 \Rightarrow 50x^2 = 4050 \Rightarrow 50x^2 - 4050 = 0$$

Thus, the equation above is presented as an example of equation of 2nd degree (p. 56).

3. Lately, it was presented to the students the definition of Equation of 2nd degree, complete and incomplete, coefficients, the meaning of roots, the deduction of Bhaskara’s formula through the method of completing squares and the geometric method by Al-Khwarizmi.
4. After the definitions it followed the development of two workshops, below it follows the two problems applied in the workshops.

*“Problem 1: My brother and I collect stamps. I have a quantity of stamps and he has four times this quantity. Knowing that if I elevate to the square the number of stamps I have, we will have the same amount of stamps, how many stamps does my brother have?”* (p. 77).

5. It was asked for the students to solve the problem using the method of completing squares and the geometric method by Al-Khwarizmi.

As a brief description of the forwarding [...] The problem was read and, as proposed, the students should solve it through the method of completing squares. In Figure 2, Coutinho (2016) presents the correct solution of a student following the method of completing squares.

+ • Solução utilizando o método de completar quadrados.

$$x^2 = 4x$$

$$x^2 - 4x = 0$$

$$x^2 - 4x + 4 = 0 + 4$$

$$(x - 2)^2 = 4$$

$$x - 2 = \pm\sqrt{4}$$

$$x - 2 = +2$$

$$x - 2 = -2$$

$$x_1 = 4$$

$$x_2 = 0$$

R: 16 figurinhas.

Figure 2.

Correct solution by one of the students about problem 1 extracted from Coutinho (2016)

“Without further difficulties, the students were able to read and interpret the problem, and also translate it to an equation of 2nd degree” (p. 80).

“Problem 2: The square of a positive number in addition to twelve times this number results in 189. What number is it?” (p. 80).

6. It was asked to the students to solve this problem using the method of completing squares and the geometric method by Al-Khwarizmi. In Figure 3, Coutinho (2016) presents the correct solution by a student following the method of completing squares.

+ • Solução utilizando o método de Al-Khwarizmi

$$x^2 + 6x + 6x$$

$$x^2 + 12x$$

$$189 + 36 = 225$$

lado = 15  
 $x + 6 = 15$   
 $x = 9$

Figure 3.

Correct solution by one of the students about problem 2 extracted from Coutinho (2016)

“Without further difficulties, the students knew how to transform the enunciation in a mathematical sentence” (p. 81).

About the choice of the problem and its introduction: By paying attention to aspects associated to the choice of the problem and its preparation for the introduction in the classroom facing the teaching objectives, we identify the researcher reveals theoretically the distinction between problem and exercise by pointing out that “[...] the realização of an exercise is resumed to the execution of skills and technique which were learned by the student. For the realization of a problem which is being done for the first time, strategies and procedures already known are sought after” (Coutinho, p. 22). Knowing this definition is a positive indicative for the good employment of Problem Solving, approaching what was proposed by Echeverría (1998) about

the dualism between these two concepts and clarifies the comprehension of the problem with the possibility of moving to the teaching of contents such as equations of 2nd degree as pointed out by Schroeder and Lester Jr. (1989), Onuchic and Allevato (2011) and Proença (2018).

About the problems and their characteristics in face of Algebra: in the introductory situation proposed by Coutinho (2016), it is referred to a qualitative problem, which according to Echeverría (1998) requires observing, judging and manipulating mathematical objects, with the end of reaching a solution being numerical or not. In a second moment, after having done the teaching via/through problem solving, Coutinho (2016) does a workshop in which presents two problems with the objective of working unusual methods, with the objective of working the unusual methods, being that of completing squares and that by Al Khawarizmi. The situations called by Coutinho (2016) as problems 1 and 2, applied in the workshop, consist, in our view, in exercises, once, according to Echeverría (1998), required the systematization of techniques immediately presented, even being necessary the translation from natural language to the mathematical language. Both the exercises presented at the second moment go back to the rhetorical of Algebra, in which only words were used to express both the situations and the means of solving them (Lins & Gimenez, 1997). However, the initial and final situations present symbolic Algebra characterized by an analogical-fundamentalism, in which recurring geometric representations constitute an intermediate stage in the visual determination of algebraic representations (Miorim, Miguel & Fiorentini, 1993).

About the teaching approach of Problem Solving: Coutinho (2016) assumes the teaching of Mathematics via/through problem solving justified by the reason of “the student becoming a participant in his own learning, [...] [creating] methods and strategies of Problem Solving, not restricting himself to [...] routine and uninteresting exercises, which value the learning by training and memorization” (Coutinho, p. 20). Thus, it was expected a conduction of the teaching and learning of Equations of 2nd degree, similar to the ideals of Schroeder e Lester Jr. (1989), Onuchic and Allevato (2011) and Proença (2018). However, the posture followed was the teaching of Mathematics for the problem solving, divergence that was also shown in studies by Proença (2018a) and Proença and Maia (2018), according to which, in both, one of the proposals of teaching analyzed there was a theoretically coherent choice, but it followed the teaching for problem solving. In the situations described in the proposal by Coutinho (2016), it is not evident the participation and the mobilization by part of the students happened, because the first worked only to represent and define the Equations of 2nd degree, without the attempt to give meaning to the obtained equation in detriment of other methods or the roots. Furthermore, in the workshop stage, were not valued the possibilities of the students proposing



strategies which could be articulated to other representations. This result is in the direction of what was pointed out in the bibliographic study by Proença and Maia-Afonso (2020) which revealed some teaching proposals of professional master's were more aligned to the approaches about and for problem solving, because they focused only on the act of solving problems.

In an analog manner, from the reading units, the research by Kuroiwa (2016) was described and analyzed based on fragments presented in chart 4.

Chart 4.

*Description of the proposal by Kuroiwa (2016)*

<b>Kuroiwa (2016)</b>
<b>Objective:</b>
“[...] aiding the teachers in the important task of teaching mathematics, and more specifically, the Equation of 2nd degree, promoting effective learning, in which there is more interaction and complicity between student and teacher” (p. 20).
<b>Teaching Proposal:</b>
<p>Proposal developed in a state school in the city of Presidente Prudente, São Paulo, with students from the 1<sup>st</sup> grade of High School and 9<sup>th</sup> grade of Elementary School. Divided in three stages: the first constituted in an Diagnostic Assessment with the objective of “<i>verifying if the students are capable of understanding and applying the concepts of equation of second degree, if they remember the mathematical methods and procedures of its solution, if they apply adequately the algorithms and operations. We also want to verify if they are able of analyze the information and interpret the proposed questions, using different strategies and resources, from intuition to algorithms, mainly the formula of Bhaskara and then obtain the correct solutions</i>” (p. 53, our stress); the second constituted in the application of an activity with the “purpose of resolve the difficulties presented in the Diagnostic Assessment, following the method by Polya” (p. 69); the third constituted in the reapplication of the evaluation with the objective of realizing the “verification of the development of the students with the acquisition of knowledge and the development of competences and skills [...] not presenting them the solution of the Diagnostic Assessment done by them” (p. 76).</p> <p>Below it is found the suggestion of conduction of solution of one of the problems from the proposal, referent to the second stage, following the procedures by Polya:</p> <p>“The area of a square with size equal to 5, added of eight times its side is equal to 65. Use the method of completing the square, using the geometric form and obtain the measurement of the size of the new square” (p. 70).</p> <p><i>Interpretation and comprehension of the enunciation of the problem.</i></p> <p>What is the data of the problem?</p> <p>By doing the reading, we understand that: the side of the square has the value of 5 and if we add eight times this side, we will have a total area of 65.</p> <p>We might question: initially, what is the kind of figure in question? Multiplying by eight times its size, what figure will it become? What is the characteristic of the square? And of the rectangle? How to calculate their areas? What is asked?</p> <p>Obtaining the measure of the new square which the figure will form by the method of completing the square.</p> <p><i>Planning a method.</i></p>

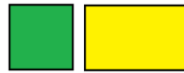
What objectives do we need to reach?

Starting from the values we have, we try to assemble a new square partitioned next to the largest rectangle in two equal parts and making it available, as a manner of obtaining a larger square. It is verified then the lack of a small square to complete it, which sides are equal to the half of the side of this rectangle. We add the value of this area to both sides of the equation.

Through the area calculation, we extract the square root and obtain the value of the side of this new square.

*Execution of the plan.*

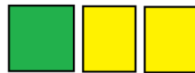
Interpreting as area of a square and the area of a rectangle, we can write in the following form:



$$5^2 + 8 \cdot 5 = 65$$

$$25 + 40 = 65$$

Dividing the rectangle in half we have two rectangles with the half of the initial value, this way:

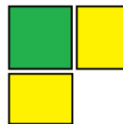


$$5^2 + 4 \cdot 5 + 4 \cdot 5 = 65$$

$$25 + 20 + 20 = 65$$

Each rectangle must be arranged in a manner it stays juxtaposed to two sides of the square.

In this composition the area remains the same:



$$25 + 20 + 20 = 65$$

To complete the square we need to add a square in the corner of the figure of measurement  $4^2 = 4 \cdot 4$  which is equal to 16.



$$25 + 20 + 20 + 16 = 65 + 16$$

Resolving, it is perceived we have:  $81 = 81$  (true sentence)

Since the new area is equal to 81, the size of the side of the new square is  $\sqrt{81} = 9$

*Validate the solution.*

The side of the square will be  $5 + 4$  which in fact is equal to 9, therefore the solution is true (p. 72).

About the voice of the problem and its introduction: Kuroiwa (2016) opted to present in two moments of diagnostic assessment situations considered by Echeverría (1998) as exercises, having in sight the objective was “verifying if the students are capable of understanding and applying the concepts of equation of second degree [...] if they apply adequately the algorithms and the operations and if they are capable of analyzing the informations and interpreting the proposed questions” (Kuroiwa, 2016, p. 53). In the second state, opted to present an exercise that, differently of those present in the diagnostic assessment, required a systematization of

techniques evidenced by the method of completing squares, but required some procedures as the translation for the mathematical language (Echeverría, 1998). The approach of solution previously chosen by Kuroiwa (2016) goes into the approach of teaching about Problem Solving aligned to the proposal by Polya, in which the students assume themselves as problem solvers for which they must develop new strategies or adapt strategies which they already have in their repertoire.

About the problems and their characteristics in face of Algebra: the main proposal involving Problem Solving demonstrated in the field of algebraic education as being fundamentalist-analog only for requiring being solved by means of completing squares with the recurrence in areas as a visual resource of the algebraic structures (Miorim, Miguel & Fiorentini, 1993). However, the situations as a whole revealed a letterist practice to ease them according to Lins and Gimenez (1997). At a certain point Kuroiwa (2016) affirms that “for the student it is much more instigating to know how the formula was obtained, [...] which demonstrates and justifies its emergence and that are exposed, if there are, other means of solution for the proposed situation” (Kuroiwa, 2016, p. 19). These affirmations, indirectly refer to a fundamentalist-structural concept of Algebra in which, once such affirmations are appropriated by the students, the idea was resumed in capacitating the students to identify and applying certain structures in different contexts that happen to succeed (Miorim, Miguel & Fiorentini, 1993).

About the teaching approach of Problem Solving: Kuroiwa (2016) utilizes as basis for his proposal the theory by Polya, what according to Schroeder and Lester Jr (1989) is characterized as teaching about problem solving. Such affirmation is evident to acknowledge he aims to encourage the students to rely on strategies which make it possible the generalization and replication of other situations, as an example a correlate problem. However, according to Kuroiwa (2016, p. 87) these actions did not promote the interaction and effectiveness of the learning because “[...] there were moments in which the application of this methodology was difficult; by the negative attitudes of some students and the lack of collaboration”. Close to the results disclosed by Gonçalves and Proença (2020) about students’ difficulties in the conceptual and procedural aspects of equations of 2nd degree, Kuroiwa (2016) highlights the difficulties by recurring to the formula of Bhaskara remained, such as:

[...] mistakes in calculations, by not remembering the follow up of the procedure of solution, by the lack of attention as the signal of equal or of minus before the letter b, by the forgetfulness of the denominator of the equation [...] mistakes in the square root of a number [...] (Kuroiwa, 2016, p. 59) (Kuroiwa, 2016, p. 59).

In such affirmations it might be inferred it is connected to the chosen approach, having in sight there is need from the students to follow strictly the stages of solution, besides having previously appropriate different strategies of Problem Solving and their potentialities in face of the content, fact which appears not occurring in the proposal. In contrary, if the adopted approach were that of teaching Equations of 2nd degree via/through problem solving, as suggested by Onuchic and Allevato (2011) and Proença (2018), the students would take the autonomy in investigating, planning and articulating allied strategies to their previous knowledge without the need of following scripts already defined and strategies a little standardized, being able to have more interaction mainly in the recurrence to group work.

### **Final Considerations**

The objective of this article was to analyze proposals of teaching Equations of 2nd degree in postgraduate research focused on Problem Solving and the conception of Algebra. Firstly, the focus given to problem solving demonstrates that in both professional master's dissertations there is higher recurrence to implementing exercises both related to the systematization of techniques by mean of repeating operations and mathematical properties, and by exercises which require procedures such as the translation of the natural language to the mathematical language. Only the initial teaching proposal by Coutinho (2016) demonstrated itself as a problem, although it did not occur a discussion about the importance of the roots in the articulation between the built solution and the situation explored in the situation as recommended in the official documents (Brasil, 1998; 2018).

In this manner, the analysis of the approach of problem solving showed that although Coutinho (2016) had aimed at teaching Equations of 2nd degree via/through problem solving his posture was of teaching for problem solving. We identified this occurred because there was a rupture between the objective of teaching and practice in the classroom, having in sight it was not evident the participation, mobilization of knowledge and the valorization of the proposal of strategies by part of the students so they could be articulated to what was desired to be thought. On the other hand in Kuroiwa (2016), we identified it followed the teaching about problem solving. In his turn, based piously his practice in the determined approach, but it was not enough to reach his goals, having in sight there was lack of engagement by the students and the persistence of mistakes linked to procedural knowledge.

About the conceptions of Algebra, we showed that in both researches the main activities appealed to a fundamentalist-analogical view, because it was given certain importance to the imposition of the geometrical method of solution by Al-Khwarizmi, in which the action of

complementing squares make the geometrical representations an aid in the interpretation of situations involving these algebraic.

However, the focus of problem solving adopted the usage of problems in the function of exercises, because the approaches of teaching Equations of 2nd degree were about and for problem solving. In this context, the analysis revealed the conceptions of Algebra are to be developed. In face of this, we highlight that our study contributes in the sense of denouncing that it still falls to the usage of exercises as if they were problems. besides the theoretical reference of one of these researches suggesting the usage of a problem as a starting point.

Lastly, it is considered important not only to amplify studies and proposals of teaching with problem solving, because we obtained only two researches - two professional master's – involved, but still following the usage of problems in the classroom in a coherent manner to the learning of Equations of 2nd degree. Therefore, it is necessary realizing studies which pay attention to the organization of teaching Equations of 2nd degree, which widen the investigations already done by Pereira, Doneze and Proença (2023) and Pereira, Doneze and Proença (2022), about the organization of teaching-learning of these topics in the didactic books in face of the BNCC and about the elaboration of problems in themselves. For such need, it is suggested the organization of conceptual teaching-learning of Mathematics proposed by (2021), where the three approaches of teaching (about, for and via) might be employed during a didactic sequence.

### References

- Bicudo, M. A. V. (2014). Metanálise: seu significado para a pesquisa qualitativa. *Revista Eletrônica de Educação Matemática*, 9, 7-20.
- Brasil. (1998). Parâmetros curriculares nacionais: 3º e 4º ciclos. Ministério da Educação (MEC).
- Brasil. (2018). Base Nacional Comum Curricular: Educação Infantil e Ensino Fundamental. Ministério da Educação (MEC).
- Coutinho, R. P. (2016). Uma aplicação da Resolução de Problemas no ensino das equações do 2º grau [Dissertação de Mestrado Profissional em Matemática em Rede Nacional, Universidade do Estado do Rio de Janeiro, Instituto de Matemática e Estatística]. <http://www.btdt.uerj.br/handle/1/4823>.
- Echeverría, M. D. P. (1998). A solução de problemas em matemática. In: J. I. Pozo (org.). *A solução de problemas: aprender a resolver, resolver para aprender*. (pp. 44-65). ArtMed.
- Florentini, D.; Miorim, M. A. & Miguel, A. (1993). Contribuição para um Repensar... a Educação Algébrica Elementar. *Pro-Posições*, Campinas, 4(1), 78-91.
- Gil, A. C. (2008). *Métodos e Técnicas de Pesquisa Social*. Atlas.

- Gonçalves, B. M. & Proença, M. C. de. (2020). Análise dos conhecimentos conceitual e procedimental de alunos do primeiro ano do Ensino Médio sobre equação do 2º grau. *Revista Sergipana de Matemática e Educação Matemática*, 5(2), 209-228.
- Kieran, C. (2004). The core of algebra: Reflections on its main activities. In: K. Stacey ; H. Chick & M. Kendal. (Ed.). *The future of the teaching and learning of algebra: The 12th ICMI study*. (pp. 21-33). Springer.
- Kuroiwa, E. T. N. (2016). Uma abordagem peculiar da Equação do Segundo Grau no Ensino Fundamental e Médio [Dissertação de Mestrado Profissional em Matemática em Rede Nacional, Universidade Estadual Paulista Júlio de Mesquita Filho, Instituto de Biociências, Letras e Ciências Exatas]. <https://repositorio.unesp.br/handle/11449/145534>
- Lins, R. C. & Gimenez, J. (1997). *Perspectivas em Aritmética e Álgebra para o século XXI*. Papyrus Editora.
- Lovatto, P. A., Lehnen, C. R., Andretta, I., Carvalho, A. D., & Hauschild, L. (2007). Metanálise em pesquisas científicas: enfoque em metodologias. *Revista Brasileira de Zootecnia*, 36, 285-294.
- Onuchic, L. de la R. & Allevato, N. S. G. Pesquisa em Resolução de Problemas: caminhos, avanços e novas perspectivas. *BOLEMA: Boletim de Educação Matemática*, 25(41), 73-98.
- Pereira, F. F. & Proença, M. C. de. (2023). Ensino-Aprendizagem de Equações de 2º grau via Resolução de Problemas: uma experiência a partir de uma trajetória hipotética de aprendizagem. *Revista Paranaense de Educação Matemática*, 12, p. 427-446, 2023.
- Pereira, F. F.; Doneze, I. S. & Proença, M. C. de. (2023). Livros Didáticos do PNLD e a BNCC: Análise da Organização do Ensino de Equações de 2º Grau. *Perspectivas da Educação Matemática*, 16, p. 01-20, 2023.
- Pereira, F. F.; Doneze, I. S. & Proença, M. C. de. (2022). A elaboração de problemas de Equação de 2º grau nos livros didáticos de Matemática. In: Congresso Iberoamericano de Educação Matemática, 2022, São Paulo. IX CIBEM. São Paulo: Akademy, 2022, 1, p. 2898-2908.
- Proença, M. C. (2021). Resolução de Problemas: uma proposta de organização do ensino para a aprendizagem de conceitos matemáticos. *Revista de Educação Matemática*, 18, 1-14.
- Proença, M. C. (2018). Resolução de Problemas: encaminhamentos para o ensino e a aprendizagem de Matemática em sala de aula. *EdUEM*.
- Proença, M. C. (2018a). O ensino de matemática por meio da resolução de problemas: metanálise de propostas nos 6º e 7º anos do ensino fundamental. *Educação Matemática Pesquisa*, 20(1), 496-517.
- Proença, M. C. D., Campelo, C. D. S. A., & Santos, R. R. D. (2022). Problem Solving in BNCC: reflections for its insertion in the curriculum and in Mathematics teaching at Elementary School. *REnCiMa. Revista de Ensino de Ciências e Matemática*, 13(6), 1-19.
- Proença, M. C & Maia-Afonso, É. J. (2020). Resolução de Problemas: análise de propostas de ensino em dissertações de mestrado profissional. *Revista Paranaense de Educação Matemática*, 9(18), 180-201.

- Proença, M. C. & Maia, É. J. (2018). O ensino de matemática por meio da resolução de problemas: análises de propostas desenvolvidas no Ensino Médio. *Educação Matemática em Revista*, 23(57), 92-112.
- Sangiorgi, O. (1961). *Matemática para a quarta série ginásial*. Ed. Nacional.
- Sangiorgi, O. (1967). *Matemática Curso Moderno*. Ed. Nacional.
- Santos, J. A., de Oliveira, G. S., & Borges, T. D. D. F. F. (2021). Metanálise como método para desenvolver pesquisas científicas. *Cadernos da FUCAMP*, 20(48).
- Severino, A. J. (2007). *Metodologia do Trabalho Científico*. Cortez.
- Schoenfeld, A. H. (2020). Mathematical practices, in theory and practice. *ZDM: The International Journal on Mathematics Education*, 52(2), 01-13.
- Schoen, L. H. (1994). Ensinar a álgebra elementar focalizando problemas. In: A. F. Coxford & A. P. Shulte. *As idéias da álgebra*. (pp. 135-144). Atual.
- Schroeder, T. L. & Lester Jr, F. K. (1989). Developing understanding in mathematics via problem solving. In: P. R. Trafton & A. P. Shulte, (eds.) *New directions for elementary school mathematics*. (pp. 31-42). NCTM.