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Prospective mathematics teachers interacting on VMTwG in a task concerning translation

Licenciandos em matemática interagindo no VMTcG em uma tarefa sobre translação

Futuros profesores de matemáticas interactuando en el VMTcG en una tarea sobre traslación

Futurs professeurs de mathématiques interagissant sur VMTaG dans une tâche concernant la translation

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Abstract

This study is part of a research project aimed at learning in virtual environments. Through a case study, interactions by prospective mathematics teachers in a task on translation in a synchronous device integrated to GeoGebra are illustrated and analyzed. The theoretical framework of the investigation emphasizes the importance of the task and two possible ways of reasoning (ascending or descending) through collaborative interactions in DGE. The analysis was based on written interactions, on-screen constructions, tables, and graphs generated on the VMTwG platform. The focus is on the way in which future teachers built what was proposed based on doubts and emerging ideas. The importance of designing tasks that improve the understanding of transformation and functional relationship in working with isometries is stressed.

Keywords: Prospective mathematics teachers, Online interaction, Isometrics, Translation.

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Resumo

Este estudo integra um projeto de pesquisa voltado ao aprendizado em ambientes virtuais. Mediante estudo de caso ilustram-se e analisam-se interações de licenciandos em matemática em uma tarefa sobre translação em um dispositivo síncrono e integrado ao GeoGebra. O marco teórico da investigação ressalta a importância da tarefa e duas possíveis formas de raciocinar– ascendente ou descendente – mediante interações colaborativas em AGD. A análise foi baseada nas interações escritas, nas construções em tela, nas tabelas e nos gráficos –expostos na Figuras – gerados na plataforma do VMTcG. Destaca-se a forma com que os futuros professores construíram o que foi proposto a partir de dúvidas e de ideias emergentes. Ressalta-se a importância do *design* de tarefas que aprimorem o entendimento de transformação e de relação funcional no trabalho com isometrias.

Palavras-chave: Formação inicial de professores, Interações *online*, Isometrias, Translação.

Resumen

Este estudio forma parte de un proyecto de investigación dirigido al aprendizaje en entornos virtuales. A través de un estudio de caso, se ilustran y analizan las interacciones de futuros profesores de matemáticas en una tarea de traducción en un dispositivo síncrono integrado a GeoGebra. El marco teórico de la investigación enfatiza la importancia de la tarea y dos posibles formas de razonamiento (ascendente o descendente) a través de interacciones colaborativas en AGD. El análisis se basó en interacciones escritas, construcciones en pantalla, tablas y gráficos generados en la plataforma VMTcG. Se destaca la forma en que los futuros docentes construyeron lo propuesto a partir de dudas e ideas emergentes. Enfatiza la importancia de diseñar tareas que mejoren la comprensión de la transformación y la relación funcional al trabajar con isometrías.

Palabras clave: Formación inicial de profesores, Interacciones online, Isometrías, Traslación.

Résumé

Cette étude fait partie d'un projet de recherche visant l'apprentissage dans des environnements virtuels. A travers une étude de cas, les interactions de formation initiale des enseignants dans une tâche de traduction dans un dispositif synchrone intégré à GeoGebra sont illustrées et analysées. Le cadre théorique de l'investigation met l'accent sur l'importance de la tâche et sur deux modes de raisonnement possibles (ascendant ou descendant) à travers les interactions collaboratives en AGD. L'analyse s’est basée sur des interactions écrites, des constructions à l'écran, des tableaux et des graphiques générés sur la plateforme VMTaG. L’accent est mis sur la façon dont les futurs enseignants ont construit ce qui était proposé à partir de doutes et d'idées émergentes. L'accent est mis sur l'importance de concevoir des tâches qui améliorent la compréhension de la transformation et de la relation fonctionnelle dans le travail avec les isométries.

Mots-clés : Formation initiale des enseignants, Interactions en ligne, isométries, Translation

Prospective mathematics teachers interacting on VMTwG in a task concerning translation

This study is part of a wider research project[3] analyzing the way in which some subjects - mathematics teachers and undergraduates- interact and learn in virtual learning environments. We focus on the learning of concepts related to isometries in Virtual Math Team with GeoGebra (VMTwG). In particular, we are showing emerging aspects in synchronic interactions by a couple of undergraduates in a task about translation.

The research, carried out within the context of Group of Studies and Research in Information and Communication in Mathematics Education (GEPETICEM) is part of a wider project[4] financed by CNPq. The research was organized in four main moments: 1) questionnaire, 2) initial probing, 3) conceptual development[5] and 4) elaboration of tasks (Silvano & Bairral, 2021), implementation and analysis of activities at VMTwG. In this article we bring results belonging to moment 4, in particular, of the application.

One way to analyze the learning of mathematics of a subject is through his or her interaction with another subject, with mathematics and with technology. The innovating aspect of the study we are showing here is in the analysis of graphs generated in the VMTwG platform itself (see below, Figures 10 and 11) and in the approach of a topic that has been little explored in the curriculum of Elementary Education (Ng & Sinclair, 2015), Secondary Education (Assis, 2020; Barbosa, 2014) and college (Delmondi & Pazuch, 2018). Another factor that justifies the relevance of this study is that VMTwG is still little used in Brazil (Menezes & Bairral, 2021). In a (post) pandemic scenario which demands planning activities online, this study is also of value.

We begin the article by noticing some particularities and ways of reasoning in dynamic geometry environments (DGE) and, starting from a revision of the literature, we sum up some theoretical aspects of geometric transformations, in particular isometries. Along the study, we apply some exploratory activities adapted from Assis (2016), in platform of VMTwG which allow the creation of lines of reasoning and production of conjectures and justifications of proofs. In the analysis we are showing one of those tasks.

Some particularities and ways to reason in DGE

[3] Part of research project “Participar, descobrir e interagir em ambientes virtuais: Potencializando novas formas de aprendizagem matemática” (“Participating, discovering and interacting in virtual environments: Potentiating new ways of mathematics learning”) financed by CNPq and Faperj.
[4] Approved by the Ethics committee under Number 916/17
[5] Results of the 3 first moments can be seen in Silvano& Bairral (2020)

With the DGE it is possible to build geometrical figures from different information given -Euclidean characteristics, measures, etc. -using personal or mathematical strategies. These environments favor visualization of figures of different positions and the modification of their Euclidean characteristics instantly, as some researchers have pointed out (Arzarello et al., 2002; Salazar & Almouloud, 2015).

Some activities performed in DGE may influence the exploration, the construction of concepts, the validation or refutation of conjectures, among others. The DGE allow a better visualization, they improve the understanding of the construction process and enable users to create conjectures and put them into situation of verification and proof (Barreira & Bairral, 2017). These authors highlight that DGE should be used together with efficient and adequate teaching goals, to orient the learners to perform the proposed tasks correctly.

Nevertheless, we are not discouraging the use of paper and pencil in class. We also learn with them: we perform other types of register, different from the ones we do in DGE. So, in class, there can be some work that integrates DGE technologies and resources that generate static constructions: paper, pencil, ruler, etc. Starting from the analysis of two geometric figures, it is possible that, when comparing them in an environment without movement, we observe only the measures that were done for that drawing. When we use an DGE, we can consider different configurations, sizes, and positions. This possible diversity in the objects to consider involves the subject in his or her process towards discovery and learning.

Another aspect to consider about DGE is the way to propose a task and reason about it (Bairral & Marques, 2016). The type of task and the ways to justify are characteristics that put mathematics in movement, as construction, visualization and manipulation of geometric objects are possibilities fostered by the use of DGE (Powell & Pazuch, 2016).

There are two ways to reason and deduct mathematical properties in DGE (Arzarello et al., 2002): **ascending** thinking -from practice to theory -and **descending** thinking: from theory to practice. When we work with ready-made constructions and we use the atmospheres as motivation and introduction for the group of students to define formally a new concept from the exploration and analysis of what is done, we call it **ascending thinking**. When we start some new contents, showing the definition, example, and applications, studying in depth each variation and process towards the solution, and then, only then, we move to the construction, we call this descending thinking. In other words, in the ascending way, concepts emerge from the situation that is being analyzed, and in the descending way, the subject applies his or her knowledge.

When we go over the movements that have been done in the constructed figures, we can also observe two important aspects: unchanging and changing (Bairral &Barreira, 2017). During the construction of any triangle, for example, we should respect its properties, like the sum of internal angles being 180º and, going deeper, analyze the location of its notable points: barycenter, the point where its medians intersect; incenter, the point where bisectors meet; orthocenter, point of intersection of the altitudes; the circumcenter, the point where 3 perpendicular bisectors from the sides of the triangle intersect or meet. When we move one of its vertices, we can notice a change in its area, its perimeter, its angles, in the location of its notable points and the type of triangles – acute angle, right angle or obtuse angle triangles. We call **variants** the characteristics that change. Nevertheless, even if the location of the notable points changes, we can always locate them, and we can find out that no matter the size of the three angles, their sum will always be 180º -and we call those properties **invariant aspects**. The invariant elements are related to the construction, and the different movements done at the free points favor their identification and understanding.

Particularly, the study of geometric transformations gives the teaching of geometry a dynamic feature it didn't have in Euclid's static version (Veloso, 2012) and it allows, among other characteristics, the understanding of (in)variant aspects of the figure and its images due to specific transformations. In VMTwG, learning is not seen as a product, but as a set of aspects (examples, concepts, conceptions, suggestions, constructions on screen, et.) that develop within the collective group. Although the contributions are individual, the result is a group product, it is not just the product of one subject only. Different groups develop their own different ways to interact, and therefore, to learn (Oliveira & Bairral, 2020, p. 304).

Some previous studies

Among the different studies previously carried out at VMTwG, we highlight eight that are relevant to our study (Figure 1), shadowing in green the results more mathematics in nature, and in gray the ones related to the design of the tasks and processes related to reasoning and proof, relevant for our analysis.

Table 1.

Research that has used VMTwG

Author(s)	AimedPublic	Theme	General Objetive	Results

Salles & Bairral (2012)	Practicing teachers (continuin geducation)	Taxi cab geometry	Analyze interactions among teachers at VMTwG in a problem of Taxicab geometry.	The categories of interlocution listed by Powell (2006) for off-line dynamics can also be identified at VMTwG (chat). The teachers learned about metrics in a non-conventional geometry as they used combinatory reasoning strategies to analyze the minimum number of ways.
Barbosa (2014)	Secondary School students	Transformations on a plane focusing on matrixes.	Understanding how students develop ideas about transformations on a plane through tasks involving matrixes.	Identify the transformation metaphor as change. Transformations on a plane were understood from physical, bodily changes. The metaphor link-matrix-identity is a neutral element and emerged from stimulations by the researcher. The neutral element of the multiplication of real numbers provoked the understanding that the transformation may not change, when the transformation matrix is the identity matrix.
Marques & Bairral (2014)	Mathematics undergraduate courses	Varignon Theorem	Reflect on the process of development of reasoning, through conjectures and proof strategies.	Adopting a more dynamic and processual approach for the production of proof in DGE. This interactive dynamic can make prospective teachers more motivated for the use of software as minimizers of difficulties usually found for mathematics proving.
Bairral & Marques (2016)		Notable points of the triangle.	Analyzing *online* interactions among prospective mathematics teachers at VMTwG.	The nature of the task affects the reasoning ways of subjects. The elaboration of justification and proof is still a challenge in virtual scenarios like VMTwG.
Powell & Pazuch (2016)	Practicing teachers (continuing education)	Quadrilaterals and their bisectors	Analyze how justifications were built, based on learner's own conjectures and perceptions of relations among geometric objects, without measuring.	The justifications were based on the properties of mathematic objects, in the relations among objects and the relations among relations. The theoretical justifications and the conditions in which they were made generate implications for the training of mathematics teachers and their teaching practice.
Barreira & Bairral (2017)	Mathematics undergraduate courses	Quadrilaterals	Illustrate a pair of undergraduate students while they interact on VMTwG to construct a quadrilateral and analyze its nature.	The convincing ways on the nature of the quadrilateral built were basically based on Properties known by the learners about the polygon that was being analyzed. The results prompt the type of task and reasoning strategies used by the interlocutors.

Oliveira & Bairral (2020)		Proposition of task on GeoGebra	Analyze interactions between prospective teachers at a task proposed with GeoGebra.	Reflections *online* patterned on the formulation and solution of new problems can go hand in hand.
Brito (2022)	Mathematics undergraduates and Continuing Education	Similarity of triangles	Analyzing the learning of similarity through interactions at VMTwG.	The importance oft ask *design*, sliding control and semiotic mediation

Among the studies shown in Table 1, the new version of VMTwG was used only by Brito (2022). The current version works well in smartphones and allows for the generation of graphs, as we can see in our analysis.

Salles & Bairral (2012) analyzed the mathematics reasoning from interactions on the whiteboard and written chat. These authors observed the four properties of interlocution (evaluation, information, interpretation, and negotiation)[6] and stressed the role of negotiating interlocution for a better understanding of the ideas of the subjects.

Marques & Bairral (2014) talked about the creation of conjectures and processes of proof in DGE. They stress two phases: conjecturing, in which the students get involved in the exploration of figures, and proofs, which is a product of exploration, but in which subjects try to validate their hypotheses, usually in a collective way. Explorations generated questionings and statements (certainties), and from all these, conjectures were made to start a process of proof.

Bairral & Marques (2016) stress the use of DGE, specially highlighting as positive points the visualization, the construction, and the interaction online. The VMTwG platform was introduced, and the interactive activity took place in different classes: one where there was a triangle previously built with the three notable points; the other one where there was no construction. In the room where there was a built triangle, the participants moved the figure considerably more, besides using the tools of GeoGebra to validate their conjectures. In the class where the triangle had not been built, participants took longer with the construction, and

[6] In **evaluating** interlocution, students state without intervention from the teacher, whose role it is to admit an **interpretive** interlocution, positioning him/herself, trying to understand the student's thought. In an **informative** interlocution, the participant simply informs something to generate curiosity or doubts. In a **negotiating** interlocution, subjects seek a shared solution to the problem and interact in a sequence of questionings (Powell, 2003).

they moved the figure less. The researchers indicated different results in terms of mathematic discoveries in each class.

Powell & Pazuch (2016) analyzed the way in which prospective teachers in continuous training built justifications based on their own conjectures and perceptions of relations among geometric objects. The researchers stated that these theoretical justifications and the conditions in which they took place generated implications for the process of training of those prospective teachers and their practice.

Bairral & Barreira (2017) stressed specificities of a DGE, as the possibility to drag figures in order to explore them, the easiest and fastest construction, the visualization of the construction in different angles and shapes, the creation of conjectures and the possibility to prove them, and the reorganization of teaching in a classroom. The authors pointed out variant and invariant characteristics in the constructions and ascendent and descendent aspects when thinking mathematically. Finally, they stressed the production of data of VMTwG, in particular, the analysis of interactions by means of the *replayer*.

Contrary to the other studies, Oliveira & Bairral (2020) analyzed prospective teachers' interactions in the course of a proposal of a task with GeoGebra in VMTwG. They stressed that reflections over the formulation and the solution of new problems can go hand in hand and take place together. They also stressed that the VMTwG proves very fruitful in the proposal of tasks, as it generates a process by feedback through the reflections related to the solution of the situation being analyzed, and with possible emerging statements and contents.

Brito (2022) analyzed the learning process of prospective teachers of the similitude of triangles and stressed the importance of the design of the tasks, picking the concept of semiotic mediation (Bussi & Mariotti, 2008). The role of sliding control was important in some tasks, especially the ones involving the comparison and measurement of the sides.

Isometries in previous studies: transformation, function, punctual and global viewings.

Besides Barbosa's work (2014) mentioned in Figure 1, other studies, focusing on isometric transformations outside of the environment VMTwG (Assis, 2016, 2020; Ng & Sinclair, 2015; Silva & Almouloud, 2021; Veloso, 2012), deserve our attention and constitute a theoretical support in our research.

Concepts of transformation and function are important in mathematics. The concept of transformation, as it is explained by the secondary school students in Barbosa's research (2014), was associated to changes, the transformations on the plane being understood as physical, bodily changes. Curiously, this was an idea of MA, one of the prospective teachers who were participating in our study, mentioning that isometry is the feature of muscular contraction, exercising in a static form (Annex 1). The answer is inciting, and also ambiguous, as it associates the contraction (movement) to a static exercise.

The notion of function, in geometry, takes a decisive role in the simplification and the fixing of a terminology that, in the case of geometric transformations, has been rather blurry. For Veloso (2012), the idea of geometric transformation as a defined function in all the points in space, whether it is a plane -a tridimensional space -or some other type of set, is decisive for the understanding of many topics in geometry. We agree on this relevance and, from Silva & Almouloud (2021), the planned activities to work with VMTwG are aimed at two areas of significance of geometric transformations, namely: the transformation considered as a punctual application of the plane over itself (functional object); and the transformation considered as a functional tool, in order to bring out the invariants or solve the problem. A look at the group of transformations -fourth level presented in Silva & Almouloud (2021) -can be seen in Assis (2020), in which the author analyzes some secondary school students doing compositions in an instigating way -nonhierarchical- in dynamics favored by touches on screen in specific tasks.

Geometric figures can be transformed through manipulation of their properties, taking into consideration their particular elements (Silva & Almouloud, 2021). Nevertheless, another idea that needs to be explored is that of the image being interpreted as transformation: identifying the image of the figure by the transformation T of the plane, that is, determining an isometry that transforms a figure into another (Veloso, 2012).

The study of isometries involves the analysis of the preservation of distances and the notions of "situated between" midpoint, segment, semi-straight line, triangle, angle, amplitude, parallelism and perpendicularism (Veloso, 2012, p.21). In this analysis, the identification of fixed points constitutes an important help in the study of a topic or a problem involving transformations. It is in such a way that activities in GeoGebra can be great allies, as they favor the exploration and handling of fixed and free points.

The analysis of geometric transformations in DGE involves global looks -described in terms of the feature of a given figure – and punctual observations, viewed in terms of the mapping of one part of a figure to the other (Ng & Sinclair, 2015). Therefore, it is important that (prospective) mathematics teachers understand the orthogonal symmetry (or axial or reflection) not only as the symmetry on the object – in which the role of the axe of symmetry is explored in geometric figures, their characteristics in elements in nature and their influence in the world of art – but also the mathematical object orthogonal symmetry, where its definition and mathematical properties have to be studied (Silva & Almouloud, 2021).

As our focus is on the learning (online and synchronous) of isometries with GeoGebra, we took advantage of the tasks elaborated in Assis (2016, 2020) and used some of them. Although they are for GeoGebra on tablets and smartphones, their design was adapted for VMTwG, and the restriction of use for some icons was a noticeable factor in some of the tasks. Assis works with rotation, translation, and symmetry, which, according to the author, are little considered in the curriculum.

From the review shown here, we summed up, in Table 2, some aspects that provide some theoretical support for our study.

Table 2.

Founding theoretical aspects

Mathematic aspects in negotiating interactions	Aspects of semiotic mediation in task design	Particularities of VMTcG
-Transformation as function. -Relations among objects and the dynamic of relations among relations. -Visualization, construction, and manipulation ofgeometric objects as intrinsic processes. -Ascending or descending reasoning. - Global and punctual approaches. -Exploring phase and proof construction. -Variants and invariants.	-Different problem statements generate different mathematic discoveries. -Importance of sliding control, icon restriction, etc. -Exploration and handling of fixed and free points. -Figures in different sizes and positions. -Previous constructions in files, to share.	-Collaborative interactions. -Joint work and feedback. -Collective production. -Nature of joint work: writing, construction, observation, mouse control. -Use of *replayer* to analyze details.

Contextualization and data production

The research is situated in VMTwG[7], an online platform, with a written chat, and tools of GeoGebra, for synchronous activities in small groups. In the chat it is possible to see if the participant has the control, it allows the construction and the use of GeoGebra. It is also possible to post and follow up the text messages in the chat. Figure 1 illustrates these aspects.

7 At: https://vmt.mathematicalthinking.org/

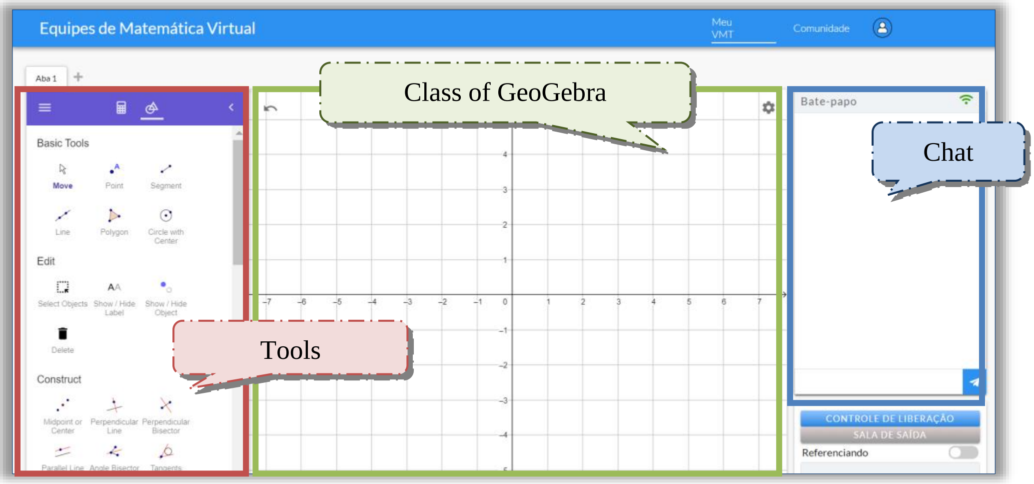


Figure 1.

Screen in a class of VMTwG (Capture of the Screen)

The planning and the field work took place synchronously, due to the Covid-19 pandemics. Ten prospective mathematics teachers (at the end of their university undergraduate studies) and five M.A students at a professional program at IFES, during the time from September to December 2020 participated in the study.

As usual, in our pedagogic intervention there is a moment – called familiarization – when the participants have a chance to situate and orient themselves, getting acquainted with the ambience, by making constructions and getting to know the scenario from a simple activity in terms of constructions and previous concepts. After this warming up moment, three activities were applied:

- **Activity[8] 1 –Quadrilateral**: a task was carried out, involving translation[9] of geometric figures (Figure 4)
- **Activity 2 – Square**: from activity 1, a task was suggested to work with rotation[10] and translation, and comparing both activities, 1 and 2 (Annex 2)
- **Activity 3 – Curious questions**: from a sheet of exercises done in class with paper and pencil. The participants were asked to choose one activity on VMTwG.

[8] In this paper we are using task and activity as synonymy.
[9]Given an MN vector, translation is said,
defined by MN, the geometric transformation T that corresponds, to each point P of the plane, the point P', which is the end of the PP' oriented (or vector) segment and has P as its source.
[10]Be a C point and a guided angle φ. It is said rotation R of center C and angle φ the geometric transformation th at corresponds, to each point P of the plane, the point P' = R(P).

Planning and structuring the analysis

We are focusing on the first activity, the objective of which was working with a geometric transformation, translation, by means of a vector. Each chat is planned for a maximum of two hours in time. We separated two possibilities: the first one, with any vector, and the second, with the radius of a circumference as vector.

ACTIVITY 1 (Adaptedfrom ASSIS, 2016)

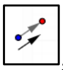
1. Build any quadrilateral.
2. Select a tool .
3. Touch the built quadrilateral and, right away, two different points in the working area.
   a) Observe what happened and comment.
   b) Move the original figure freely, selecting either the whole figure or one of its vertices. Register your observations.
4. Build a circumference, select a tool , then select the quadrilateral, the center of the circumference and a point on the circumference.
   c) Move the point on the circumference and observe what happens. Elaborate hypotheses (conjectures) and justify their validity.

Figure 2.

Activity 1 – Quadrilateral

The analysis was based on the tables and graphs – shown here in the Figures – generated on the VMTwG platform. We analyzed the chat and the constructions to observe the teamwork during the activities, how the participants constructed what was proposed to them and the hypothesis created during the activities on the movement of the figures, including the attempts to prove those conjectures.

The work at VMTwG takes place in small groups. Each class (group) is considered a unit of analysis and is considered unique, as its participants are unique. The data illustrated and discussed here are from the graphs generated and the constant reviewing – use of *Replayer* – of the interactions and constructions of the participants. The graphs and the *replayer* are resources of VMTwG.

The constructions are made together at GeoGebra, and the participants comment and debate on their remarks, answers, and doubts in the chat application. Therefore, learning at VMTwG is seen as the association (Çakir et al., 2009) of the ideas of the subjects in different spaces of the environment, mainly the written chat and the constructions and observations that are shared at the board with the GeoGebra. The interaction is the communicative process that potentiates this collaborative learning (Bairral & Marques, 2016).

The 15 participants were divided into 4 groups of a maximum of 4 participants. In all, the activity 1 was applied in 8 rooms. Besides the initial answers given by the prospective teachers (Annex 1), particularly, with their doubts cleared, the choice of room "AVA_6_quadrilateral" (Figure 3) was made from the contribution of the participants to the chat, to think the activity in a different way, as the researcher (second author to this text) realized the difficulty one prospective teacher had about the variation of the translated vector. The aim of the activity was to compare the radius of the circumference, being one vector (figure 5, vector GH) to the vector built before (Figure 3, vector EF), as the undergraduate students said they were the same. Then, a task was proposed with a fixed radius so they could compare a new vector (Figure 3, vector IJ). Carrying out the whole thing took approximately one hour.

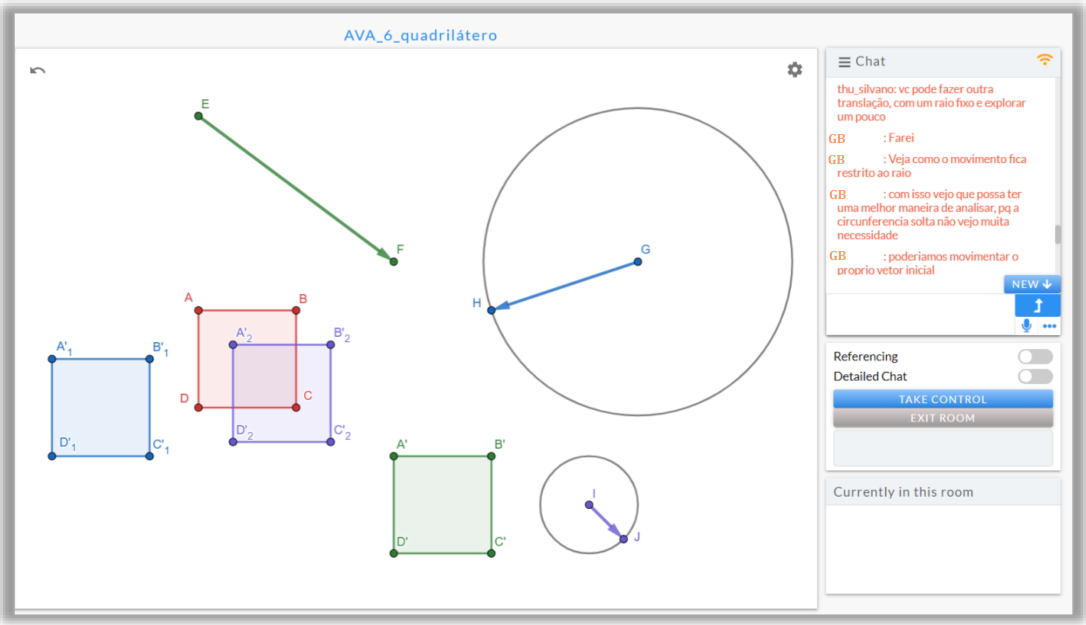


Figure 3.

Resolution of new activity proposed at the VMTwG (Capture of screen)

The data – written registers, constructions, and movements on GeoGebra, graphs and *replayer* – are generated directly by the VMTwG. The illustrating figures along the article are generated by the researchers in documents that can be edited, in order to facilitate the textual composition of the analysis. In the platform, it is possible to generate tables and graphs with all the information of the room, or part of it, from what needs to be analyzed (Figure 4)

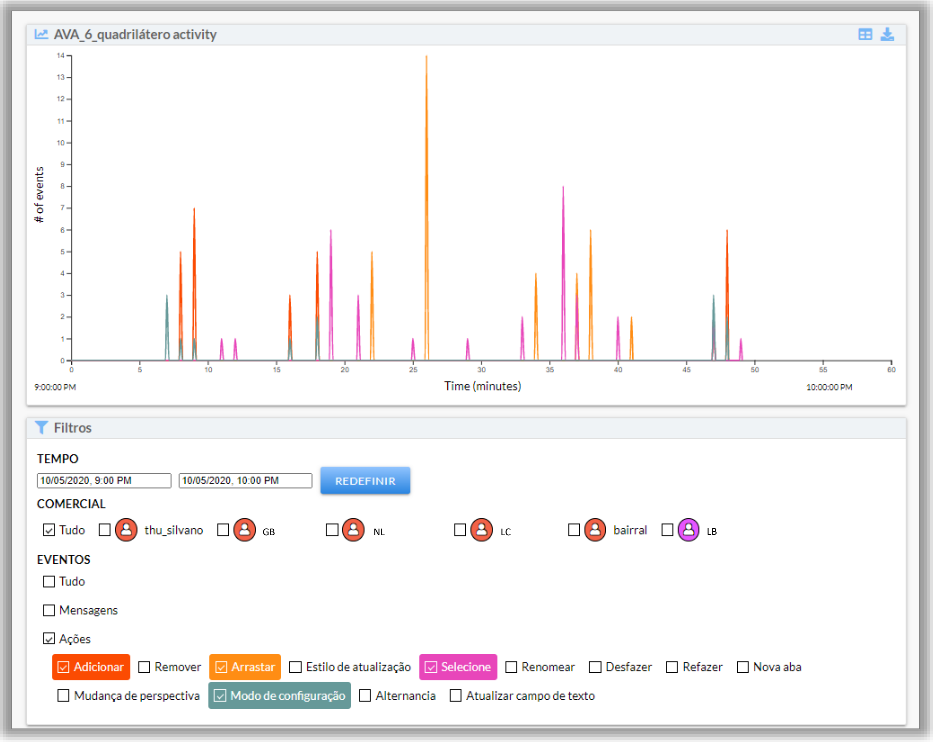


Figure 4.

Graph generated at VMT, with filters (Capture of screen)

From Figure 6 it is possible to observe which are the main and most used actions by the participants (thu_silvano, GB, NL, LC, bairral e LB) at the platform. We can see that the participants add or create the geometric figures and they drag or move their constructions, according to what is required by the activity, and they are always focusing on selecting and dragging, that is, exploring the construction. We can also filter in order to see who has moved more on the platform and check who has participated.

VMT platform has different spaces with specific functionalities. Besides the statistics space (Figure 6), there is a space called *Replayer*, which enables to review completely the interactive activity, step by step, and at the desired speed. This space appears like a video, where we can accelerate in the desired speed. This space appears as a video, in which we can accelerate or slower down, advance, play Pause or Play, and, finally, passing the cursor over the bar on the video, it is possible to see with a few details which action happened at that moment, as illustrated in Figure 5.

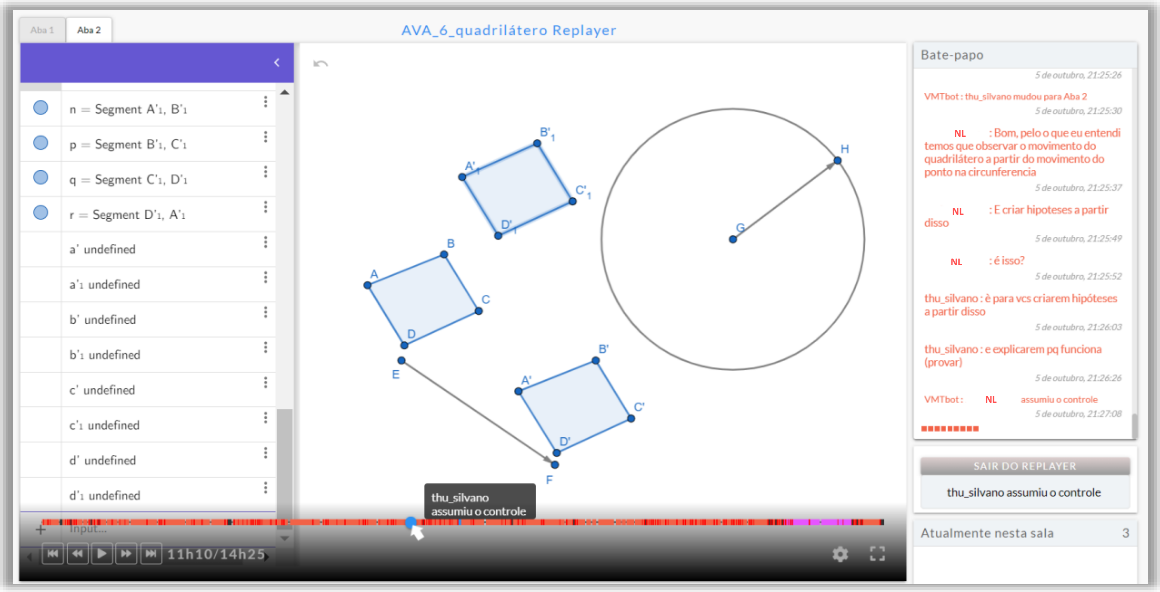


Figure 5.

Screen that registers the activity happening seen at the replayer *(Capture of the screen)*

With the *Replayer* we can find out some detail that may have been overlooked during the activity and even analyze, simultaneously, the constructions, as in this version of VMTwG it is not possible to see the constructions of the white board, live -it is necessary to update the page or “take over the control”.

Analysis of the case: GB and NL, pair work.

The research is configured as a case study, mainly exploratory in nature, because the participants did not know VMTwG and had little experience learning with isometries (Annex I). They had studied Linear Transformations in Algebra I and II, and Theory of Rings in Algebra III.

The analyzed room proved special. It was composed of a pair of undergraduates of both genders, who offered interesting challenging answers: GB (*I don’t know what it’s about, but the name is familiar*) and NL (*I must admit this topic was the hardest, most complicated to answer for me, I had to search a little in order to give an answer. The results took me to one side of geometric isometry, that is, they are similar images in a sequence with the same distance between the points*)[11]. The aim is to analyze in more detail the understanding and conceptual development of these undergraduates in the proposed activity, to guide new activities and their elaboration (Berg, 20006).

[11] See Analysis in Silvano & Bairral (2020)

Our analytical process does not establish comparisons of learning performance between the classes. In new tasks and implementations, in our research group, we let ourselves be guided by the analysis of different classes, in such a way as to potentiate everybody's learning, including the researchers. In the rooms, the attitude of the researchers is one of accompanying the discussions and intervening only when asked, or when they perceive there is some conceptual or construction error. We always let interactions flow as naturally as possible, as in that process, learning keeps happening and re-dimensioning itself as it goes along, following its own needs.

The help of generated graphs.

Using the statistics tool provided by the VMTwG platform, we generated graphs to filter the participants individually, separating in general actions (Figures 6 and 7) and messages written in the chat (Figures 8 and 9).

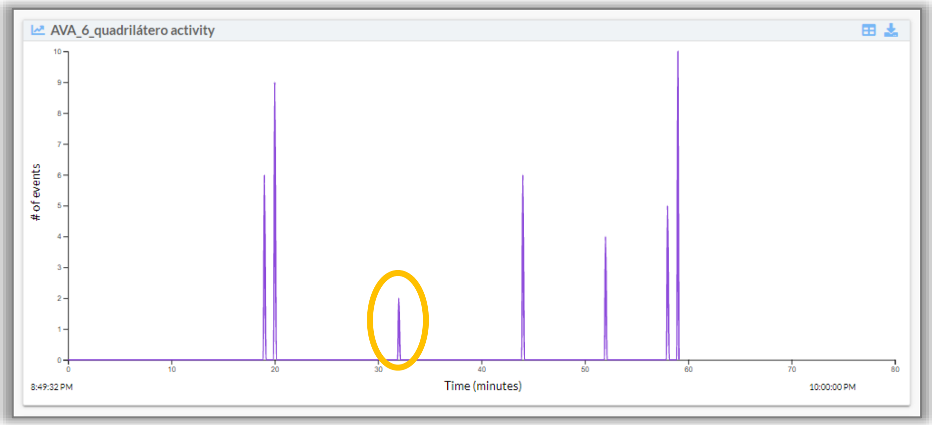


Figure 6.

Graph showing actions of participant GB at VMTwG (screen capture)

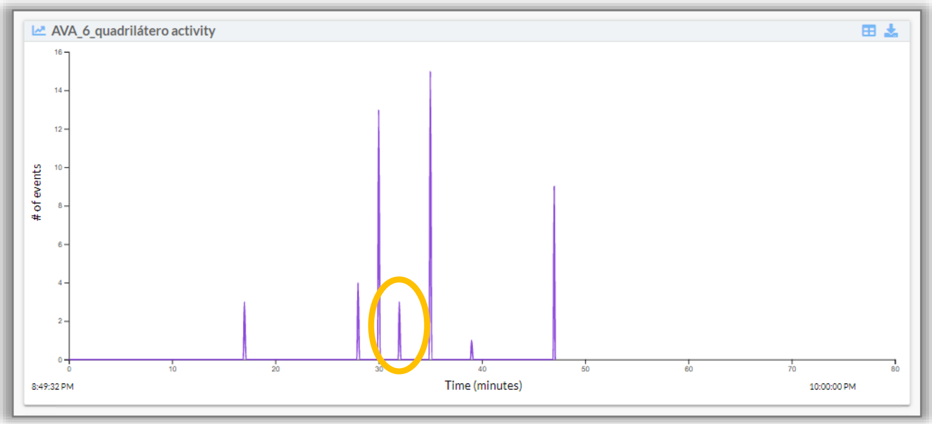


Figure 7.

Graph with participant NL's actions at VMTwG (screen capture)

The graph in Figure 6 represents the actions of student GB, while Figure 7 shows the actions of NL20. We can see that GB moved on the whiteboard longer, as opposed to NL, who moved there for a shorter while and was followed by fewer interactions. Nevertheless, there were few movements and creations during the activity, which led us to believe that the participants performed the construction that had been asked of them but did not explore different visualizing or positioning perspectives of the geometric forms.

Figures 6 and 7 provide no information about learning, but they allow the researcher to observe the interactive development, i.e., variations, access, temporal intervals, and amount of time spent. For instance, by observing the intersection of the interval (approximately at 33 minutes) highlighted in yellow in Figures 6 and 7, it is possible to capture that both undergraduates performed actions at similar times.

From that intersection, we were able to verify that both participants moved the same point in the construction, point H (Figure 5) for the observation of the behavior of the vector. When analyzing the intersection on the graph, we could see that, following the statement of the enquiry, proposing the exploration of the different cases of transformation, the students moved the vector in order to study the behavior of the geometric figures and to create conjectures from those movements. Nevertheless, they did not produce simultaneous constructions or movements, as the VMTwG only allows one participant at a time to control the mouse to build or move the constructions.

Besides the constructions, the activity proposed the undergraduates to write their remarks on the chat. The chat statistics (for all the time of the activity) is illustrated in Figures 8 and 9.

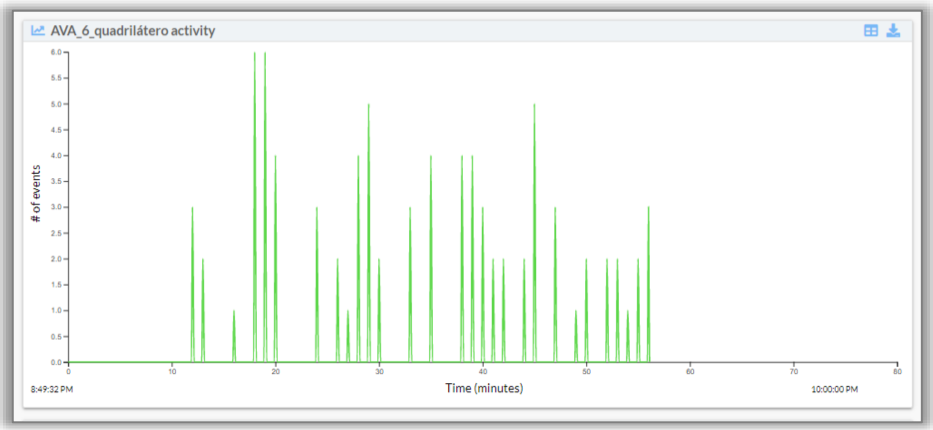


Figure 8.

Graph showing GB's written messages on the chat at VMTwG (screen capture)

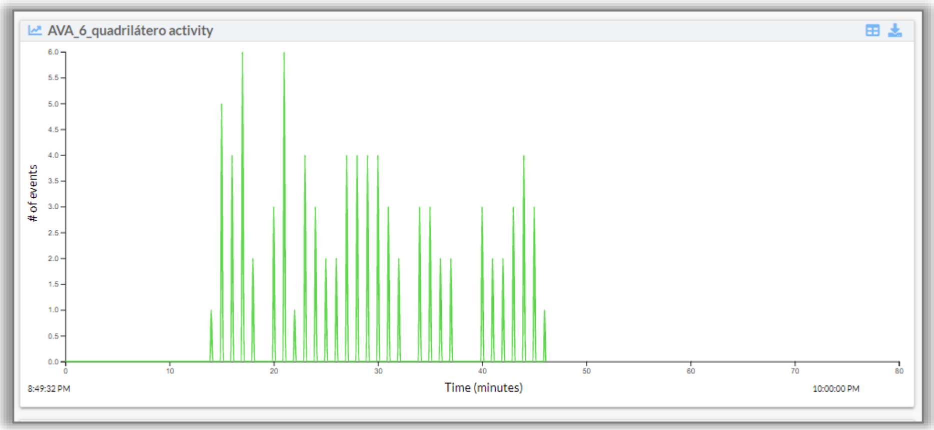


Figure 9.

Graph showing NL's messages at the VMTwG chat (Capture of the screen)

Figure 8 shows the messages written by GB, and Figure 9 shows ML's written messages. In both figures, we can see more variations and movements. That is, the activity took place on the chat most of the time. There, the students made remarks on the constructions (Table 3, line 15), they expressed and cleared out their doubts, and, as was proposed, they made conjectures and tried to justify them. As can be seen in Figures 8 and 9, GB spent more time, which can be explained by the fact that NL had to leave the activity earlier.

In comparing Figures 8 and 9, it is not easy to identify intersections, as both students participated in the written chat at all times, as opposed to what happened at the construction chat (white board), which can only be moved by one student at a time.

After a graphic analysis of the written chat, we went on to a detailed observation of each message, paying attention to the way in which the participants answered the questions and how they planned to perform the constructions – sometimes together and other times individually.

Table 3.

Fragment of VMT chat

10	GB[12]	You guys done with the first part?
11	Mediator	You done, GB?
12	NL	There, now it's gone

[12] The subjects will be identified by the initials of their first and last name.

13	Mediator	I can't see it here yet
14	Mediator	Now I got it
15	GB	I can see there was a translation of the quadrilateral in relation to the created vector.
16	GB	How about you guys?
17	NL	Yep, I saw that too
18	GB	You got something else to add?
19	NL	I don't
20	NL	So now we've got to construct a circumference?
21	Mediator	Yes

Interacting and learning about translation

After a while spent on greetings and doubts while waiting for the participants in order to begin the activity, we could see that GB made an individual construction, not with his teammate, as had been proposed. Nevertheless, after the first construction, both participants started to interact on the chat, commenting on the constructions already made and the ones they still had to make, and sharing their remarks about what had taken place in the first phase of the activity, when GB said: "*I can see there was a translation of the quadrilateral in relation to the created vector*" in line 15, and, agreeing with him, NL states: "*Yep, I saw that too*".

Table 4.

Fragment of VMTwG chat

23	GB	Can I take over the control?
24	NL	Sure you can, did you get what I did?
25	GB	Yes, NL, it appeared on my screen
26	Mediator	I got it too
27	NL	As we move the point in the circumference, the quadrilateral keeps rotating, see?
28	GB	I didn't quite understand the proposal of question (d)
29	GB	Did anybody understand that?
30	NL	It disappeared
31	NL	Well, from what I gather, we are supposed to observe the movement of the quadrilateral from the movement of the point in the circumference
32	NL	And create some hypotheses from there
33	NL	That it?
34	Mediator	You guys create hypotheses from there,

35	Mediator	And explain why it works (you prove it)
36	GB	If we put a translation in "function" of the vector and then right away we put this vector fixed in a circumference, the quadrilateral continues in function of the vector, right?
37	NL	I think so
38	GB	Because the movement I do with the quadrilaterals through the circumference, it's also possible to do that without the circumference, right?
39	GB	You gonna try
43	NL	I am trying to make the movement of the quadrilateral and without the point of the circumference, and I didn't get to
44	NL	Then I believe it is related, yes, with the circumference
45	Mediator	How come it is possible to move without the circumference?
47	GB	Can I take over the control?
48	NL	GB had suggested that it would be possible to move the quadrilateral without the point in the circumference and I tried to, but I couldn't
49	NL	Yes, you can
55	NL	GB, we've Only got to analyze the quadrilateral that was created from the relation of the first quadrilateral, the center of the circumference and the point in the circumference.
56	GB	I don't know if you saw that, but in my point of view, the circumference doesn't make that much difference, unless we put a fixed radius.
57	GB	Then you can get to make a comparison of the differences
58	Mediator	I couldn't get to see that, but to me, the circumference makes a difference
59	Mediator	I didn't get to understand your idea
60	NL	Me neither, I couldn't get that very well
61	GB	NL, just as the first one, there is a translation
62	NL	Yes, and now this translation is associated to a circumference

As can be seen on Table 4, there were some initial doubts about the request to create a hypothesis on the construction and putting it to the test, as when GB states (line 28): "I didn't quite understand the proposal of question (d)" "Did anybody understand that" (line 29) and, right away, there is NL's explanation: "Well, from what I gather, we are supposed to observe the movement of the quadrilateral from the movement of the point in the circumference […] And create some hypotheses from there" (lines 31, 32).

From the hypothesis suggested by GB: "If we put one translation in "function" of the vector and then we put that vector fixed in a circumference, the quadrilateral continues in function of the vector, right" (line 36), a discussion started related to the vector that made the translation, which belonged to the circumference.

"Because the movement I do with the quadrilaterals through the circumference, it's also possible to do it without the circumference, right? (line 38). "I don't know if you saw that, but in my point of view, the circumference doesn't make that much difference, unless we put a fixed radius" (line 56). Participant GB understood that, because the radius of the circumference was not fixed and able to be augmented and diminished along with the point of the circumference being moved, that vector did not differentiate itself from the other one, because the other one also allowed to be moved freely on the plane. NL and the mediator could not see that possibility. As NL had to leave early for personal reasons, the mediator continued with GB.

The sequence of this discussion is reproduced in Table 5.

Table 5.

Fragment of the VMT chat

87	Mediator	I understood
88	Mediator	Really, the movements are quite similar
89	Mediator	if the radius is not fixed
90	Mediator	You can get to make another translation, with a fixed radius, and explore a little bit
91	GB	I'll do it
92	GB	See how the movement keeps restricted to the radius
93	GB	With this I can see that I'm going to get a better way to analyze, 'cause the loose circumference, I don't see the need
94	GB	We could move the initial vector itself
95	Mediator	With the fixed radius, is there a way we could position the translated quadrilateral with the original?
96	Mediator	In the same position
97	GB	No
98	Mediator	Yes
99	Mediator	It's an interesting difference between the circumferences
100	GB	Yes yes
101	Mediator	It's great
102	Mediator	Would you like to make any other remark?
103	GB	But it's a difference we didn't find in the first construction and then after we created the circumference
104	GB	I am quite satisfied
105	GB	It was really interesting

After the explanations and remarks regarding vectors, the mediator suggested participant GB to build a circumference, with a fixed radius that would be a vector, and that he did the translation of the initial quadrilateral by means of that vector. With this proposal, new questions arose in order to improve the exploration of the construction, as on line 95, when participant GB is questioned on whether it is possible to position the translated quadrilateral on top of the initial quadrilateral, as there is this possibility in both previous cases.

From probing to graphs, to writing, to constructions on screen and to emerging ideas in interactions.

Table 6 sums up GB and NL's pair work in the activity that focused on translation, particularly on the attention to vectors. Although our aim here is to illustrate synchronous interactions, we consider it worthwhile to save his and her ideas at the moment of probing. We could observe it was a productive moment for both. Their learning was displaced from something unknown (or complicated) to powerful ideas in the study of geometric transformations (Silva & Almouloud, 2021; Veloso, 2012), as the ones on functional relation (GB -L 36) and the one about the image generated by a transformation (NL-62). Besides, the terminology involved, although it was known to them, emerged from the statement of the task through icons and forms of questioning. The typology of the task is not neutral, and its constitutive elements take on a role in the participants' learning (Bussi & Mariotti, 2008; Powell & Pazuch, 2016).

The goal of the activity was that the participants would differentiate the two vectors. We wanted them to observe that the vector being a fixed radius in the circumference, it only moved in a circular way (invariant element), as opposed to the vector created from any two points, which can move in any direction and change in size - ideas that only arose from the interactions between GB and NL. When we performed the transformations and moved the vector, we could see that both the original and the figure generated by the transformation were not altered. When the vector is not fixed to the circumference, figures can be generated from other transformations. Problematizing this and figuring which other possibilities could take place would be another interesting idea, for a conversation at some other chat or informal conversation.

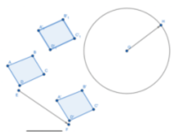
The interactions in this activity were based on ascending reasoning (Arzarello et al., 2002), that is, the participants performed constructions, handled, and questioned, so they could explore the emerging concepts later on, without applying what they knew directly (Table 6).

This way of reasoning may have been affected by the nature of the task, which was new to them (GB, L28). This interpretation should be further explored.

After remodeling the activity and some new explorations, GB declared that he was satisfied with the activity, and that using VMTwG had been interesting. Besides, when asked if he would use this task in class, he said it would be a good proposal, provided some changes were made, like what was done after his doubt. We can see the importance of learning being problematized together with the developments for professional performance of the people involved (Powell & Pazuch, 2016).

Table 6

Summing up of GB and NL's pair work

Moment 1: Probing	**Moment 2:** Implementation and analysis of activities at VMTwG				
Main ideas (written on paper)	Aim	Doubt	Screen capture (Fig. 5)	GB	NL
GB: **I can't tell** what it's about, but I know the name is familiar NL: **I have to admit this topic was the hardest, the most complicated to answer** for me. I had to search a little in order to give an answer. The results took me to one side of	Summary of statement (annex 1) 1. Touch the built quadrilateral and, right away, two different points in the working area. a) Observe what happened and comment. b) Move the original figure freely, selecting either the whole figure or one of its vertices. Register your observations. c) What did you observe when you moved one of the ends of the arrow (vector)	GB - L28: I didn't quite understand the proposal of question. (4d). Did anybody understand that?		L15: I can see there was **a translation of the quadrilateral in relation to the created vector**. L36: If we put a translation **in "function" of the vector** and then right away we put this vector fixed in a circumference, the quadrilateral	L17: Yep, I saw that too. L31-32: Well, from what I gather, we are supposed to observe **the movement of the quadrilateral from the movement of the point** in the circumference. [...]And create some hypotheses from there. L43: I am trying to make the movement of the quadrilateral and without the point

geometric isometry, that is, they are similar images in a sequence with the same distance between the points.	Build a circumference, select a tool, then select the quadrilateral, the center of the circumference and a point on the circumference. Move the point on the circumference and observe what happens. Elaborate hypotheses (conjectures) and justify their validity.			continues in function of the vector, right?	of the circumference, and I didn't get to L62: Yes, and now this translation **is associated** to a circumference.

Final Considerations

Using DGE is one more way of teaching and learning mathematics, whether online or offline. When analyzing interactions on VMTwG, this study is methodologically, conceptually, and didactically innovating.

Methodologically, this study innovates when analyzing from graphs generated in its own platform. Although the graphs do not provide information about the language of the participants involved, they allow the researcher to observe the interactive development as a whole, that is, the variations, access, use of the whiteboard or the chat, amount of time, etc.. From the generation of graphs, it is possible to select the elements to be analyzed, as we could see in the case that we selected.

Conceptually, this study acts on the approach to the somewhat limited contents in the teachers of mathematics' continuing education. When potentializing synchronous interactions in DGE, it stimulates creativity, construction, and analysis of geometrical figures, turning the themes more exciting to a generation of people surrounded by technology (Delmondi & Pazuch, 2018).

When we think of generating hypothesis and put them to test (justification), we imagine an algebraic environment, which is what usually pervades mathematics writings. Nevertheless, we saw, along the analysis, that this process was quite flowing and dynamic. The suggested hypothesis generated various explorations on the board of the GeoGebra, to validate or invalidate any such conjecture. It generated also quite a lot of dialogue in the written chat among the participants who turned to it to clear out their doubts or explain their ideas.

Elaborating and putting conjectures to test has not been a usual practice in our University, in spite of the efforts of some teachers. In the case presented here, undergraduates focused on the moment of exploration from the emerging conjecture in dialogue NL -L31 and GB -L36. Would some longer time be needed to perfect the justification of NL L32 or GB-93? Were the mediator's statements, like in lines 85, 95, 99, cues for them and dispensed them from the need to prove? Although a chat may not be enough to detail a procedure of proof (Marques & Bairral, 2014), the whole can be potent enough to minimize the difficulty of prospective teachers to generate their own proofs. We need to study further, also with longitudinal analyses among the classrooms.

Didactically, our contribution lies in the design of the tasks for online scenarios. In the analyzed activity, it is interesting to see how small details can change an activity completely, how we learn from our students, besides teaching them. There is a relationship based on an exchange of information in the online class. Each activity carried out at VMTwG brought out, for both the participants and ourselves, researchers, new perspectives for teaching and learning isometric transformations in DGE.

Through reflection we can produce all the isometries on the plane (Silva & Almouloud, 2021), and the four basic isometries -translation, rotation, reflection, and axial symmetry and sliding symmetry- can be obtained as compositions of reflection (Veloso 2012). Here we focus on translation, but there is room for activities that explore this composition in a way that more details can be verified in the learning through tasks with DGE, including the improvement of understanding of concepts of transformation and functional relation.

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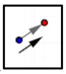
Annex I

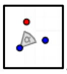
CONCEPT OF ISOMETRY	
ANSWERS RELATED TO MATHEMATICS:	
CM13	They are geometric transformations, i.e. they are geometric figures or solids that maintain the distances among points. They can change in sense, Direction, rotate, translate without changing their initial features.
FM	Talking Geometry, they would be repetition of figures, or pieces of them.
JS	Geometric transformation that conserves mathematical properties of the figures.
LA	In mathematics, isometry is a geometric transformation where the original figure and the transformed one are congruent.
LS	Equal measures, geometry.
PB	Process that enables the shifting of a figure or the enlargement or reductionof figures (to obtain similar shapes).
TS	In geometry, they are transformations that do not alter the angles and the distance among the points of the figure, but the position of the figure can change. For instance, through translation.
TA	They are transformations of images.
ANSWERS THAT EXPRESS DOUBTS:	
GB	I can't tell what it's about, but I know the name is familiar.
LC	I Guess it's a kind of geometric transformation.
MS	I remember having studied about isometries in the subject of Group Theory, and I remember there were some concepts, like reflection, symmetry and translation related to that term. I believe it's directly related to matters of position of an object, parallel to its geometry on the plane.
NL	I have to admit this topic was the hardest, the most complicated to answer for me. I had to search a little in order to give an answer. The results took me to one side of geometric isometry, that is, they are similar images in a sequence with the same distance between the points.
ANWERS WITH IDEAS OUTSIDE OF MATHEMATICS:	
MA	Isometry is the feature of muscular contraction, if you exercise it still.

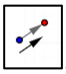
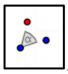
13Os sujeitos serão identificados pelas iniciais de seu nome e sobrenome.

Annex II

ACTIVITY 2 (Adapted from ASSIS, 2016)

1. Build a square using the tool “regular polygon”.
2. Select the tool . With the selected icon, touch on the square andon one of its verteces. Chose an angle and sense (clockwise or anti-clockwise) Comment onwhathappens.
3. Create a vector and select the tool. With the selected icon, touch on the original square and vector. Comment on what happens.
4. Move the original figure freely. Comment on what you observe which you deem interesting.
5. If we take 2 equal squares:

❖ On the 1st squarewe use the tool , and with the result, we use the tool,

❖ On the 2nd we use the tool and with the result, we use the tool . Will the end constructions be equal? Justify.

   a. If you were to apply this activity with a group of secondary students, would you do it differently? Comment.

   b. Which concepts did you use to solve this activity? Do you have any doubts? Comment.