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Professional teacher knowledge when teaching geometric transformations: an analysis of class situations

Conhecimento profissional do professor ao ensinar transformações geométricas: uma análise de situações de aula

Conocimiento profesional del profesor al enseñar transformaciones geométricas: un análisis de situaciones de clase

Connaissance professionnelle de l’enseignant lors de l’enseignement des transformations géométriques : une analyse des situations de classe

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Abstract

This article aims to *identify and understand how the teacher’s knowledge impacts their practice regarding the content of geometric transformations*. To this end, we analyzed qualitatively, in June 2019, a class episode in which such content was taught to an 8th-grade class at an elementary municipal school in the city of São Paulo. To this end, we used the Knowledge Quartet (KQ) tool as the main theoretical and methodological reference. We raised and categorized 21 critical events, and discussed five of them. The analysis showed a convergence between theoretical research and the practice of the teacher who teaches the content of geometric transformations and indicated gaps in teaching knowledge and the use of closed exercises with reduced challenges for teaching the content. The KQ proved to be an appropriate tool for such analysis, to highlight such aspects so that both the initial and continuing teacher education can be structured according to the demands observed in practice.

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Keywords: Geometry teaching, Reflective practice, Professional practice, Mathematics teaching.

Resumo

Este artigo tem como objetivo *identificar e compreender como os conhecimentos do professor impactam na sua prática em relação ao conteúdo de transformações geométricas*. Para tanto, foi analisado qualitativamente, em junho de 2019, um episódio de aula em que tal conteúdo foi ministrado para uma turma do 8.º ano do Ensino Fundamental de uma Escola Municipal da cidade de São Paulo. Para tal, foi usada como principal referencial teórico e metodológico a ferramenta *Knowledge Quartet* (*KQ*). A partir dela foram levantados e categorizados 21 eventos críticos, sendo 5 deles discutidos nesse artigo. A análise deles evidenciou uma convergência entre as pesquisas teóricas e a prática do professor que ensina o conteúdo de transformações geométricas, e indicou lacunas no conhecimento docente e a utilização de exercícios fechados com desafios reduzidos para o ensino desse conteúdo. O *KQ* se mostrou uma ferramenta adequada para tal análise, com o intuito de destacar tais aspectos, de modo que tanto a formação inicial quanto a continuada de professores possam ser estruturadas de acordo com as demandas observadas na prática.

Palavras-chave: Ensino de geometria, Prática reflexiva, Prática profissional, Ensino de matemática.

Resumen

El objetivo de este artículo es *identificar y comprender cómo el conocimiento del profesor impacta su práctica en cuanto al contenido de transformaciones geométricas*. Para ello, se analizó cualitativamente, en junio de 2019, un episodio de clase en el cual este contenido fue enseñado a alumnos de 8.º año de primaria de una Escuela Municipal en la ciudad de São Paulo. Para ello, se utilizó la herramienta *Knowledge Quartet* (*KQ*) como principal marco teórico y metodológico. A partir de esta herramienta se identificaron y categorizaron 21 eventos críticos, de los cuales 5 se discuten en este trabajo. El análisis mostró una convergencia entre los estudios teóricos y la práctica del profesor que enseña transformaciones geométricas, y reveló brechas en el conocimiento docente y en la implementación de ejercicios cerrados con desafíos reducidos al enseñar dicho contenido. El *KQ* se mostró como una herramienta adecuada para tal análisis en tanto permitió destacar tales aspectos, lo que hace que la formación inicial y continua de profesores pueda estructurarse de acuerdo con las demandas observadas en la práctica.

Palabras clave: Enseñanza de la geometría, Práctica reflexiva, Práctica profesional, Enseñanza de las matemáticas.

Résumé

Cet article vise à *identifier et à comprendre comment les connaissances de l'enseignant impactent sa pratique par rapport au contenu des transformations géométriques*. Pour cela, un épisode de classe a été analysé qualitativement, en juin 2019, dans lequel ce contenu a été donné à une classe de la 8ème année de l'école primaire d'une école municipale de la ville de São Paulo. Pour cela, l'outil *Knowledge Quartet* (*KQ*) a été utilisé comme cadre théorique et méthodologique principal. À partir de là, 21 événements critiques ont été soulevés et catégorisés, dont 5 ont été discutés dans cet article. Leur analyse a montré une convergence entre la recherche théorique et la pratique de l'enseignant qui enseigne le contenu des transformations géométriques, et a indiqué des lacunes dans les connaissances pédagogiques et l'utilisation d'exercices fermés avec des défis réduits pour l'enseignement de ce contenu. Le KQ s'est avéré être un outil adéquat pour une telle analyse, afin de mettre en évidence ces aspects, de sorte que la formation initiale et continue des enseignants puisse être structurée en fonction des exigences observées dans la pratique.

Mots-clés **:** Enseignement de la géométrie, Pratique réflexive, Pratique professionnelle, Enseignement des mathématiques.

Teacher's Professional Knowledge When Teaching Geometric Transformations: An Analysis of Classroom Situations

This article refers to a stage of research in which aspects related to teaching the mathematical content of geometric transformations were analyzed. This specific topic of geometry deals with changes in the positioning (isometric transformations) or dimensions (homotheties) of a given figure, in relation to an initial figure (Wagner, 2007).

We chose this specific geometry content because it can help students develop skills to transform concrete reasoning into abstract reasoning to contribute to the learning not only of other mathematical topics but also of other subjects. In addition, through geometric transformations, we can work with content such as numbers and measurements, similarities and differences, and regularities between structures at different levels of depth, which means that this content can be explored with different age groups (Lage, 2008; Medeiros & Gravina, 2015). Still, like other topics in geometry, teachers do not work on this content often in class for reasons such as the lack of time, lack of knowledge on the subject, lack of adequate teaching resources, or simply because they believe that it is irrelevant (Lage, 2008; Maia, 2014; Marschall & Fioreze, 2015; Medeiros & Gravina, 2015).

The first of the aspects analyzed that refers to the teaching of geometric transformations involved research already carried out on the subject. Thus, in 2017, we carried out a systematic literature review in which 30 articles were analyzed and allocated into ten analytical themes, which were grouped into four analytical units: formative processes, teaching practice and/or teacher knowledge; geometric thinking and relationship with students; geometric and technological knowledge; didactic resources and methodological strategies. Such articles showed that teachers' knowledge about geometric transformations is superficial and results, consequently, in a lagged teaching of this concept. We also detected a lack of articles that deal with tasks aimed at teaching geometric transformations and other didactic materials focused on this topic (Delmondi & Pazuch, 2018).

Then, in 2018, we analyzed official documents from the state of São Paulo, resulting from the implementation of the *Programa São Paulo Faz Escola* [São Paulo Makes School] in 2008. The main materials resulting from this program are the *Currículo Oficial* and *Cadernos do Aluno* and *Cadernos do Professor do Estado de São Paulo* [Official Curriculum and Students' Notebooks and Teachers' Notebooks of the State of São Paulo]. The analysis of those materials showed that, although the *Currículo Oficial do Estado de São Paulo* foresees the mathematical content of geometric transformations with proportional emphasis to the other contents, the tasks proposed by the *Cadernos do Aluno* and *Cadernos do Professor do Estado de São Paulo* that deal with this content are superficial and not very challenging nor close to the students' context (Delmondi & Pazuch, 2019a, 2019b).

Based on such findings, we thought pertinent to deepen research on the nature of the tasks designed to address geometric transformations in the classroom. Thus, at the beginning of 2019, we studied nine articles dealing with this theme that included drills mostly, i.e., closed tasks and with reduced challenges. Although this kind of exercises is important, there was a lack of challenges. We also observed that some articles had similar assignments, demonstrating a gap in the proposition of new, more contextualized and attractive tasks for students addressing the content of geometric transformations (Delmondi & Pazuch, 2020).

With an overview of the literature, the analysis of documents and curriculum resources, and tasks that address the mathematical content of geometric transformations, a master's research aimed to look into the teacher's knowledge mobilized when approaching the content of geometric transformations in the classroom, and on the teacher's knowledge mobilized in the interaction with mathematical tasks on geometric transformations. Thus, the subsequent stage of the research above consisted of analyzing the practice of the teacher who teaches this content to verify how all the theoretical aspects described above occur in practice, and whether there is convergence or divergence between them. This article approaches this specific stage.

So, our objective is to *identify and understand how teachers' knowledge impacts their practice regarding geometric transformations.* By completing this step, we intend to establish a theoretical overview of how this approach has been taking place in Brazilian education and encourage new research that can contribute to improving the teaching and learning of geometric transformations and teachers' professional practice based on their awareness of possible gaps in their theoretical and methodological knowledge and how they impact the classroom, as well as the necessary actions so that they themselves can improve them, without the need for external interventions. In this way, we intend to contribute to the teaching of geometric transformations content and cooperate with the education of teachers who teach mathematics.

Knowledge Quartet: a theoretical tool to analyze teaching practice

Teachers' professional knowledge is one of the recurrent research themes in teaching education and practice. Shulman (1986) discussed three dimensions of teacher knowledge: content knowledge, pedagogical content knowledge, and knowledge of the curriculum. In

1987, these dimensions were expanded, including theorization about the students, teaching, and school management.

One of his contributions was the definition of pedagogical content knowledge (PCK), characterized as “[...] the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learns, and presented for instruction” (Shulman, 1987, p. 8) – as one of the teachers’ necessary domains.

In the context of Mathematics Education, Ball et al. (2008) expanded Lee Shulman's theoretical approach, delimiting domains of *mathematical knowledge for teaching* (MKT), subdivided into *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK).

The MKT expanded the domains of teachers’ professional knowledge for Mathematics teaching by making a distinction between the domains of *mathematical knowledge for teaching*. In contrast, the "*Knowledge Quartet*” (KQ) focuses on the classification of situations of *mathematical knowledge in teaching* (Rowland, 2013), a case to be treated in the data analysis of this article. In particular, when relating KQ and MKT, Rowland (2013, p. 22) points out that “[...] the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching”.

Rowland (2013) informs that the theory that emerges from MKT, in particular, aims to undo and clarify what was previously somewhat elusive and theoretically undeveloped in terms of professional knowledge. The KQ was theoretically configured from professional practice, observing and analyzing the teacher's work in the classroom to highlight the situations in which mathematical knowledge is mobilized (Gumiero & Pazuch, 2020).

Gumiero and Pazuch (2020) also indicate a timeline, including that Shulman's works represented a milestone in research on professional teaching knowledge during the 80s and 90s and extend to the present day. The authors explain that the studies by Ball and his collaborators were widely disseminated, especially in 2008. Regarding KQ, the first article was published in 2005, and focused on Mathematics teaching. However, this theoretical tool has been gaining notoriety since 2013, when research involving KQ was developed by researchers from different groups, in addition to the authors themselves (Gumiero & Pazuch, 2020).

The KQ consisted of an analysis with pre-service teachers working with children aged 3 to 11 years, from the planning of their classes to their execution, which was video recorded for later exploration. By observing the data, the researchers identified significant actions related to subject matter knowledge and pedagogical mathematical knowledge, which they characterized with codes. In the first study, they generated 17 codes. Subsequently, 21 codes, each with a specific name, indicate the type of action performed. KQ comprises four dimensions – *foundation, transformation, connection,* and *contingency* (Gumiero & Pazuch, 2020; Rowland, 2013; Rowland & Turner, 2017; Turner & Rowland, 2008).

The dimensions and the respective codes that form them, according to Rowland (2013), are shown in Table 1:

Table 1.

KQ dimensions and their respective codes (Rowland & Turner, 2017, p. 106).

Dimensions	Codes
Foundation **knowledge and understanding of mathematics per se and of mathematics-specific pedagogy; beliefs concerning effective mathematics instruction, the nature of mathematics, and the purposes of mathematics education**	- Awareness of purpose - Identifying errors - Overt display of subject knowledge - Theoretical underpinning of pedagogy - Use of mathematical terminology - Adheres to textbook -Concentration on procedures
Transformation **The presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations**	- Teacher demonstration (to explain a procedure) - (Mis)use of instructional materials - Choice of representations - Choice of examples
Connection **The sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks**	- Making connections between procedures - Making connections between concepts - Making connections between representations - Anticipation of complexity - Decisions about sequencing - Recognition of conceptual appropriateness
Contingency **the ability to make cogent, reasoned and well-**	- Responding to students' ideas - Deviation from agenda

informed responses to unanticipated and unplanned events	- Teacher insight during instruction - Responding to the (un)availability of tools and resources

The KQ is a theoretical tool, relatively recent and under development, for the analysis of the practice of teachers who teach mathematics. This suggests that we can explore and present its potentialities and challenges based on international research and encourage the development of new studies on this topic (Gumiero & Pazuch, 2020). The literature synthesis produced by the referred authors confirms that few studies in the American continent about KQ, particularly in Brazil.

The study of *foundation* situations (Gumiero & Pazuch, 2021a) reveals a new code in which classroom situations are recognized and valued and can contribute to the teacher him/herself expanding professional knowledge when teaching geometric concepts. The *contingency* situations in geometry classes were generated from teachers’ difficulties teaching geometric concepts to Basic Education students. In other words, diversion from the planned agenda or the unavailability of resources can generate new knowledge (Gumiero & Pazuch, 2021b).

It is essential to clarify that the KQ can constitute moments of reflection on the planning and execution of classes based on its own dimensions and codes. Considering the nature of the situations of the teacher's practice and the mathematical (geometric) knowledge related to classroom situations, we emphasize the importance of this article. In the next topic, we will discuss the context and research methodology.

Methodology

The nature chosen to carry out this research was qualitative. When we consider human relations, we deal with extremely complex aspects, which, combined with the educational context, are intertwined with issues related to teaching, learning, political, and social contexts. This network of relationships can hardly be measured quantitatively, highlighting our choice (Bicudo, 2012). This nature also allows an approximation with the object of study, in a horizontal relationship between researcher and researched, so that the subjective aspects inherent to the subject of the research are considered to obtain the results and, in this way, present greater possibilities of reflecting a broader reality (Moraes, 2018).

The data was collected in June 2019, at a Municipal School[3], based on a 50-minute class on the content of geometric transformations, for 30 students attending the 8th grade of Elementary School. The data were collected and recorded through face-to-face observation of the class and videos, focusing on the teacher's performance. The teacher has a degree in Mathematics and seems to have a good relationship with teaching, always seeking to update and appropriate various teaching tools to enhance students' learning of mathematical content.

For the collection and treatment of video data, we used the research method by Powell et al. (2004). This model of description, transcription, and video data analysis is aimed at teaching and learning Mathematics.

Using video recordings has some disadvantages, such as the selectivity of some actions in the classroom due to equipment limitations, and the failure to record specific subjective aspects and the context in which the class is taking place. However, unlike being limited to observing classes, it allows data to be revisited and analyzed in greater depth, reducing the possibility of hasty conclusions. That is why we chose this resource for our research. To minimize its limitations, the video recording was associated with face-to-face observations, which aimed not to be restricted to the look but sought to carry out a detailed perception of the context in question, trying to assimilate subjective aspects that could not be captured by the camera. The apprehended perceptions were registered in writing in the most detailed way possible to avoid losses in the collected material, added by photographic records. Thus, observation and video recording were used as complementary data collection instruments, in which the disadvantages of one are covered by the advantages of the other.

The data treatment suggested by Powell et al. (2004), used as a parameter for this research, presents the following steps:

1. *Carefully observing the video data*: they must first be observed in general, so the researcher can become familiar with the collected data.
2. *Describing the video data*: describe, as impartially as possible, the observed data, in particular actions that complement and contextualize the registered speeches.
3. *Identifying critical events*: identify relevant points, which vary according to the objectives of each research.
4. *Transcribing*: Transfer to paper the recorded speeches and complementary actions, which can help interpret the data.
5. *Coding*: identify and categorize sections where critical events may be contained.

[3] Participants and/or those responsible for this study signed a consent form authorizing the collection of information.

6. *Building the plot*: relate description, transcription, critical events, and codification to interpret the collected data.
7. *Composing the narrative*: present the observed data, which was decomposed, analyzed, and recomposed.

The authors point out that such steps are not linear. They can be performed in another order or simultaneously, and may not necessarily all be used, according to each research objective.

Thus, after collecting the data, they were first transcribed, simultaneously adding descriptions that could complement them. Then, the data were observed in an overview, and then, the critical events were defined according to the transcription of the class information. According to Powell et al. (2004), a teaching situation in which students present a mathematical explanation, or some type of argumentation that may be meaningful for a research question concerned with constructing mathematical justifications or demonstrations and proofs by students can be identified as a critical event. In other words, the critical event can be a remarkable situation in a mathematics class, subject to analysis.

Subsequently, critical events were associated with one or more evident KQ codes. Finally, the plot and narrative were composed, a process that will be outlined below, through the analysis and discussion of one of the critical events.

Results and Discussion

The observed class consisted of introducing the concepts of the axis of symmetry and geometric transformations, with an emphasis on reflection. The teacher started the class by discussing the concept of symmetry with the students, presenting reflection as a process in which the original image and the reflected image are symmetrical. To exemplify the reflection, the teacher used images contained in nature and architecture. In parallel with the visual examples, the concepts of symmetry and reflection were introduced, however, without major formalizations. Continuing with the architectural examples, the rotation concept was also briefly presented.

Then, the teacher asked the students to take the *Caderno da Cidade*, a didactic material provided by the São Paulo City Hall, and, together with the students, read the proposed activity. The teacher carried out some of the task items, which proposed that the students trace the axis(s) of symmetry of several polygons, if they existed. While resolving these items, the teacher deepened the concept of symmetry, always in collaboration and discussion with the students. Then, definitions of some quadrilaterals were also resumed and discussed, as well as the concepts of parallel and perpendicular lines. After performing two items with the students, the teacher asked them to perform the remaining items alone.

After some time, the teacher corrected the other task items on the blackboard, together with the students. At the end of the class, she asked the students to do the next task in the *Caderno da Cidade*, in which they should draw a symmetric figure in relation to an initial figure and a given axis of symmetry. The class ended, so the students should resume the task in the next class.

After observing this overview, critical events related to the KQ codes were identified. We found twenty-one critical events related to a maximum of three KQ codes each, the most evident ones. They were named by a letter or syllable, corresponding to the dimension to which they fit (*F* for foundation, *T* for Transformation, *Co* for Connection, and *Ct* for Contingency), and by a number, which corresponds to the order in which the critical event occurred in the class within that dimension. To facilitate identifying and visualizing the KQ dimensions and codes, we used a color scheme, based on Gumiero and Pazuch (2021a), as shown below:

Table 2.

Identification of KQ dimensions by color (Gumiero & Pazuch 2021a, p.6)

Caption	
Foundation	
Transformation	
Connection	
Contingency	

Table 3 presents the correspondence of each of the critical events with the KQ codes, the dimension to which they belong, and the number of critical events related to each of the dimensions.

Table 3.

Relationship of critical events with KQ dimensions and codes

KQ dimension	Number of occurrences	Nomenclature of the Critical Event	KQ code
Foundation	6	F1	Use of terminology Overt display of subject knowledge
		F2	Use of terminology
		F3	Overt display of subject knowledge
		F4	Use of terminology Overt display of subject knowledge
		F5	Use of terminology Awareness of purpose Overt display of subject knowledge
		F6	Use of terminology Overt display of subject knowledge
Transformation	4	T1	Choice of examples
		T2	Choice of examples
		T3	Choice of representations Teacher demonstration
		T4	Choice of examples Teacher demonstration
Connection	6	Co1	Decisions about sequencing
		Co2	Decisions about sequencing Making connections between procedures
		Co3	Making connections between concepts
		Co4	Making connections between procedures
		Co5	Making connections between concepts Making connections between procedures
		Co6	Making connections between concepts Making connections between procedures
Contingency	5	Ct1	Responding to students' ideas
		Ct2	Teacher insight
		Ct3	Responding to students' ideas
		Ct4	Deviation from agenda
		Ct5	Teacher insight

Five of the mentioned Critical Events were discussed in this article in two situations. In the first one, we discussed Critical Events F1, F2, F3, and F4, related to the dimension *Foundations* of the KQ, because this dimension directly contemplates several aspects of the teacher's knowledge. In the second situation, we present the Critical Event Ct3 to demonstrate, so much so that, although the situation of *Contingency* related to *Response to the students' ideas* was the most evident, it is possible to identify other KQ codes and dimensions within it, as such codes and dimensions also relate to the knowledge mobilized by the teacher when teaching the content in question. These situations are presented below.

Situation 1

The first situation to be discussed refers to the dimension *Foundations*, i.e., how the teacher addressed the content in question with the students. Next, we present the opening section of the class, when the teacher introduces to the students the content to be addressed during the class. In this excerpt, the moments in which the teacher addresses the concept of geometric transformations are evidenced, as well as other concepts necessary for understanding this content, contemplating the first four critical events identified, referring to the dimension *Foundations*[4].

> *Teacher*: Today I'm going to teach about symmetry, right? And we're going to talk about different types of symmetry. So, the first thing I want you to tell me, what I want you to tell me, initially, is: when I say the word symmetry, what comes to mind?
> *Student* 1: Equality
> *Student* 2: Geometry
> *Teacher*: Equality
> *Student* 3: Symmetric
> *Teacher*: Symmetric... symmetry, symmetric, ok. What else?
> *Student* 4: A kind of measurement
> *Student* 5: A ruler
> *Teacher*: Is symmetry a type of measurement? I don't know... What did you say?
> *Student* 5: A ruler
> *Teacher*: Why?
> *Student* 6: I remember...
> *Teacher*: Pardon?
> *Student* 6: It's that thing, what's the name?
> *Student* 7: Oh, I remember equality
> *Teacher*: Equality, symmetrical. If I say something is symmetric...
> *Student* 7: It's because it's equal.
> *Teacher*: And that is equal? That's right! Symmetry is... They are things... In fact, it is a geometric transformation, and when the geometric transformation takes place in

[4]The other two critical events referring to the dimension *Foundations* occurred in other moments of the class, and, due to the limitations of a scientific article, they will not be discussed here.

> symmetry, which is symmetry, this figure is not modified. It changes places, but it is not modified. It is... And there are some kinds of symmetry [...].

The highlighted section refers to the Critical Event F1. Two codes contemplated by the KQ *Foundations* dimension could be identified in that event. The first of those codes refers to the *Use of terminology*. Regarding this code, the most evident finding is that, although the terminologies used refer to the content in question, they are not used correctly, which indicates a gap related to the second code observed in the Critical Event F1, the *Overt display of subject knowledge*.

In the transcription of the teacher's speech, symmetry is presented as a geometric transformation, which is incorrect information. Geometric transformations, such as reflection, can occur on an axis of symmetry and result in symmetric figures. However, not all geometric transformations will necessarily result in symmetric figures. Still, although the geometric transformation concept is not incorrect, it is incomplete. In a geometric transformation, the initial figure remains in the same position, and a new figure is constructed, congruent to the initial figure,i.e., that preserves its dimensions.

The teacher continues explaining through the example as follows:

> *Teacher*: If we observe a flower, right? If I draw a line on this flower, dividing this flower in two, what happens?
> [...]
> *Student* 8: It will separate it in two.
> *Teacher*: So, imagine, what would remain? Half on one side, half on the other. If we think about the human body, right? And we draw a line, dividing us. Everything I have on one side, don't I have on the other?
> *Student* 9: No!
> *Student* 10: How not?
> *Teacher*: It's like I got it... A mirror, and reflected, everything I have on one side, I have on the other side, right? This is the first type of symmetry we are going to talk about, which is reflection symmetry [...]. When we draw a line, let's think of a face, yes, it has to pass exactly through the middle. If I take two objects, and I put one here, and an equal one here [indications with hands], the distance between them must be the same, right? To be symmetrical, the distance between the axis, that is my reference, must be the same [...]. So, I can have the reflection of a point [teacher goes to the board] so, look, there's a point here. Oh, here I have a point A, I can have a reflection of a point, I can have a reflection of a straight line in geometry, I can have a reflection of a plane, right? Let's think about this point here. If one calls this my axis of symmetry, its distance from my symmetric must be the same. So, look, I'm using the grid on the blackboard as a reference, one, two, three, four, five; one, two, three, four, five. This one is going to be my A', which is my symmetric, right? This here is reflection symmetry, which is the one I'm talking about, it's the same one that we draw a line on the face, if I put a mirror here, everything that's here, there's on this side, right? Reflection symmetry [...].

The previous section presents two more critical events related to the *Foundation* Dimension. Critical Event F2 contemplates the code *Use of terminology*. Just like in Critical Event F1, the teacher keeps referring to symmetry as a geometric transformation, when, in fact, the transformation in question is reflection. At Critical Event F3, the code identified was again *Overt display of subject knowledge*, which is not incorrect, but rather incomplete. The concept of symmetry, that the distance between a point and its symmetric with respect to the axis of symmetry is the same, is correct. However, information is missing that this must occur with all points of the figure in question. This point will be relevant to the second situation discussed in this article, which will be presented later.

Returning to the class, the teacher goes on with the explanation, presenting other geometric transformations to the students:

> *Teacher*: I don't just have reflection symmetry, I have rotation symmetry. Look here, around a point, if you think about it, isn't this up here? [...] If you rotate specific degrees, we can determine this and we will study it, we go back to the initial figure, so this here is another type of symmetry, rotation symmetry, different from the other one, reflection [symmetry], right? That the reflection is the one that reflects the same figure, as if it were a mirror, and we have rotation symmetry, which is another geometric transformation [...]. We are going to talk today about reflection symmetry in our activity, but there is reflection symmetry, there is rotation symmetry, that is around a point, right? And there is translation, that the figure moves, but it is not... It doesn't even rotate around a point, and it isn't reflected either, translation, it just slides on the sheet, right?

The introductory moment of the class ends with a superficial presentation of two other geometric transformations, characterized as Critical Event F4. Again, in this critical event, we observed the codes *Use of terminology* and *Overt display of subject knowledge*, referring to the dimension *Foundation*. As pointed out, "symmetry" does not indicate a geometric transformation. Reflection will always result in symmetrical figures since it is a geometric transformation carried out from an axis of symmetry. However, the occurrence of symmetry in the rotation will depend on the angle and the figure in question, and the axis of symmetry taken as a reference for this, since this is not a geometric transformation carried out from an axis, but from an angle. Therefore, it does not always result in symmetric figures. This applies to the translation, which will depend on the figure and the axis of symmetry taken as a reference for verifying the obtainment of symmetric figures. We also emphasize that reflection is a geometric transformation whose purpose is to obtain symmetric figures and, although symmetry can be observed in rotation and translation, this is not the purpose of these geometric transformations.

The excerpt discussed shows gaps in teachers' knowledge of the content of geometric transformations, especially regarding terminology, resulting in delayed teaching of this mathematical content to their students. However, it is not possible to deduce whether the teacher really conceives this content in this way, or whether she made a mistake when she taught it to the students. In the first case, it would be necessary to go over those concepts with the teacher to construct a more appropriate designation for them. In the second case, a more thorough class preparation, resuming the concepts to be taught could, if not solve, at least reduce the errors observed.

Situation 2

The second situation to be presented refers to a specific critical event – the Ct3. It was chosen because, despite the *Contingency* dimension being the most evident, highlighting in it the code *Responding to students' ideas*, it also includes other codes of other dimensions, as we will present below.

At Critical Event Ct3, the teacher corrected the task proposed by *Caderno da Cidade*, described above, which asked the students to trace the axis(s) of symmetry of several polygons, if any. The excerpt highlighted in Critical Event Ct3 corresponds to the discussion of the existence or not of symmetry axes in a parallelogram. One of the students indicated that the parallelogram had an axis of symmetry, so, the teacher asked her to go to the blackboard to indicate how she had performed the task, as shown below:

> *Teacher*: How did you draw it? Can you do it for us? Come here. [student goes to the blackboard to do the task, tracing a diagonal of the parallelogram] [...] Look, remember that I said that a point and its symmetric point, they are always perpendicular to the axis, they always form a 90° angle? So I'm going to do it like this, *look*: I'm going to draw a line perpendicular to that axis you drew, right? [...]. Look, you drew this here as the axis of symmetry, look, your axis of symmetry is this line here. Then I took it, and drew a line perpendicular to its axis. Then, I'm going to do this: I'm going to select this line where I traced two points in the polygon. I'm going to select two points that go over my axis, from this line that's perpendicular to my axis. So, I'm going to take this point over here, and this point over here, okay? I'll call it point A and point B here. Are they symmetrical?
> *Student*: No.
> *Teacher*: Why?
> *Student*: [silence]
> *Teacher*: What measure is not equal?
> *Student*: [silence]
> *Teacher*: The axis distance is not the same. So, look, if you think that this here is your axis of symmetry, you drew a line that is perpendicular to your axis of symmetry, you took two points of the figure that are on top of that line, is the distance you have between them the same? No. So, that orange line is your axis of symmetry?

Student: No.

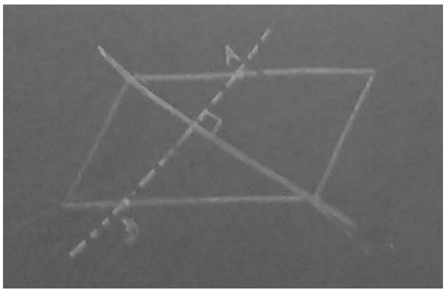

Figure 1.

Discussion of the axis of symmetry of the parallelogram (1) (Teacher's drawing)

In this excerpt, it is possible to identify several codes related to more than one KQ dimension. First, the teacher chose *Use of instructional materials* (dimension *Transformation*), i.e., from *Caderno da Cidade*, to consolidate a previously presented concept. Probably she chose it while preparing the class, when she took *Decisions about sequences* she intended to implement during her class, and she could also carry out *Recognition of conceptual appropriateness* proposed by the instructional material at students' level of knowledge, these two codes of the *Connection* dimension.

However, the most obvious code in this critical event is the *Responding to students' ideas*, allocated in the *Contingency* dimension, which justifies its initial rating. This code can be seen when the teacher, due to a resolution proposed by a student, develops a response and a discussion about the presented idea. In this same action, it is also possible to verify that the teacher used a resolution proposed by a student to demonstrate that the hypothesis elaborated could not be validated according to the definition of the axis of symmetry previously presented by the teacher.

Thus, it is possible to verify the presence of the code *Teacher demonstration*, belonging to the dimension *Transformation*. For this demonstration, the teacher had to perform the *Choice of representations* (another code of the same dimension), by drawing a line perpendicular to the supposed axis of symmetry, as well as points A and B on that line, to show that they were not equidistant from the supposed axis. Then, during the demonstration, the teacher was *Aware of the purposes* intended with it, and *Used mathematical terminology* during her explanations. Furthermore, the teacher had to be able to *Identify errors* committed by the student, which was only possible because of her *Overt display of subject knowledge*. The last four codes presented are found in the dimension *Foundation*.

Let us now observe the unfolding of this discussion:

Teacher: No. If I draw it this way here [on the other diagonal of the parallelogram], the same thing will happen. What I can do? So let's think about another [axis of symmetry] now. If I think like this, look, vertically, will that work?
Student: No.
Teacher: Why?
Student: [silence]
Teacher: [teacher realizes that if she draws a line exactly through the center of the parallelogram, perpendicular to the vertical axis, the points will be symmetric, so she draws a new line, higher up, without mentioning or discussing it with the students] Look, the distance I have, I traced this here as my axis, look. Now I changed its direction, right? Can I call this the axis of symmetry? I got two points, right? I drew a line perpendicular to my axis, took two points on the figure. This distance that I have here, is it the same distance that I have here?
Student: No.
Teacher: No. So, could this be my axis? No. If I draw this horizontally, the same thing will happen. What does that mean? This figure here, does it have an axis of symmetry? No. Do you realize that the distance I have here is not the same distance I have here, right? And I called this here the axis of symmetry. I'm going to draw a line in the middle, will it work? Oh, teacher, apparently it works here [in the center of the figure, in the first drawn line described above], but does it work in the whole figure? No. So this here is not your axis of symmetry, just like this line here is also not an axis of symmetry, because when I draw a perpendicular on my axis, I take two points of the figure, is the distance to the axis the same? No, so this one doesn't have an axis of symmetry.

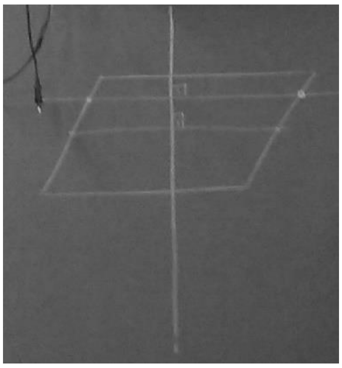

Figure 2.

Discussion of the axis of symmetry of the parallelogram (2) (Teacher's drawing)

In this excerpt, we can see that the teacher, when drawing a horizontal line exactly in the center of the parallelogram, perpendicular to the vertical axis, would have only two symmetric points. Although the teacher had mobilized, until now, a knowledge of the content, even knowing that the parallelogram does not have axes of symmetry, this situation was not foreseen by her. So there was a *Deviation from agenda*, thus characterizing a situation of *Contingency*. We could also notice that, although she knew about the non-existence of axes of symmetry in the parallelogram, she could not explain why those two points, even equidistant, could not determine an axis of symmetry.

Thus, contrary to the previous section, we could notice a gap in the *Overt display of subject knowledge* (*Foundation* dimension), which could have been corrected if there had been, during the planning of the lesson, an *Anticipation of complexity* of the task (*Connection* dimension), through the teacher's previous task execution. Thus, other aspects could have been further explored in relation to this concept – such as the emphasis that, for an axis of symmetry, all the points of the figures in relation to the axis must be symmetric.

According to the analysis carried out, it was possible to identify, in a single critical event, all the dimensions of the KQ, contemplated by 12 of its 21 codes. Through the identification and analysis of these codes, some characteristics of the teacher's knowledge during the given class could be identified. Regarding the *Foundation* dimension, we could observe that although the teacher knew the content in question, she showed she did not fully master it, even though she herself had identified such a gap during her action. The *Transformation* dimension was evidenced by the materials and examples used by the teacher during her explanation, however, although they were foreseen by the teacher, by the way they were presented, they did not seem to have been previously structured. Although this usually happens, as such resources were used in response to the students' elaborations, they could have carried out more structured anticipation of students' thoughts when carrying out the proposed task - including wrong resolutions, as presented, to draw up action plans for such situations, a practice that corresponds to the *Connection* dimension. This anticipation aims to reduce as much as possible *Contingency* situations, if they occur, allow them to be conducted more safely by the teacher, and be added to their personal repertoire, so that they already know how to act, in case they happen again in the future situations (Serrazina, 2017).

Therefore, the KQ is a tool that can contribute to teaching practice from different perspectives. Besides allowing teachers and their peers to identify gaps to be improved in their practice, it is a tool that can help those teachers' educators in structuring and focusing of both initial and continuing education, as it gives them subsidies to improve teaching performance based on demands in practice. With regard to the critical events analyzed in this article, we could think of a teacher’s continuing education course that emphasized the importance and aspects that class planning should cover and, when discussing such planning, act on the gaps described, which emerged in each of the KQ dimensions. Another possibility would be to address such gaps through the elaboration, selection, adequacy, and resolution of tasks with the teachers, through which all the dimensions and codes of the KQ that emerged in the analyzed critical event can also be contemplated.

Final Considerations

This article aimed *to identify and understand how the teacher's knowledge impacts their practice regarding geometric transformations*. Twenty-one critical events and their analysis were raised and categorized. Our stages of analysis evidenced that the knowledge of the content and the pedagogical knowledge of the teacher impact positively or negatively their professional practice in relation to the content of geometric transformations.

Concerning previous discussions on this mathematical content, we could observe convergences between theoretical studies and teaching practice. In these studies, we indicated gaps in teaching knowledge of the mathematical content of geometric transformations, which, although to a lesser extent, could also be observed in practice. And, although the teacher sought to contextualize the content to be addressed, this did not occur, as it culminated in tasks classified as exercises, closed and not very challenging, like most of the tasks observed in the literature and in the teaching resources already analyzed in the articles mentioned.

As limitations, we highlight the analysis of a single class episode. We emphasize that our option owes to the global context of the period of the master's research that triggered this article, the Covid-19 pandemic. At first, to reach more consistent reflections on the proposed themes, we intended to analyze more than one class episode. However, only one, part of this article, could be collected in person, before the pandemic started. Considering the losses that a data collection carried out remotely could contain for the proposed objective, we decided to keep the analysis of only one class episode. Although this option does not provide a comparison of the data collected with other situations, which may or may not be similar to the one presented here, we believe this analysis could contribute to achieving the proposed objective all the same. In future research, we intend to resume this stage, analyzing more class episodes and comparing and discussing the collected data.

To carry out such analyses and reflections, we chose the KQ tool. Using the KQ, through its codes and dimensions, made it possible to develop the proposed objective and highlighted some particularities of this tool. Although the authors present dimensions and codes in a particular order, they may occur in a random sequence, and a previously identified code may appear again later.

Furthermore, not all dimensions and codes will necessarily be identified in a given classroom situation. According to Gumiero and Pazuch (2020), some of them may be more recurrent than others, which depends also on the objectives intended by the researchers when carrying out such analyses. Also, the analysis carried out here showed that, although in a critical event, a specific code predominates, several others can be identified in that same critical event.

Research that uses the KQ tool for the analysis of teaching practice can have different purposes, such as knowing and developing the potential of that tool, contributing to the conception of a new methodological approach to the analysis of teaching practice, or analyzing and seeking to understand how teachers' knowledge is mobilized during their performance (Gumiero & Pazuch, 2020). This article is bound to this last purpose.

This study is one of the ways to contribute to teachers' professional practice so that they become aware of such gaps, so that based on this reflection and the experiences constituted throughout their careers, they can carry out the necessary transformations in teaching practice. In this way, we intend to contribute to the teaching of geometric transformation content and cooperate with the education of teachers who teach mathematics.

We recommend further studies that continue to defend the epistemology of teachers' practice, with emphasis on argumentation and proof processes in geometric contents – in particular, geometric transformations – in Basic Education, as advocated by curricular guidelines. We also defend that studies related to integrating dynamic geometry in teaching actions can expand the possibilities of approaching contents in teaching and provide visualization lenses for students.

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