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# A praxeological approach to the study of fractal geometry in high school textbooks

## Uma abordagem praxeológica do estudo da geometria dos fractais em livros didáticos do ensino médio

## Un enfoque praxeológico para el estudio de la geometría fractal en los libros de texto de secundaria

## Une approche praxéologique de l'étude de la géométrie fractale dans les manuels scolaires du secondaire

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### Abstract

This text is part of a larger research that characterized didactic and mathematical praxeologies related to how Fractal Geometry is approached in high school textbooks. The research aimed to analyze four collections containing three books each. The chosen books were the most adopted among the five most populous cities in the state of Paraná and were all approved by Plano Nacional do Livro Didático (PNLD) of 2018. The analysis of the produced data was carried out from the perspective of praxeological organizations - didactics and mathematics - with the Anthropological Theory of The Didactic (ATD) as a theoretical-methodological reference providing opportunities to investigate the mathematical and didactic choices made by the authors of the selected books. For this article, we will focus on the analyses of the four books adopted in the First Year of High School in the state of Paraná. As a result of the analyses, it was possible to point out that the object of knowledge Fractal Geometry is present in four out of the twelve textbooks analyzed, either theoretically or in the exercises. Such presentation

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occurs in association with other thematic units of Base Nacional Comum Curricular (BNCC) that are out of the scope of Geometry, such as Numbers and Algebra. With regard to the categorized Task Types, they are presented through explorations of fractals such as the snowflake and Koch curve, Sierpinski carpet and triangle, but in general, the proposals for the study of this theme were predominantly used as a means for teaching mathematics subjects other than Fractal Geometry itself.

***Keywords:*** Fractal Geometry, Anthropological Theory of the Didactic, Textbooks.

## Resumo

Este texto é parte de uma pesquisa maior que caracterizou praxeologias didáticas e matemáticas da abordagem do conteúdo Geometria dos Fractais em livros didáticos do Ensino Médio. Tal pesquisa visou analisar quatro coleções compostas por três livros cada, aprovadas pelo Plano Nacional do Livro Didático (PNLD) de 2018, que foram as mais adotadas entre as cinco maiores cidades do estado do Paraná em termos populacionais. A análise dos dados produzidos foi realizada sob a ótica das organizações praxeológicas – didática e matemática –, com a Teoria Antropológica do Didático (TAD) como referencial teórico-metodológico, no intuito de investigar escolhas matemáticas e didáticas dos autores das coleções. Para o presente artigo, nos ateremos às análises dos quatro livros selecionados para o Primeiro Ano do Ensino Médio no estado do Paraná. Sendo assim, as análises nos possibilitaram apontar que o objeto de conhecimento Geometria dos Fractais se faz presente, seja de modo teórico ou durante os exercícios, em quatro dos doze livros didáticos analisados. Tal apresentação ocorre de modo articulado com outras unidades temáticas da Base Nacional Comum Curricular (BNCC) diferentes da Geometria, como Números e Álgebra. No que diz respeito aos Tipos de Tarefas categorizados, esses ocorrem através de explorações dos fractais como o floco de neve e a curva de Koch, o triângulo e o tapete de Sierpinski. Porém, de modo geral, a proposta de estudo do tema foi predominantemente abordada como um meio para o ensino de outros assuntos da matemática diferentes da própria Geometria dos Fractais.

***Palavras-chave:*** Geometria dos Fractais, Teoria Antropológica do Didático, Livros Didáticos.

## Resumen

Este texto es parte de una investigación más amplia que caracterizó las praxeologías didácticas y matemáticas del enfoque de contenido de Geometría Fractal en los libros de texto de secundaria. Tal investigación se propuso analizar cuatro colecciones compuestas por tres libros cada una, aprobadas por el Plano Nacional do Livro Didático (PNLD) de 2018 y que fueron las más adoptadas entre las cinco mayores ciudades del estado de Paraná, en términos de habitantes. El análisis de los datos producidos se realizó desde la perspectiva de las organizaciones praxeológicas - didáctica y matemáticas - teniendo como referente teórico-metodológico la Teoría Antropológica de la Didáctica (TAD), brindando oportunidades para investigar las elecciones matemáticas y didácticas de los autores de las colecciones. Para este artículo nos ceñiremos a los análisis de los cuatro libros seleccionados para el Primer Año de Secundaria en el estado de Paraná. Así, frente a los análisis, fue posible señalar que el objeto de conocimiento Geometría de los Fractales está presente, ya sea teóricamente o durante los ejercicios, en cuatro de los doce libros de texto analizados. Tal presentación ocurre de forma articulada con otras unidades temáticas de la Base Curricular Común Nacional (BNCC) diferentes a la Geometría, como, por ejemplo, Números y Álgebra. En cuanto a los Tipos de Tareas categorizados, estos se presentan a través de exploraciones de fractales como el copo de nieve y la curva de Koch; el triángulo y la alfombra de Sierpinski, pero en general predominó una propuesta de estudio de este tema como medio para la enseñanza de otras materias de las matemáticas distintas a la propia Geometría de los Fractales.

***Palabras clave***: Geometría de Fractales, Teoría Antropológica de lo Didáctico, Libros didácticos.

## Résumé

Ce texte vise à caractériser les praxéologies didactiques et mathématiques de l'approche de contenu de la Géométrie Fractale dans les manuels scolaires du secondaire. Il a été proposé d'analyser quatre collections composées de trois livres chacune, approuvées par le Plano Nacional do Livro Didático (PNLD) de 2018 et qui étaient les plus adoptées parmi les cinq plus grandes villes de l'État du Paraná, en termes d'habitants. L'analyse des données produites a été réalisée du point de vue des organisations praxéologiques - didactiques et mathématiques - ayant la Théorie Anthropologique de la Didactique (TAD) comme référence théorique et méthodologique, offrant des possibilités d'enquêter sur les choix mathématiques et didactiques des auteurs des collections. Pour cet article, nous nous en tiendrons aux analyses des quatre livres sélectionnés pour la première année du lycée. Ainsi, au vu des analyses, il a été possible de relever que l'objet de connaissance Géométrie des Fractales est présent, soit théoriquement, soit lors des exercices, dans quatre des douze manuels analysés. Une telle présentation se fait de manière articulée avec d'autres unités thématiques du National Common Curricular Base (BNCC) différentes de la Géométrie, comme les Nombres et l'Algèbre. En ce qui concerne les types de tâches catégorisés, ceux-ci sont présentés à travers des explorations de fractales telles que le flocon de neige et la courbe de Koch; le triangle et le tapis de Sierpinski, mais en général, une proposition pour l'étude de ce thème prédominait comme moyen d'enseigner d'autres matières de mathématiques différentes de la géométrie des fractales elle-même.

***Mots-clés*** : Géométrie des Fractales, Théorie Didactique Anthropologique, Livres didactiques.

**A Praxeological Approach to the Study of Fractal Geometry in High School Textbooks**

Our initial interest in studying Fractal Geometry emerged as a result of the guidelines prescribing the inclusion of this subject in the basic education school curriculum in the state of Paraná, Brazil. Such addition was guided by Diretrizes Curriculares do Estado do Paraná (Paraná State Curriculum Guidelines, in free translation) or simply DCE (Paraná, 2008), which required schools to approach contents related to non-Euclidean geometry notions. Therefore, we chose to study some of the non-Euclidean types of geometry when we had our first contact with the characteristics of Fractal Geometry because we noticed that this object of knowledge allows us to explore several mathematical subjects other than only Fractal Geometry itself, which in turn aroused our curiosity about how those types of geometry were approached in the textbooks adopted for high school in the state of Paraná.

When a teacher adopts a textbook, it helps them in activity planning and becomes a supporting tool to clarify any unclear points, among other things. For students, the textbook is a tool that is available at any time to help them understand mathematical topics, solve problems, do exercises and offer examples that help them to understand a given subject, thus playing the role of a helper in teaching.

Therefore, we were intrigued by how Fractal Geometry is approached in textbooks, and after reading and discussing the Anthropological Theory of the Didactic (ATD) in Grupo de Pesquisa em Ensino de Geometria (GPEG – Geometry Teaching Research Group, in free translation), we decided to reflect on the praxeological organization – didactic and mathematical – of this type of geometry that could be proposed based on the textbooks used in Paraná state high school.

In this paper, the ATD is used as a tool to study and analyze Fractal Geometry from a mathematical point of view and the didactic choices contained in high school mathematics textbooks. Our work is focused on analyzing the didactic praxeology, or didactic organization (DO), and the mathematical praxeology, or mathematical organization (MO), which contain elements that will support the questions raised in our investigation.

Thus, the research question is defined as follows: Can Fractal Geometry be found in the high school textbooks adopted in Paraná state? If so, what are the teaching proposals for this content?

As a way of clarifying the results produced by this investigation, we will present the analyses of the four textbooks selected for the first year of high school because they were the publications in which we encountered more references to Fractal Geometry.

In the next section, we will describe the principles underlying Fractal Geometry and the Anthropological Theory of the Didactic that caught our interest throughout our studies.

## About Fractal Geometry

Some phenomena and images found in nature hold mysteries and beauty that cannot be understood or explained by Euclidean Geometry. Even so, they are observed, admired and studied by several researchers. Salvador (2009) states that:

> In the Geometry of the world we live in, we carefully observe the winding shapes of paths, shorelines, valleys, mounts, clouds, the human vascular system, leaves, tree branches, bushes, how broccoli and cauliflower are shaped, how bread and cheese are riddled with holes. At the nanometric level of objects, we find rough formations that have self-similar structures, in which small parts of the object are or seem to be tiny replicas of the whole (Salvador, 2009, p. 1, our translation).

Ancient geometers, including Euclid, considered that natural shapes were perfect in their aspects and beauty. Benoit Mandelbrot (1924 - 2010) saw such shapes from a different viewpoint: he researched "the geometry of objects as a shape that repeats itself inside of itself and that always looks similar, regardless of the magnification or reduction of its image, which led to the introduction of the concept of fractal (Salvador, 2009, p. 2, our translation).

Mandelbrot defined a fractal object based on three main characteristics: self-similarity, fractal dimension and infinite complexity. With regard to self-similarity, Barbosa (2022) stated that this characteristic aimed to explain the layout of irregular and fragmented shapes, and to present the surprising impact of the order contained in disorder. Furthermore, the aforementioned aspects allow us to visualize order and patterns where one could only see irregularity, unpredictability and chaos. Additionally, the dimension of a fractal, unlike what occurs in Euclidean geometry, is not necessarily an integer; it can be a fractional number.

Nevertheless, understanding and determining the dimension of a fractal requires calculations that are sophisticated, albeit not very complex, which we will not approach in this text. Regarding infinite complexity, the processes that generate fractals can be recursive, having an infinite number of iterations attributed to it. In other words, we can magnify a fractal figure as many times as we want to, but we will never reach the final image of a fractal perfectly.

For example, Figure 1 shows different levels of Kock Snowflake:

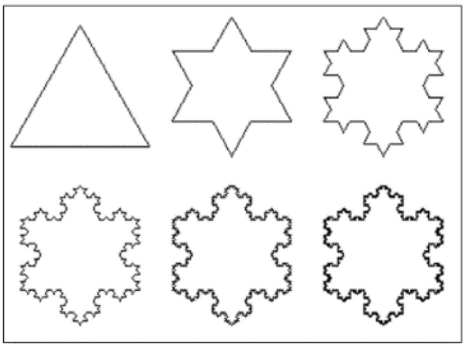


Figure 1.

*Koch Snowflake levels (Pescini, 2021, p. 22)*

The inclusion of Fractal Geometry study in schools, as well as approaching it in an educational context, enables students to have contact with the subject, thus helping improve their knowledge and leading them to develop critical thinking towards a new type of geometry. On this matter, Piccolli (2006, p. 7) said that teaching mathematics without a clear context might end up encouraging only memorization and formulas, without any real meaning.

Therefore, we decided to carry out an investigation into the presence of the fractal content in the textbooks adopted in the state of Paraná, which will happen through didactic and mathematical praxeological studies based on the Anthropological Theory of the Didactic (ATD).

## The Anthropological Theory of the Didactic in this Research

The Anthropological Theory of the Didactic (ATD), proposed by Yves Chevallard, was devised in the scope of the didactics of mathematics, and it allows us to analyze situations related to the teaching and learning of school mathematics, particularly. It offers support so that we can investigate and model mathematical work. This theory considers that every human activity puts into practice an organization that Chevallard called praxeology or praxeological organization (Bittar, 2017).

From the perspective proposed by the ATD, knowledge emerges from human activity. For this reason, mathematical knowledge results from mathematical activities. According to Casabò (2001, apud Kaspary & Bittar, 2014, p. 39-40), “studying conditions for the production and diffusion of mathematical knowledge requires, therefore, the ability to describe and analyze certain types of human activities that are performed under particular conditions.

In this research work, we will focus on how Fractal Geometry is approached and on the mathematical activities proposed by textbooks. In order to analyze and describe this and any other mathematical practice, the ATD provides us with tools that are clearly operative

(Chevallard & Bosch, 1999, p. 4). It all results from the composition of a model called praxeology or praxeological organization. This model comprises the following elements: task type, technique, technology and theory.

Task Type is determined by a proposition composed of a verb related to a set of tasks of the same type, which is represented by the letter T. For example: applying Bhaskara's quadratic formula is a task; developing the sum and product formula is another task similar to the previous one. In this sense, we are talking about tasks of the same type: finding the roots of a second-degree equation. The task type is described or defined by an action verb (find) and a complement (the roots of an equation). Thus, we notice that the complement is necessary so that the type of task is clearly defined.

In order to solve a given task type, we should select a technique $\tau$, which is a way or a model chosen to perform the task, an action or a step-by-step procedure with the objective of fulfilling the proposed task type. The technique, in turn, is justified by a technology $\theta$, which underpins and supports its use because it lends credibility and reliability to the chosen technique. The theory $\Theta$ aims to justify and fully explain the technology. Finally, these four elements compose the praxeological quartet, which is represented by the notation $[T, \tau, \theta, \Theta]$ (Chevallard, 1999).

The ATD offers elements so that we can use it as a research methodology and as a teaching methodology, through the investigation and exploration of tasks, in order to devise praxeological models that could guide the development of mathematical activities.

For the analysis of textbooks, Chevallard (1999) allows the identification of praxeological organizations encompassing both mathematical organization and didactic organization. The analysis of a mathematical praxeology is based on the elements surrounding the study of mathematics and its identification. In other words, it is the verification of the praxeological quartet: task types, techniques, technologies and theories.

The didactic organization can be understood as the way through which the mathematical praxeology is built and organized. The didactic praxeology can be structured by the quartet and described by didactic moments, or moments of study, as proposed by Chevallard (1999) and summarized as follows: 1) it is characterized by the first contact with the praxeology in question. It is important to emphasize that such organization can and must be reviewed over the school term. In most cases, the didactic moment is not supposed to explore the mathematical object excessively, which could be done at other moments; 2) it is the time to explore tasks, when a technique starts to be developed. Chevallard (1999) stresses that the study of a given task type in itself would not make sense if it were not related to the development of techniques, which is the purpose of all mathematical activities; 3) the constitution of a theoretical-technological environment [$\theta,\Theta$]; 4) this moment has the objective of studying and exploring the technique that has been developed in order to make it more efficient and reliable; 5) the moment to formalize the mathematical knowledge that has been studied by harnessing what proved to be more efficient for the construction of the praxeology; and 6) this moment is connected with the moment of institutionalization. It is the time to reflect on the efficiency of the techniques that have been attributed and to investigate and verify what has been studied so far.

It is essential to clarify that such moments do not necessarily follow a specific order: they do not happen sequentially, and they might be repeated several times. Therefore, in order to understand how Fractal Geometry is approached in schools, it is crucial to analyze the didactic organization proposed by the collections of textbooks that have been adopted.

## The ATD and Fractal Geometry

When studying aspects related to the ATD and Fractal Geometry, we understood that the latter can be studied from the point of view of the former. In addition to the aforementioned aspects, the ATD allows us to analyze and model ostensive and non-ostensive objects that are present in mathematical work, and such analysis may be used to approach the mathematical topics suggested in the textbooks.

> We will use the term ostensive object [...] to refer to every object of sensitive nature, of a certain materiality, and which, because of this, has a perceptible reality for human beings. This is the case of any material object and, especially, particular material objects like sounds […], graphics […] and gestures. Non-ostensive objects are, therefore, all "objects" which, just like ideas, intuition or concepts, exist institutionally – in the sense that existence is attributed to them – without, however, being seen, said, heard, perceived or shown by themselves: they can only be evoked or invoked by the proper manipulation of certain ostensive objects related to them (a word, a phrase, graphics, writing, a gesture or a long speech) (Bosch & Chevallard, 1999, p. 10, our translation).

In this sense, when we consider that representations of fractals are ostensive objects, we could mention the construction of objects like Koch Curve and Sierpinski Triangle performed by mathematics software, and also their construction by using manipulable materials and drawing instruments. Such representations ensure a certain materiality and sensitivity to the referred object, producing a perceptible materiality in face of its characteristics and properties.

On the other hand, when describing a fractal, Benoit Mandelbrot determined that self-similarity, fractional dimension and infinity complexity are the main characteristics of the object. Therefore, after reflecting on said characteristics, we understand that the infinite iterations of self-similarity and the fractal object in its ideal of infinite complexity only become understandable when the fractal object is in a condition of non-ostensive object.

Thus, with regard to the fractal generating process, we understand the non-ostensivity of the final object, since it might be recursive, with an infinite number of iterations, and it can be amplified as many times as we desire to without ever reaching its final image. Consequently, the consideration that a fractal is a non-ostensive object results from the fact that when we think of a fractal object and imagine its umpteenth iteration, we acknowledge the state and the notion of a non-ostensive object because we cannot conceive – both visually and mentally – the representation of the object in this state.

The importance of such reflection allows us to comprehend that we will not obtain a final representation of a fractal as it is. A fractal has characteristics that need to be understood beyond its ostensivity, thus providing students with a reflection on these characteristics helps them understand the particularities and differences among the objects that pervade mathematics and their specific properties.

In the following section, we will discuss the textbooks selected to produce the data for this investigation.

## Selected Materials

Our investigation encompassed the four most adopted book collections by public schools located in the five most populous cities in the state of Paraná. The cities are Curitiba, Londrina, Maringá, Ponta Grossa and Cascavel, and they all adopted books that are part of Plano Nacional do Livro Didático 2018 (PNLD 2018 – Textbook National Program, in free translation) for high school. We established that one of the search criteria would be the main high school textbook collections adopted by schools in the cities. Besides, we considered as a hypothesis the largest circulation of the publications used by teachers and students, so that we could be as close to reality as possible in terms of the didactic support available for teachers and students. However, in this text, we will only include the analyses of the books adopted for the first year of high school.

According to our search results, we decided to analyze the following collections: Contato Matemática (Souza & Garcia, 2016), Matemática – Contexto & Aplicações (Dante, 2016), Quadrante Matemática (Chavante & Prestes, 2016) and Matemática Ciência e Aplicações (Iezzi et al, 2016) because these were the four most adopted collections by the five most populous cities in the state of Paraná. The Contato Matemática collection, published by Editora FTD, was the most used by schools according to the data we gathered. In order to optimize the analysis, we established abbreviations for each textbook, which are presented in Table 1.

Table 1.

*Abbreviations for the adopted textbooks (Pescini, 2021, p. 57)*

| **Collection** | **1st Year** | **2nd Year** | **3rd Year** |
|---|---|---|---|
| Contato Matemática | LD 1.1 | LD 2.1 | LD 3.1 |
| Matemática - Contexto & Aplicações | LD 1.2 | LD 2.2 | LD 3.2 |
| Quadrante Matemática | LD 1.3 | LD 2.3 | LD 3.3 |
| Matemática Ciência e Aplicações | LD 1.4 | LD 2.4 | LD 3.4 |

Based on this information, we obtained the textbooks in order to perform the production and analysis of the data.

## Data Analysis Support

The analysis of the textbooks investigated in the scope of the master's thesis encompassed the three volumes that formed each collection. Nevertheless, as we previously stated, this article will only include the four books related to the first year: LD 1.1 to LD 1.4. Through our analyses, we aimed to reveal how Fractal Geometry teaching is approached in the publications. To do so, we investigated the sections Parte Curso (Course Part, in free translation), Atividades Resolvidas (Solved Activities, in free translation) and Atividades Propostas (Proposed Activities, in free translation) contained in the books that integrate the praxeological organization of Fractal Geometry. About these concepts, Bittar says that:

> The Course Part encompasses the explanation of definitions, properties, results and solved exercises. In this section, the textbook authors offer, albeit implicitly, what they consider students at that school level must learn, and it is in this section that students should look for clues in order to solve what is requested. The analysis of the Course Part allows the identification of some task types that seem to be important in that publication – in this case, the textbook. […] Once the analysis of the Course Part was finished, we proceed to Proposed Activities. We aimed to analyze each activity by identifying what task students have to perform and what technique they are expected to use in order to solve the task, supported by the praxeology(ies) identified previously (2017, p. 371-373).

The Solved Activities section refers to the activities proposed to students with their respective solutions, so that they can be used as examples to follow while they are doing the activities. As we noticed that the fractal geometry content was included in Solved Activities, we dedicated a part of the analyses to this textbook section.

Thus, the praxeological analysis proposed in this paper was performed based on what authors expect students to do. To do so, our praxeological analysis was guided by the Teacher's Book, which provided us with support regarding the theory and the activities that are proposed in each textbook.

In view of this, our research was guided by the following question: Can Fractal Geometry be found in the high school textbooks adopted in Paraná state? If so, what are the teaching proposals for this content?

In order to devise the praxeological organization regarding Fractal Geometry teaching in the selected collections, we carried out a study into its mathematical organization (MO) and didactic organization (DO) according to the Anthropological Theory of the Didactic.

With regard to the MO, we identified the proposed task types that were present in Solved Activities and Proposed Activities in the form of exercises. We grouped the task types that shared similarities in order to acknowledge their potentialities, the techniques harnessed and the theoretical-technological aspects that allowed the justification of the techniques used.

When it comes to the DO, we analyzed its presence and the presentation of didactic moments in the suggested teaching tasks to helps us understand the construction and organization of the praxeological quartet identified in the mathematical organization. We chose to investigate the chapters that comprised the study of Fractal Geometry, since this subject has connections with other mathematical or non-mathematical objects. Moreover, we briefly analyzed the convergences and differences between what is suggested by the official education guidelines and what is present in the textbook collections we analyzed.

Next, we will describe the aspects related to the analyses.

## Data Production Analysis and Discussion

In this section, we present productions and analyses of the books selected for the first year of high school. With regard to the Fractal Geometry content found in the textbooks, we noticed that, in general, these materials contain activities focused on other mathematical contents and, especially, the Course Part about this type of non-Euclidean geometry could be identified in only one out of the four volumes analyzed: LD 1.3. We highlight that when analyzing LD 2.1, LD 2.2 and LD 2.3, we did not find it in the Course Part, Solved Activities or Proposed Activities. That is to say, no information on the topic of our investigation was found in these books. Likewise, none of the volumes for the third year of high school contained Fractal Geometry.

Regarding the types of fractals that were found, Sierpinski Triangle appears in all of the analyzed books for the first year of high school, with one activity in each book. In addition, we identified Sierpinski Carpet and Kock Snowflake in LD 1.3.

We will now analyze the sections Course Part, Solved Activities and Proposed Activities from the selected books for the first year.

Course Part

The analysis of the Course Part from the selected textbooks was guided by the Didactic Organization, following the didactic moments proposed by Chevallard (1998). Therefore, to start our analysis, we highlight that among the selected books for the first year, LD 1.1, LD 1.2 and LD 1.4 do not have the Course Part dedicated to Fractal Geometry. As a result, the analysis of this section is entirely focused on LD 1.3, which was the only first-year textbook that explored this content.

In LD 1.3, there is a brief section called *Ampliando Fronteiras* (Expanding Frontiers, in free translation), which presents contents, mathematical curiosities that were not included in other chapters, and other highlights that the author considered relevant. After chapter 10, which focuses on Trigonometry teaching, there is the first contact with Fractal Geometry.  Over two pages the book briefly introduces general aspects of this type of non-Euclidean geometry, and the first modeling in the form of praxeology is conceived through the exploration of Kock Curve, or Kock Snowflake.

First, general characteristics of fractals are introduced. For instance, their infinite complexity, with comments on aspects related to the area and perimeter of the figures. The characteristics and illustrations of the first four iterations of the fractal are presented, as shown in Figure 2.

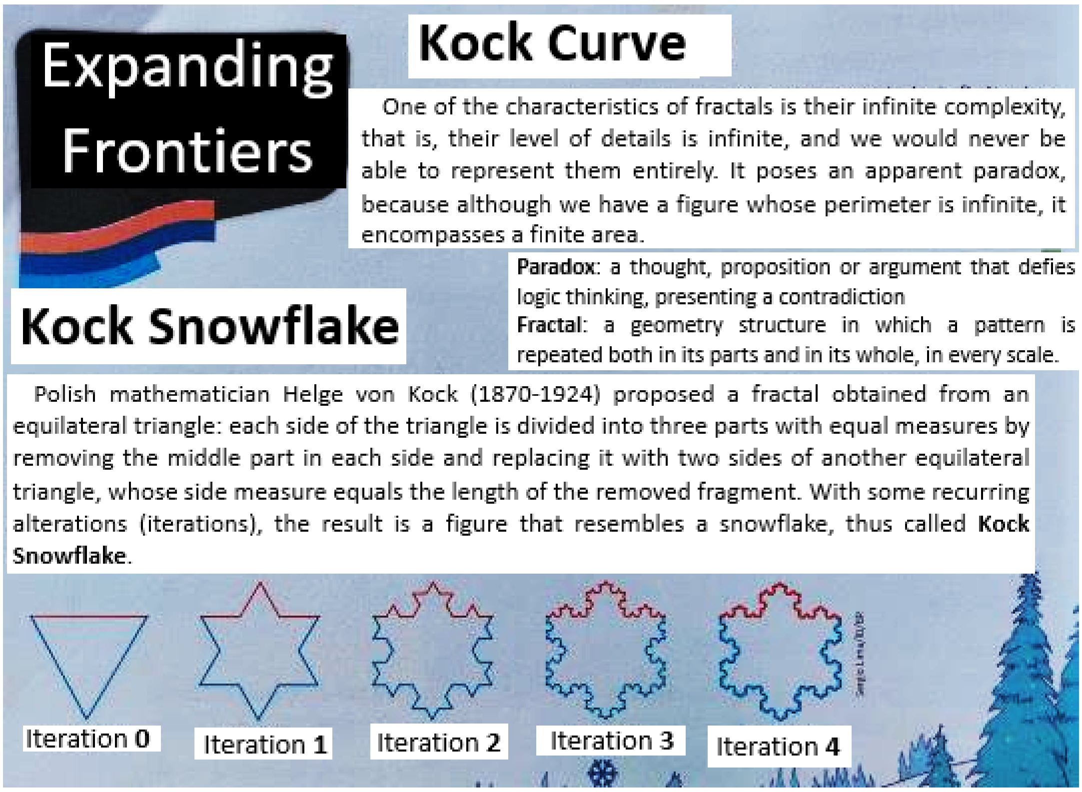



Figure 2.

*Expanding Frontiers: Koch Curve (Chavante & Prestes, 2016, p. 248)*

It is important to highlight that before the illustration of the first iterations of Kock snowflake, the construction of the fractal was described in detail through explanations of traces and fragments of the iterations. About fractal theory, the author refers to the paradox of infinite complexity in a general way, by mentioning that although said fractal figure has an infinite perimeter, its area is finite.

Then, the author proposes an exploration of the perimeter of the figure that was previously presented. However, he emphasizes that in order to construct it, he will only consider the line length presented in the images, which represent the Kock Curve. The images offered by the author can be seen in Figure 3.

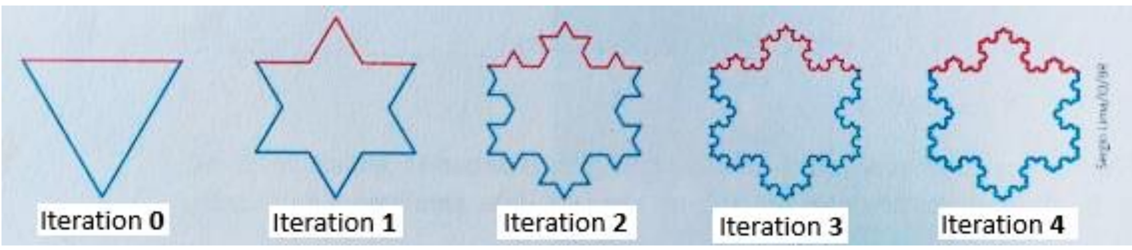
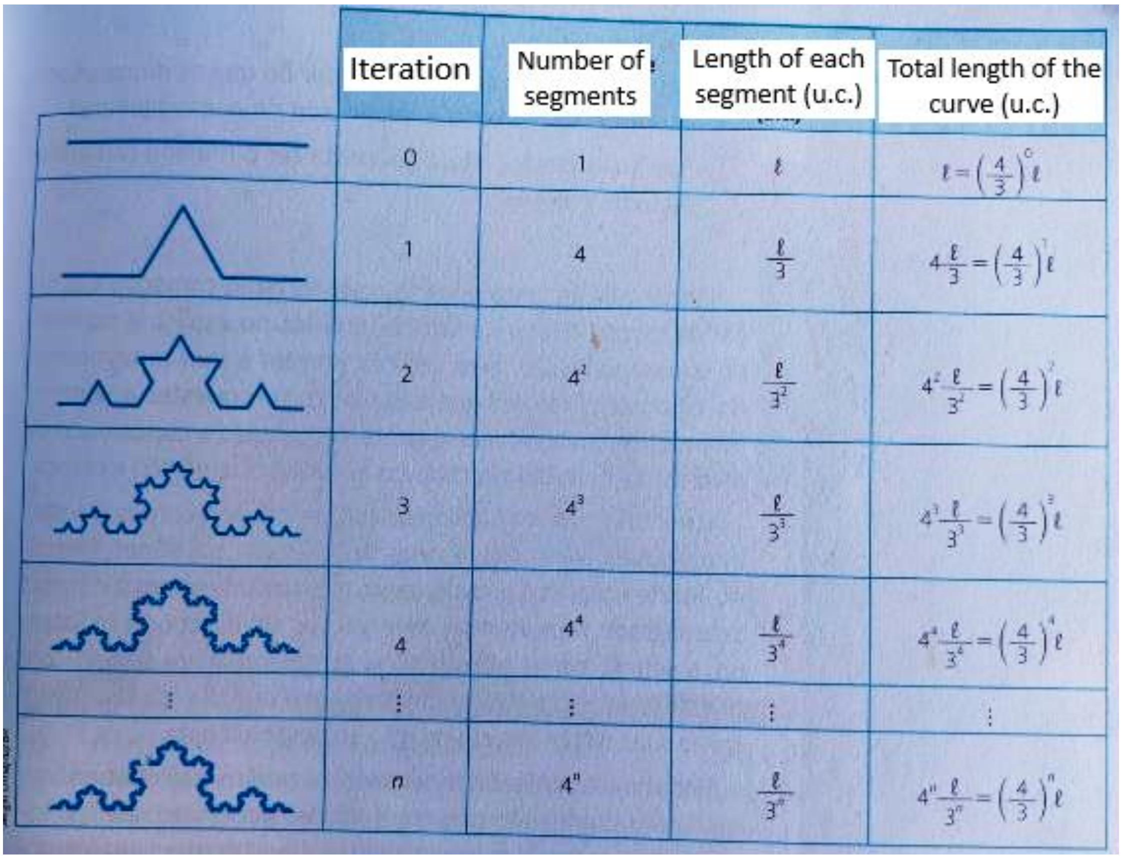



Figure 3.

*Kock Snowflake (Chavante & Prestes, 2016, p. 248)*

At that point, the author experimented with a situation involving a mathematical object that was previously presented so as to explore the concept of perimeter for this fractal. To do so, he devised a table for students to explore and understand its perimeter, as well as its formation process. The red tracing in the figures supported the construction of the table, considering an initial segment of length “l” (the letter “l”), as follows:



| | Iteration | Number of segments | Length of each segment (u.c.) | Total length of the curve (u.c.) |
|---|---|---|---|---|
| | 0 | 1 | $\ell$ | $\ell = \left(\frac{4}{3}\right)^0 \ell$ |
| | 1 | 4 | $\frac{\ell}{3}$ | $4\frac{\ell}{3} = \left(\frac{4}{3}\right)^1 \ell$ |
| | 2 | $4^2$ | $\frac{\ell}{3^2}$ | $4^2\frac{\ell}{3^2} = \left(\frac{4}{3}\right)^2 \ell$ |
| | 3 | $4^3$ | $\frac{\ell}{3^3}$ | $4^3\frac{\ell}{3^3} = \left(\frac{4}{3}\right)^3 \ell$ |
| | 4 | $4^4$ | $\frac{\ell}{3^4}$ | $4^4\frac{\ell}{3^4} = \left(\frac{4}{3}\right)^4 \ell$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| | $n$ | $4^n$ | $\frac{\ell}{3^n}$ | $4^n\frac{\ell}{3^n} = \left(\frac{4}{3}\right)^n \ell$ |

Figure 4.

*Table containing perimeter calculation: Kock Snowflake (Chavante & Prestes, 2016, p. 249)*

After presenting Figure 4, the book mentions the content Geometric Progression (GP), stating that the terms of the total length of the curve in each iteration represent the terms of a GP with ratio $\frac{4}{3} > 1$. Thus, students are provided with an example related to the paradox of infinite complexity through the total length of Kock Curve, which is finite, whereas the Kock Snowflake perimeter is infinite.

Next, after the authors presented the definitions of fractal concepts and the teaching proposal with the construction of the table regarding the fractal perimeter, they investigated the next moment, which refers to the application of what was pre-determined, such as questions and research on fractal theory.

The exploration of the techniques developed so far – related to knowledge of segment length, total length and figure perimeter –, the investigation into other examples of fractals and complementary information on mathematical paradoxes are all harnessed through tasks contained in the book, as follows.

A Do you know other types of fractals? Mention them.

B Considering the table that was presented, what is the number of segments, the length of each segment and the total length of the Kock Curve in iteration 5?

C Calculate the perimeter of the figures that represent the first four iterations of Kock snowflake, considering L = 3 cm.

D Like the Kock Curve paradox, there are other mathematical paradoxes such as Hilbert's paradox of the Grand Hotel, Achilles and the tortoise paradox, and others. Do research into it, then explain it to a classmate.

Figure 5.

*Investigation into fractals (Chavante & Prestes, 2016, p. 249)*

The moments proposed in the investigations aim to promote formalization or reflection on the mathematical knowledge that has been studied. Figure 5 allows us to notice that items A and D might have several answers, depending exclusively on students. For this reason, these activities are characterized as personal tasks. However, items B and C provide us with clues to consider so as to devise their MO, which we will discuss next.

With regard to item B, we noticed that it poses questions about the length of Kock curve in each stage, as explored in Figure 4. Since this task is related to perimeter aspects, we characterized its Task Type as *Determining the measurement of a quantity from a fractal iteration*, then we modeled the techniques based on the development of this activity as described in the Teacher's Book. The MO is presented as follows:

Table 2.

*Task Type 1 (Pescini, 2021, p. 62)*

| | |
|---|---|
| **Task type** | T1: Determining the measurement of a quantity based on a fractal iteration. |
| **Technique** | τ1a: Interpreting the table about perimeter.<br>τ1b: Replacing the number of segments in the desired stage.<br>τ1c: Replacing the length of each segment with the desired stage.<br>τ1d: Replacing the total length of the curve with the desired stage. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

Item C approaches knowledge of the fractal perimeter explored at the moment, just like item B. Nevertheless, it contains some extra information that the author had not made available up to that point: he attributed a numeric value to the segment length of the fractal object. Thus, the techniques differ from the ones harnessed in the previous item. Next, we present its MO.

Table 3.

*Task Type 1 (Pescini, 2021, p. 62)*

| | |
|---|---|
| **Task Type** | T1: Determining the measurement of a quantity from a fractal iteration. |
| **Technique** | τ1a: Interpreting the table about perimeter.<br>τ1b*: Replacing the segment length numeric value in the total curve length in each desired stage. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

We noticed that in these two Mathematical Organizations, the techniques harnessed directly depended on Figure 4, and the step-by-step procedure of the activities, without help from this pre-constructed table, would require a deep interpretation of perimeter knowledge, as well as of the figure pattern of the fractal.

Therefore, the Course Part presented in LD 1.3 is explored briefly, out of context and as a curiosity. An introduction to the subject – that is, Fractals – was done through an example and some investigative questions. Numeric, algebraic and figure-related aspects were approached, thus guiding students to have knowledge of the existence of this geometric object and of its possible mathematical exploration.

Solved Activities

During the analysis of LD 1.1, LD 1.2 and LD 1.4, it was not possible to perform the MO of the proposed activities because they are not presented in these books. The same holds true with regard to all of the textbooks recommended for the second and third years of high school.

Regarding the solved activities that LD 1.3 proposed, in the chapter titled Sequências e Progressões (Sequence and Progression, in free translation), in the segment aimed at geometric progression teaching, we found the section Atividades Resolvidas (Solved Activities, in free translation), which approaches Fractal Geometry through a task referring to Sierpinski Carpet.

The instructions on this Task present four iterations of the mentioned fractal, as we can see in Figure 6.

R18. A square is divided into nine congruent squares, then the central square is removed. In the eight squares left, the same procedure is repeated, then again successively. The result is known as Sierpinski Carpet.

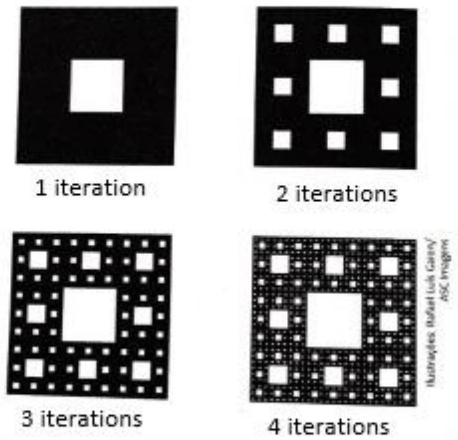



Figure 6.

*Solved Activity (Chavante & Prestes, 2016, p. 198)*

Following the illustrations, the authors pose two questions involving the aforementioned fractal. The first one refers to the sum of the squares that were removed in the four iterations illustrated in the image, and the second one encourages students to reflect on devising a law of formation related to the fractal in question.

Next, we present the questions, their solutions and the tables that represent the MO of this proposal.

a) Determine the number of squares removed in the first four iterations.

Solution

a) By using the information that has been provided, there is the following relationship:

| Iterations | Removed Squares |
|---|---|
| 1 | 1 |
| 2 | $8 \cdot 1 = 8$ |
| 3 | $8 \cdot 8 = 64$ |
| 4 | $8 \cdot 64 = 512$ |

Figure 7.

*Solved Activity – Item A (Chavante & Prestes, 2016, p. 198)*

Based on the solution provided by the authors, we performed the praxeological analysis and, next, we present it in Table 4.

Table 4.

*Task type 2 (Pescini, 2021, p. 68)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures[4] in the sequence, depending on the desired fractal level and vice-versa. |
| **Technique** | τ2a: Interpreting figure pattern.<br><br>τ2b: Numerically writing the number of squares in each level.<br><br>τ2c: Identifying the geometric progression of the fractal.<br><br>τ2d: Writing the numeric expression corresponding to the fractal iteration pattern. |
| **Technology** | Θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

With regard to item B of this Task, we have:

b) Write the law to form an exponential function that is related to the sequence formed by the number of squares removed in each iteration.

b) Notice that the sequence of values obtained in item **a**, (1, 8, 64, 512) constitutes a GP in which $a_1 = 1$ and $q = 8$.
Thus, to determine the number of squares removed after *n* iterations, we should use the general term of the GP:

$a_n = a_1 . q^{n-1} \rightarrow a_n = 1 . 8^{n-1} \rightarrow a_n = 8^{n-1}$

Therefore, the law of formation of the exponential function related to this sequence is

$f(x) = 8^{x-1}$.

Figure 8.

*Solved Activity – Item B (Chavante & Prestes, 2016, p. 198)*

It is possible to notice that this item requests the algebraic construction for the law of formation of the fractal comprising the number of squares removed in each iteration, which characterizes it as an exponential function. Next, we present the table with our analysis.

[4] We chose the word *subfigure* to represent a part of the whole.

Table 5.

*Task Type 3 (Pescini, 2021, p. 69)*

| | |
|---|---|
| **Task Type** | T3: Determining the algebraic generalization that expresses the number of subfigures contained at a random level of the fractal. |
| **Technique** | τ3a: Identifying the figure pattern of the fractal.<br>τ3b: Numerically writing the number of squares at each level.<br>τ3c: Identifying a numerical relationship between the number of squares and the levels.<br>τ3d: Algebraically writing the function that corresponds to the fractal iteration pattern. |
| **Technology** | Θ3: Understanding fractal infinite complexity |
| **Theory** | Θ3: Geometries and Functions. |

Thus, with regard to Solved Activities in LD 1.3, we encountered a proposal that encompassed two different task types with contents related to Geometries and Functions. Both tasks were found in a chapter separated from Geometry, which could be more conventional, since Fractal Geometry is an area inside the great scope of the geometry theme. The tasks encountered in this book explore aspects related to the law of formation of the fractal in question. Even though they were presented in a single solved activity, it contained two different Task Types, which provides us with a brief explanation of the fractal and of his articulation with mathematics - in that context, it was specifically related to sequence and progression.

Proposed Activities

The section dedicated to proposed activities was found in three textbooks for the first year: LD 1.1, LD 1.2 and LD 1.3. LD 1.4, in turn, did not contain any proposed activities related to Fractal Geometry.

As for the textbooks for the second and third years of high school, we have no comments to make, since they did not propose any activities comprising the content that is the object of this study.

LD 1.1 contains a task related to a sequence of figures in the chapter about Exponential Functions. The task shows three levels of the composition of a fractal, as shown in the following figure:

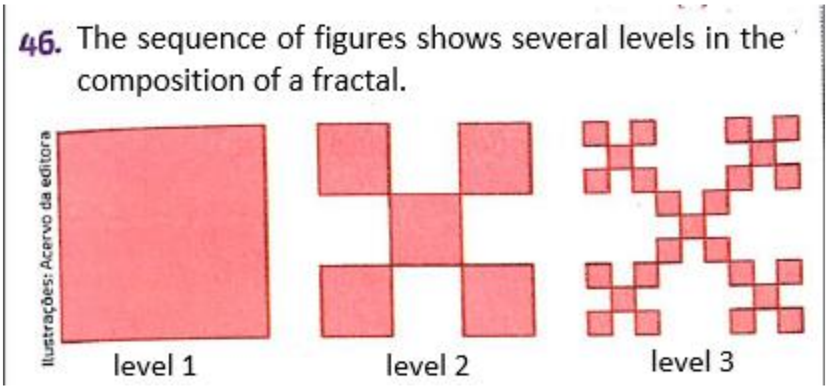



Figure 9.

*Proposed Activity 1 (Souza and Garcia, 2016, p. 151)*

The authors offer three exercises, which refer to figure and algebraic reasoning regarding the fractal in question. About item A, see Figure 10 below.

> a) On graph paper, build a figure that corresponds to the next level of the sequence.

Figure 10.

*Proposed Activity 1 - item a (Souza and Garcia, 2016, p. 151)*

While devising the MO of this task, we noticed that it could be represented by the following Task Type: *Building a figure that corresponds to the next level of a sequence*. Through the analysis contained in the Teacher's Book, it was possible to observe that the technique harnessed at that moment referred to the manipulation of ostensive objects: in the exercise instructions, the authors recommend using graph paper for the construction of the desired level of the fractal. Thus, we devised Table 6 about Task 4.

Table 6.

*Task Type 4 (Pescini, 2021, p. 70-71)*

| | |
|---|---|
| **Task Type** | T4: Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal. |
| **Technique** | τ4a: Identifying the figure pattern of the fractal.<br>τ4b: Manipulating ostensive objects – graph paper – aiming at the construction of the desired level. |
| **Technology** | θ4: Understanding fractal self-similarity. |
| **Theory** | Θ4: Geometries. |

With regard to the Task item b, which is related to knowledge of the law of algebraic formation of the fractal, the authors provide students with three answer choices:

b) What is the function that expresses the number y of squares contained in the figure of level x of the sequence?

$Y = 5^{x+1}$ $y = 5^{x}$ $y = 5^{x-1}$

Figure 11.

*Proposed Activity 1 - item b (Souza and Garcia, 2016, p. 151)*

It was possible to notice that the question that guides this Task leads students to aim at the algebraic generalization, which points out the number of subfigures at a given level, thus corresponding to Task Type 3. However, the step-by-step procedure and the theories harnessed to perform this task are different from those used in the previously presented Task Type 3. For this reason, we used the terms τ3d* and Θ3*.

Table 7.

*Task Type 3 (Pescini, 2021, p. 71)*

| | |
|---|---|
| **Task Type** | T3: Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal. |
| **Technique** | τ3a: Identifying the figure pattern of the fractal.<br>τ3b: Numerically writing the number of squares at each level.<br>τ3c: Identifying a numeric relationship between the number of squares and the levels.<br>τ3d*: Identifying the law of algebraic formation of the fractal.<br>τ3d: Algebraically writing the function that corresponds to iteration pattern of the fractal. |
| **Technology** | θ3: Understanding fractal infinite complexity |
| **Theory** | Θ3*: Geometries and Numbers and Algebra. |

As a final question for the activity, the authors recommend that students use the function that corresponds to the iteration level of the previously shown fractal by using substitution.

c) At what level of the sequence the number of squares will be:

3125? 78125?

Figure 12.

*Proposed Activity 1 - item c (Souza and Garcia, 2016, p. 151)*

It is possible to observe that this Task type shares similarities with Task Type 2, which was previously mentioned. The difference is that the previously shown task refers to the number of subfigures in the sequence depending on the desired level, but this item asks students to determine the desired level of the sequence based on the number of subfigures. Thus, we built our MO for Task Type 2.

Table 8.

*Task Type 2 (Pescini, 2021, p. 72)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures in the sequence depending on the desired level of the fractal and vice-versa. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c*: Identifying the law of algebraic formation.<br>τ2d: Substituting the number of subfigures in the law of algebraic formation.<br>τ2e*: Based on the result, verifying what level the desired object belongs to. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Numbers and Algebra. |

Among the activities proposed in the scope of Geometric Progression, we found four activities that have fractal properties. However, three of them do not cite the term in the instructions, which makes the idea implicit.

The next Task corresponds to a sequence of dots. Three of its iterations are given by the authors, as shown in the following image:

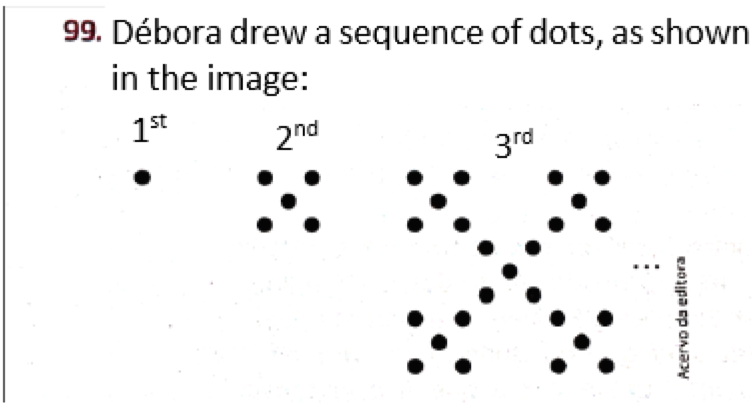



Figure 13.

*Proposed Activity 2 (Souza and Garcia, 2016, p. 224)*

Based on this information, students are asked two questions that guide the sequence according to the principles of geometric progression. The first question tells students to think about the continuity of the sequence and determine the number of dots contained in the 5$^{th}$ figure.

a) Determine the number of dots in the 5$^{th}$ figure of the sequence.

Figure 14.

*Proposed Activity 2 - item a (Souza and Garcia, 2016, p. 224)*

This Task encourages students to write the numeric expression that corresponds to the iteration pattern of the fractal based on the identification of the figure pattern, as well as the identification of the geometric progression of the fractal. As a result, we devised the following table.

Table 9.

*Task Type 2 (Pescini, 2021, p. 73)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures in the sequence depending on the desired level of the fractal and vice-versa. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c: Identifying the geometric progression of the fractal.<br>τ2d: Writing the numeric expression that corresponds to the iteration pattern of the fractal. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

In the second question of this activity, in addition to the identification of the figure pattern included in the previous item, the authors suggest the sum of the terms of a GP, as follows:

b) How many dots will Débora have drawn when she finishes the 5$^{th}$ figure of the sequence?

Figure 15.

*Proposed Activity 2 - item b (Souza and Garcia, 2016, p. 224)*

With regard to this task, students are expected to make use of the properties of finite geometric progression by manipulating the sum of its terms in order to obtain the answer. Even though this task shares similarities with the previous one, what differentiates them is the use of the sum of all terms of a finite sequence. Thus, we have the following quartet, which we called T2, and the technique harnessed in this Task is different from the previous one.

Table 10.

*Task Type 2 (Pescini, 2021, p. 74)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures in the sequence depending on the desired level of the fractal and vice-versa. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c: Identifying the geometric progression of the fractal.<br>τ2d**: Algebraically writing the function that corresponds to the sum of the terms of a geometric progression and substituting with the desired stage. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

The next Task in the scope of the geometric progression content in LD 1.1 harnesses knowledge related to quantities and measurements in addition to geometric progression. We noticed it because, in the instructions, the authors show the figures of a sequence and ask students to consider two extra figures, besides the three presented ones, which establishes a finite sequence of five terms. Then, they ask students about the area in blue and yellow, which are the colors representing the iterations of these fractals, as we can see in the figure.

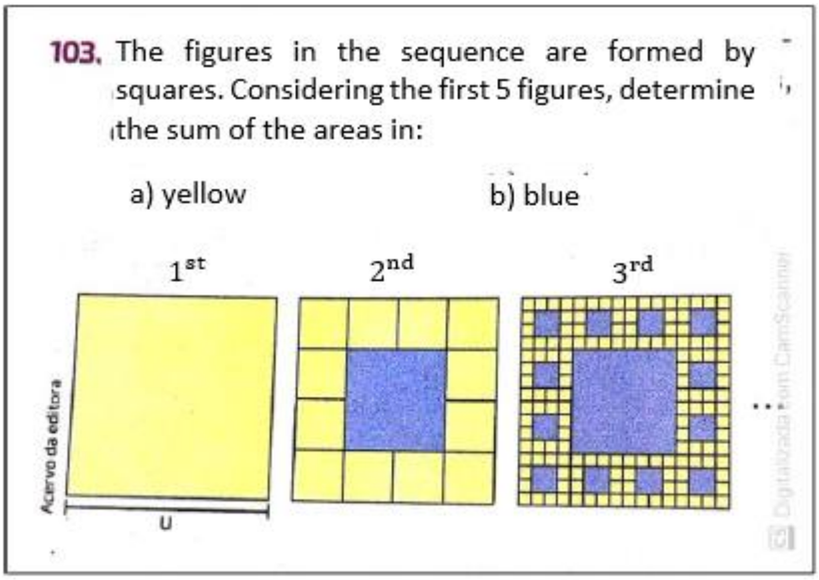



Figure 16.

*Proposed Activity 3 (Souza and Garcia, 2016, p. 224)*

Our analysis of the Teacher's Book about this Task allowed us to verify that the techniques harnessed by the authors suggested that, based on the identification of the figure pattern and the geometric progression, students should algebraically write the function that corresponds to the iteration pattern of the fractal and replace it with the desired stage. Thus, students are encouraged to determine the areas of the figures and add them. Based on this information, we present the table we devised with Task type 1.

Table 11.

*Task Type 1 (Pescini, 2021, p. 75)*

| | |
|---|---|
| **Task Type** | T1: Determining the measurement of a quantity based on a fractal iteration. |
| **Technique** | τ1a*: Identifying the figure pattern of the fractal.<br>τ1b**: Identifying a numeric relationship between the number of squares and levels.<br>τ1c*: Algebraically writing the function that corresponds to the iteration pattern of the fractal.<br>τ1d*: Substituting with the desired stages.<br>τ1e: Determining the areas of the desired subfigures.<br>τ1f: Adding the areas of the subfigures of the fractal. |
| **Technology** | Θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

Still within the geometric progression topic, the next Task implicitly involves Fractal Geometry, because, as we can observe, the exercise instructions do not refer to any fractal terms. Like the activity we analyzed before, this exercise comprises knowledge of area as well as of geometric progression.

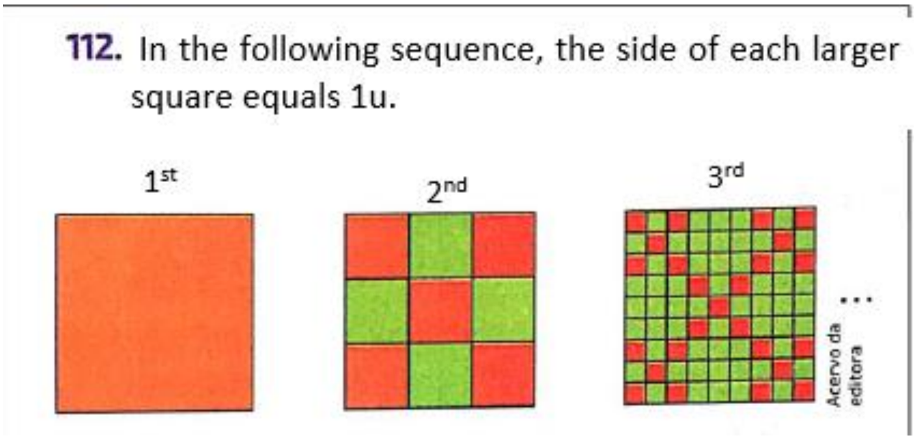



Figure 17.

*Proposed Activity 4 (Souza and Garcia, 2016, p. 224)*

The two questions that guide this activity involve concepts of area, but item *a* explores the area of two specific levels of the fractal, whereas item *b* studies the area of the infinite sequence of the fractal. The first item shows us the following:

> a) What is the area in red in the 2nd figure of the sequence? And in the 4th figure?

Figure 18.

*Proposed Activity 4 - item a (Souza and Garcia, 2016, p. 224)*

Regarding the MO, we understand that its Task Type is similar to the T1 described previously: it explores the technique in which students, in order to determine the area of the fractal object corresponding to the desired level, are asked to identify the figure pattern and the numeric relationship between the number of squares and the level first, so that they can write algebraically the function that corresponds to the iteration pattern of the given fractal. Next, they should substitute the desired level in the function so as to obtain the properties of the areas and add them.

Table 12.

*Task Type 1 (Pescini, 2021, p. 76-77)*

| | |
|---|---|
| **Task Type** | T1: Determining the measurement of a quantity based on a fractal iteration. |
| **Technique** | τ1a*: Identifying the figure pattern of the fractal.<br>τ1b**: Identifying a numeric relationship between the number of squares and levels.<br>τ1c*: Algebraically writing the function that corresponds to the iteration pattern of the fractal.<br>τ1d*: Substituting with the desired stages.<br>τ1e: Determining the areas of the desired subfigures.<br>τ1f: Adding the areas of the subfigures of the fractal. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

As for item b, the question is:

b) Considering that this sequence is infinite, calculate the sum of the area in red.

Figure 19.

*Proposed Activity 4 - item b (Souza and Garcia, 2016, p. 224)*

Since this is an infinite sequence, we understand that students cannot make use of the same techniques harnessed in the previously mentioned MO. To search for the answer to this item, they should use knowledge of limit. In this sense, Table 13 corresponds to Task Type 1 and expands its technique.

Table 13.

*Task Type 1 (Pescini, 2021, p. 77)*

| | |
|---|---|
| **Task Type** | T1: Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal. |
| **Technique** | τ1a*: Identifying the figure pattern of the fractal.<br>τ1b**: Identifying a numeric relationship between the number of squares and levels.<br>τ1c*: Algebraically writing the function that corresponds to the iteration pattern of the fractal.<br>τ1d**: Employ limit. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

The first time the term “fractal” appeared was in the first activity explored in this textbook. After that, we found three Tasks that make use of fractal concepts, but they did not use the term *fractal* to refer to the objects. The fifth and last activity is the only one that offers a brief explanation of the term “fractal”, as seen in Figure 20.

129. (ENEM – MEC) **Fractal** (from Latin *fractus* ("fraction, broken") – an object that can be divided into parts that are similar to the initial object. Fractal geometry, created in the 20th century, studies the properties and the behavior of fractals – geometric objects formed by repetitions of similar patterns.

Figure 20.

*Proposed Activity 5 (Souza and Garcia, 2016, p. 229)*

Next, the authors refer to the exploration of Sierpinski Triangle by commenting on its construction procedures and presenting three figures that represent the three first levels of the fractal, as seen in the figure below:

The Sierpinski triangle, one of the elementary shapes of fractal geometry, can be obtained by following these steps:

1. start with an equilateral triangle (figure 1);
2. build a triangle in which each side is half the side of the previous triangle, then make three copies of it;

Figure 21.

*Proposed Activity 5 – part 2 (Souza and Garcia, 2016, p. 229)*

Later, based on the Sierpinski Triangle, the authors propose a Task in which students should determine what figure belongs to level 4 of the fractal. Five answer choices are offered:

According to the described procedure, figure 4 of the sequence presented above is: c

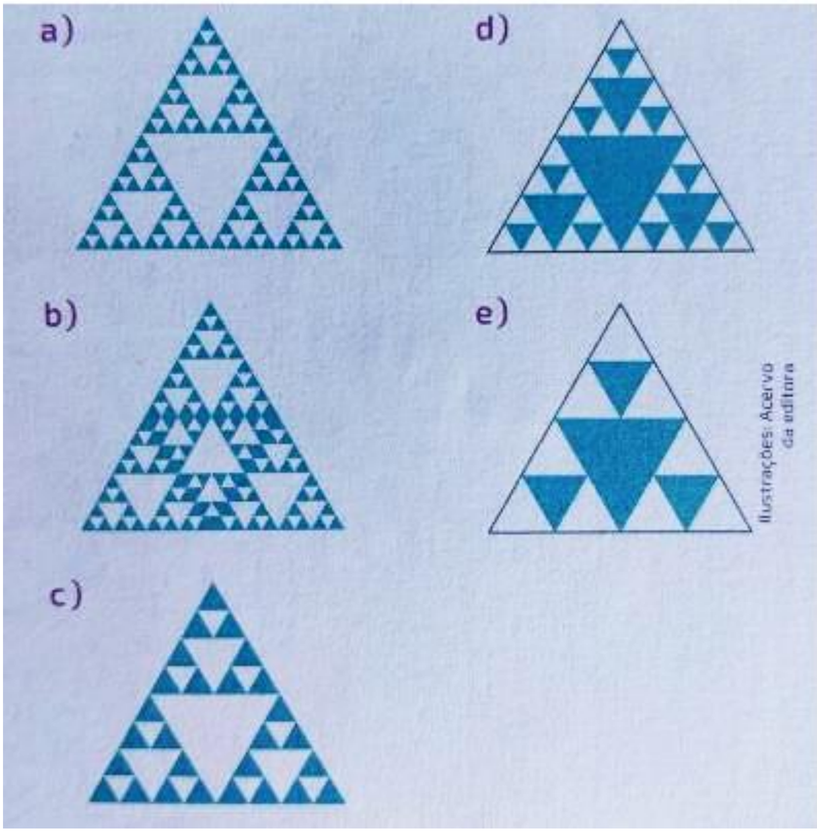



Figure 22.

*Proposed Activity 5 – answer choices (Souza and Garcia, 2016, p. 229)*

After analyzing this task, we determined its MO through Task type 4. A careful observation of how the solution to this exercise is developed in the Teacher's Book led us to notice that the technique harnessed to perform the Task refers to the manipulation of ostensive objects – in this case, paper – in order to build the desired level.

Table 14.

*Task Type 4 (Pescini, 2021, p. 79)*

| | |
|---|---|
| **Task Type** | T4: Building the figure that corresponds to the next level of a sequence. |
| **Technique** | τ4a: Identifying the figure pattern of the fractal.<br>τ4b: Manipulating ostensive objects – graph paper – aiming at the construction of the desired level. |
| **Technology** | θ4: Understanding fractal self-similarity |
| **Theory** | Θ4: Geometries. |

It was possible to observe that LD 1.1 contained five activities involving the content Fractal Geometry, with variations of seven Task Types identified in these activities – all of them encountered in the section dedicated to geometric progression. Even though fractals have been explored through other geometric contents in this textbook, we believe the authors have been explicit, albeit brief, about the knowledge related to this theory. Every presentation happened in the form of activities; there was not an exclusive moment dedicated to the exploration of the theory.

The Serpinski Triangle fractal was used in the Proposed Activity 5 in LD 1.1 and also in a Proposed Activity in LD 1.2, which we will discuss next. The analysis of LD 1.2 identified the presence of only one activity approaching the content Fractal Geometry in the whole book. The activity is presented at the end of the chapters, in a section dedicated to questions related to college admission exams in different areas of Brazil and to Exame Nacional do Ensino Médio (ENEM – National High School Exam, in free translation).

The aforementioned activity starts by showing extracts from a journal article published in *Revista do Professor de Matemática* in 2005 by Sociedade Brasileira de Matemática (Brazilian Mathematics Society, in free translation) on the theme of Fractals in High School. In the text, the author provides a brief presentation of the general aspects of a fractal and explores the construction of the Sierpinski Triangle as an example.

After the comment on the construction of the fractal, three figures illustrating Sierpinski Triangle are presented, as shown in Figure 23.

In order to continue this activity, the author presents a table containing the iterations of the fractal and how many triangles correspond to each level, thus exhibiting the function that expresses the total number of triangles at each new stage. Later, the Sierpinski Triangle is illustrated up to its fifth iteration.

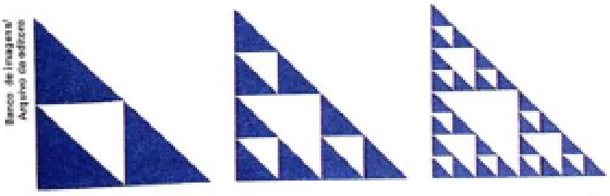
## Thinking about Enem

1. Read the following text and observe the illustrations.

"[...] A fractal is a figure that can be broken into small pieces, each one of them being a reproduction of the whole. We can't see a fractal because it is a limit shape, but the stages of its construction can give us an idea of the whole shape. Its name refers to the fact that the dimension of a fractal is not an integer.

[...] By starting with a right triangle of catheti with length L and dividing its sides in half, we obtain four congruent triangles that are similar to the original one, with a similarity ratio equals ½.

By removing the center of the triangle and successively repeating the process in the triangles that are left, we obtain – as a limit – a fractal called Sierpinski Triangle."



Fonte: SOCIEDADE BRASILEIRA DE MATEMÁTICA. Fractais no Ensino Médio. *Revista do Professor de Matemática*, n. 57. 2º quadrimestre de 2005. p. 1-8.

Figure 23.

*Proposed Activity 6 (Dante, 2016, p. 261)*

The task that comprises the aspects mentioned above is structured into five answer choices, and students have to find the only correct answer. The options involve knowledge of fractal dimension, as well as the sequence formed by the total areas of each shape in each iteration, and the area of each triangle. See Figure 25.

This procedure is a description of the iterative construction of the fractal called Sierpinski Triangle based on a right triangle, but we can also start with an equilateral triangle (as shown below).

| Iteration | 0 | 1 | 2 | 3 | ... | $n$ |
|---|---|---|---|---|---|---|
| Number of triangles | 1 | 3 | $3^2$ | $3^3$ | ... | $3^n$ |

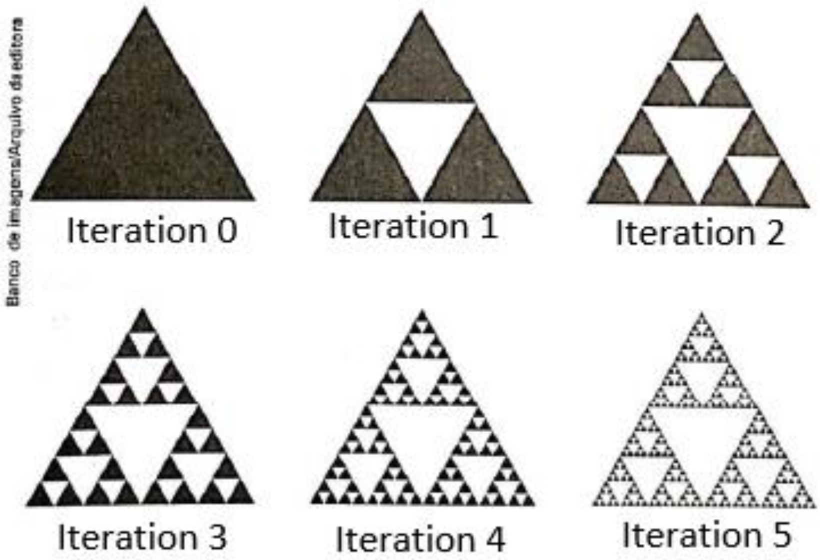



Figure 24.

*Proposed Activity 6 – part 2 (Dante, 2016, p. 261)*

The correct statement is:

a) The number of triangles grows linearly at each iteration.
b) The sequence formed by the total areas of each shape at each iteration is a geometric progression with ratio ¾.
c) The sequence formed by the areas of each triangle, at each iteration, is a geometric progression with ratio ¾.
d) The sequence formed by the number of triangles at each iteration is an arithmetic progression with ratio 3.
e) The area of each triangle tends to infinity.

Figure 25.

*Proposed Activities 6 - Tasks (Dante, 2016, p. 261)*

If we analyzed only the activities and the set in which they are presented, it is not possible to consider their MO because they are not concentrated on a specific theme, just at the end of the chapters in a set of diversified activities about college admission exams and ENEM. Nevertheless, an analysis of the answers contained in the Teacher's Book allowed us to devise the MO of the Tasks.

About the explanations for each item, see Figure 26.

**Thinking about Enem**

1. Let's analyze each option:

a) The number of triangles grows **exponentially** at each iteration. (F)

b) The sequence formed by the total areas of each shape at each iteration is a geometric progression with ratio $\frac{3}{4}$. (T)

c) The sequence formed by the areas of each triangle, at each iteration, is a geometric progression with ratio $\frac{1}{4}$. (F)

d) The sequence formed by the number of triangles at each iteration is a **geometric progression** with ratio 3.

e) The area of each triangle tends **to zero**. (F)

| Iteration | **0** | **1** | **2** | **3** | **...** | ***n*** |
|---|---|---|---|---|---|---|
| Number of triangles | 1 | 3 | $3^2$ | $3^3$ | ... | $3^n$ |
| Area of each triangle | $A$ | $\frac{1}{4}A$ | $\left(\frac{1}{4}\right)^2 A$ | $\left(\frac{1}{4}\right)^3 A$ | ... | $\left(\frac{1}{4}\right)^n A$ |
| Total area | $A$ | $\frac{3}{4}A$ | $\left(\frac{3}{4}\right)^2 A$ | $\left(\frac{3}{4}\right)^3 A$ | ... | $\left(\frac{3}{4}\right)^n A$ |

**Answer:** option **B.**

Figure 26.

*Key to Proposed Activity 6 (Dante, 2016, p. 306)*

It is important to notice that the Task Type varies in each item. Thus, to solve this Task, students should harness more than one technique.

With regard to answer choice *a*, the Task Type we identified refers to *Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal*. As a result, our MO for this Task refers to T3.

Table 15.

*Task Type 3 (Pescini, 2021, p. 82-83)*

| | |
|---|---|
| **Task Type** | T3: Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal. |
| **Technique** | τ3a: Identifying the figure pattern of the fractal.<br>τ3b: Numerically writing the number of squares at each level.<br>τ3c: Identifying a numeric relationship between the number of squares and levels.<br>τ3d: Algebraically writing the function that corresponds to the iteration pattern of the fractal. |
| **Technology** | θ3: Understanding fractal infinite complexity. |
| **Theory** | Θ3: Geometries and Functions. |

Items *b* and *c* make use of knowledge of total areas and areas belonging to each iteration. Both harness task type 1, but what differentiates them is that during the algebraic construction of the function corresponding to the iteration pattern of the fractal, one refers to the total area, whereas the other one concerns the area of each triangle.

The table containing the MO of items *b* and *c* is presented as follows:

Table 16.

*Task Type 1 (Pescini, 2021, p. 83)*

| | |
|---|---|
| **Task Type** | T1: Determining the measurement of a quantity based on a fractal iteration. |
| **Technique** | τ1a*: Identifying the figure pattern of the fractal.<br>τ1b**: Identifying a numeric relationship between the number of squares and levels.<br>τ1c*: Algebraically writing the function that corresponds to the iteration pattern of the fractal.<br>τ1d*: Substituting with the desired stages.<br>τ1e: Determining the areas of the desired subfigures.<br>τ1f: Adding the areas of the subfigures of the fractal. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

While investigating the Task related to item *d*, we identified that it asks students to think about the numeric expression that corresponds to the iteration pattern of the fractal. Thus, we devised our MO for Task Type 2 as follows.

Table 17.

*Task Type 2 (Pescini, 2021, p. 83-84)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures in the sequence depending on the desired level of the fractal and vice-versa. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c: Identifying the geometric progression of the fractal.<br>τ2d: Writing the numeric expression that corresponds to the iteration pattern of the fractal. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

In conclusion, about the MO for this activity, we understand the author suggested that students should write the function corresponding to the iteration pattern of the fractal in order to determine if each item is true or false. Thus, by developing this function, they would be able to know if the area of each triangle tends to infinity or not. As a result, our MO of this item belongs to T1.

Table 18.

*Task type 1 (Pescini, 2021, p. 84)*

| | |
|---|---|
| **Task Type** | T1: Determining the measurement of a quantity based on a fractal iteration. |
| **Technique** | τ1a*: Identifying the figure pattern of the fractal.<br>τ1b**: Identifying a numeric relationship between the number of squares and levels.<br>τ1c*: Algebraically writing the function that corresponds to the iteration pattern of the fractal.<br>τ1d*: Substituting with the desired stages.<br>τ1e: Determining the areas of the desired subfigures.<br>τ1f: Adding the areas of the subfigures of the fractal. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

With the development of these items, students will be able to reach the correct answer to this activity. We highlight that the Tasks encompassed three different mathematical organizations in a single activity, explored different techniques and fractal knowledge of Geometries and Functions.

While Reading LD 1.2, we detected only one activity which approached fractal knowledge. It was presented in the section dedicated to questions related to college admission exams and ENEM at the end of the chapters of the book. Since it is an activity, we identified a variation among three Task Types, on which we based to consider that the presentation and the exploration of the theory studied in this paper were scarce because the sections dedicated to college admission exams might not be approached in the classroom, thus preventing students from having contact with Fractal Geometry through this book.

LD 1.3 promotes the first contact with Fractal Geometry in a Proposed Activity in chapter 8 of the textbook, in the scope of Sequence and Progression teaching. The task instructions briefly present general aspects of Fractal Geometry such as the meaning of the word fractal, its predecessors and advances in the theory, and gives natural and computer-generated examples of fractal objects. A computer-generated example is shown, then Serpinski Triangle and its creator are introduced, followed by three iterations of this fractal. After that, there are two suggested tasks concerning the fractal in question, as shown in the figure below.

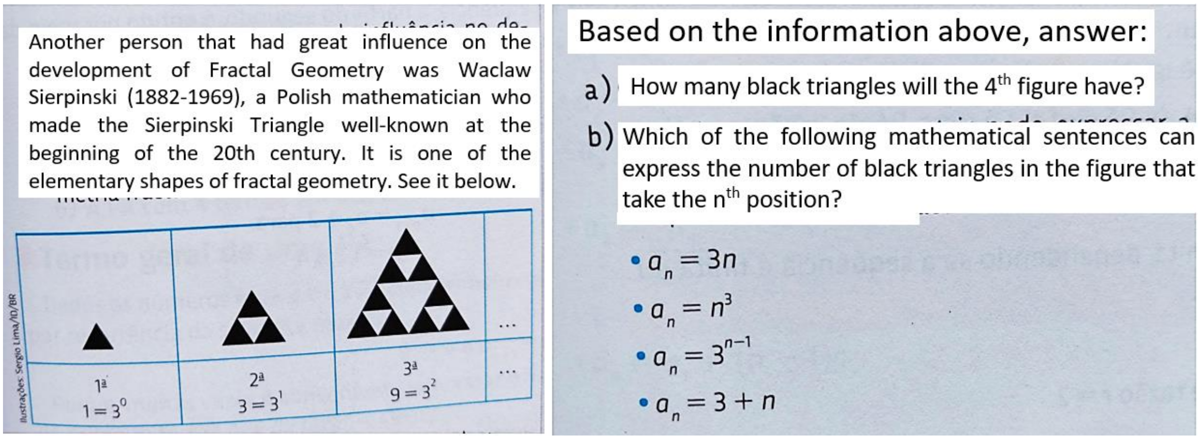



Figure 27.

*Proposed Activity 7 (Chavante & Prestes, 2016, p. 175)*

With regard to item *a*, the following table contains its MO.

Table 19.

*Task Type 2 (Pescini, 2021, p. 85)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures in the sequence depending on the desired level of the fractal and vice-versa. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c: Identifying the geometric progression of the fractal. |

| | |
|---|---|
| | τ2d: Writing the numeric expression that corresponds to the iteration pattern of the fractal. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

The following table comprises the MO related to item *b* of the aforementioned Task.

Table 20.

*Task Type 3 (Pescini, 2021, p. 86)*

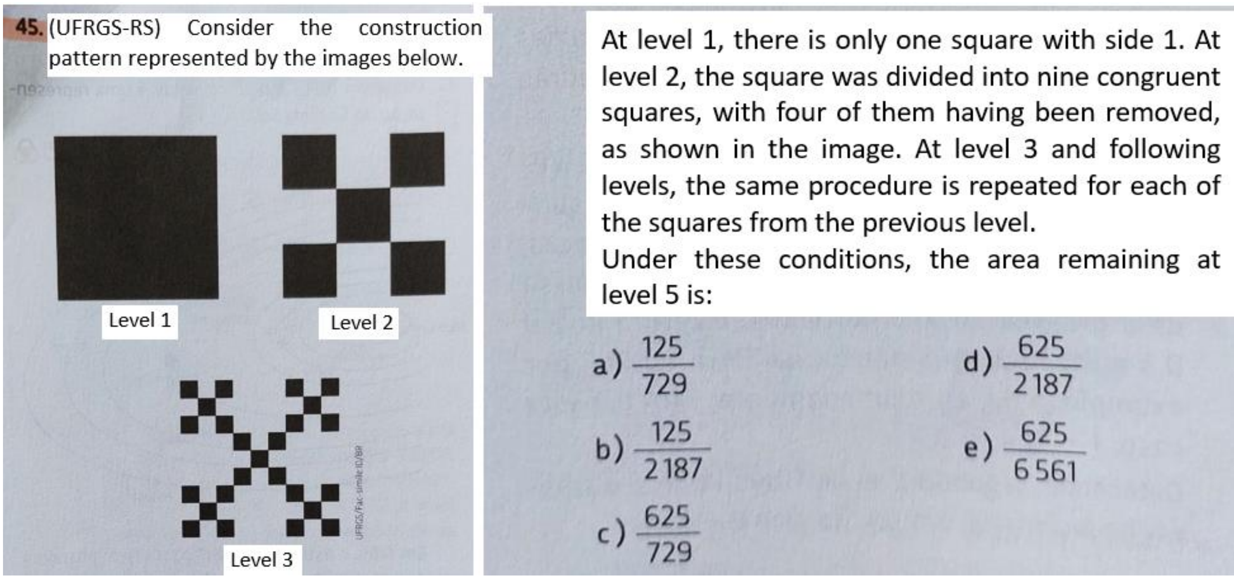
| | |
|---|---|
| **Task Type** | T3: Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal. |
| **Technique** | τ3a: Identifying the figure pattern of the fractal.<br>τ3b: Numerically writing the number of squares at each level.<br>τ3c: Identifying a numeric relationship between the number of squares and levels.<br>τ3d: Algebraically writing the function that corresponds to the iteration pattern of the fractal. |
| **Technology** | θ3: Understanding fractal infinite complexity. |
| **Theory** | Θ3: Geometries and Functions. |

It is possible to notice that item *a* asks students to find out how many subfigures belong to a specific level – level four, in this case. On the other hand, item *b* of the same activity asks them to develop a reasoning that involves generalization in order to obtain the function that corresponds to the iteration pattern of the desired fractal.

This example shows how Fractal Geometry can be approached not only by itself, but as a way of developing other mathematical contents. Thus, it could be related to and interconnected with other theories.

Still in line with the notion of linking Fractal Geometry to other mathematical contents, Proposed Activities presented a Task suggested by the authors in the section dedicated to geometric progression that implicitly refers to Fractal Geometry and was part of a college admission exam. The Task considers a shape construction pattern that is presented in three stages and describes, in the instructions, how the shapes were built.

The task question refers to the area of the figure at level five, as follows.



45. (UFRGS-RS) Consider the construction pattern represented by the images below.

Level 1 Level 2

Level 3

At level 1, there is only one square with side 1. At level 2, the square was divided into nine congruent squares, with four of them having been removed, as shown in the image. At level 3 and following levels, the same procedure is repeated for each of the squares from the previous level.

Under these conditions, the area remaining at level 5 is:

a) $\frac{125}{729}$ b) $\frac{125}{2\,187}$ c) $\frac{625}{729}$ d) $\frac{625}{2\,187}$ e) $\frac{625}{6\,561}$

Figure 28.

*Proposed Activity 8 (Chavante & Prestes, 2016, p. 192)*

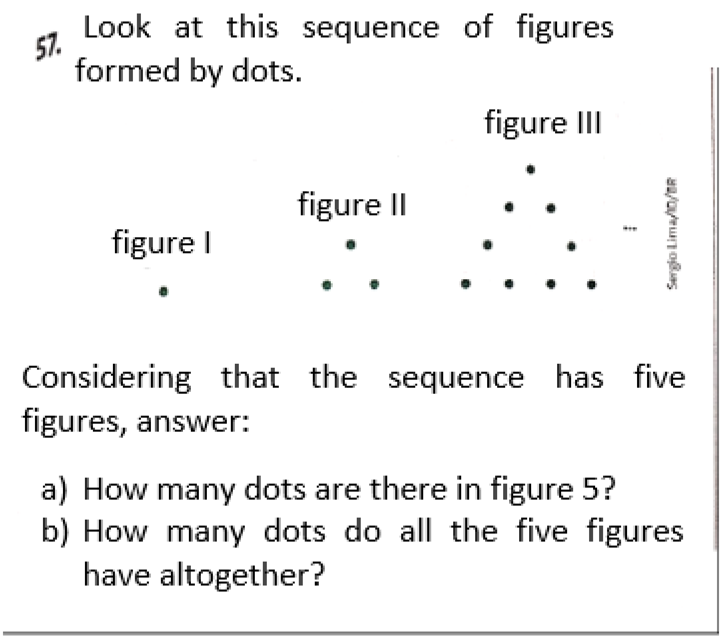
This Task consists in determining the remaining area at level 5, after successively removing squares, depending on the level of the figure. Thus, the MO is established as follows.

Table 21.

*Task Type 1 (Pescini, 2021, p. 87)*

| | |
|---|---|
| **Task Type** | T1: Determining the measurement of a quantity based on a fractal iteration. |
| **Technique** | τ1a*: Identifying the figure pattern of the fractal.<br>τ1b**: Identifying a numeric relationship between the number of squares and levels.<br>τ1c*: Algebraically writing the function that corresponds to the iteration pattern of the fractal.<br>τ1d*: Substituting with the desired stages.<br>τ1e: Determining the areas of the desired subfigures.<br>τ1f: Adding the areas of the subfigures of the fractal. |
| **Technology** | θ1: Understanding fractal infinite complexity. |
| **Theory** | Θ1: Geometries and Functions. |

Moreover, the authors refer to Fractal Geometry once more in an activity in the scope of geometric progression – again in an implicit way. Figure 29 below presents the task, and its MO is shown in detail in table 22.


57. Look at this sequence of figures formed by dots.



Considering that the sequence has five figures, answer:

a) How many dots are there in figure 5?
b) How many dots do all the five figures have altogether?

Figure 29.

*Proposed Activity 9 (Chavante & Prestes, 2016, p. 199)*

While devising the Mathematical Organization of this Task, we noticed that item *a* fits Task Type 2.

Table 22.

*Task Type 2 (Pescini, 2021, p. 88)*

| | |
|---|---|
| **Task Type** | T2: Determining the algebraic generalization that indicates the number of subfigures at a random level of the fractal. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c: Identifying the geometric progression of the fractal.<br>τ2d: Writing the numeric expression that corresponds to the iteration pattern of the fractal. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

As for item *b* of this Task, the author of LD 1.3 expects students to employ the sum of the terms of a finite geometric progression so as to answer the question. Even though this Task Type is very similar to the previous one, it contains differences, since it requires the sum of the subfigures up to level 5. Therefore, we have the following organization:

Table 23.

*Task Type 2 (Pescini, 2021, p. 88-89)*

| | |
|---|---|
| **Task Type** | T2: Determining the number of subfigures in the sequence depending on the desired level of the fractal and vice-versa. |
| **Technique** | τ2a: Identifying the figure pattern.<br>τ2b: Numerically writing the number of squares at each level.<br>τ2c: Identifying the geometric progression of the fractal.<br>τ2d**: Algebraically writing the function that corresponds to the sum of the terms of a geometric progression and substituting with the desired stage. |
| **Technology** | θ2: Understanding fractal infinite complexity. |
| **Theory** | Θ2: Geometries and Functions. |

Among the seven tasks proposed in LD 1.3 that approached Fractal Geometry, we detected the presence of four different Task Types. It is important to highlight that all the analyzed tasks were presented in the section dedicated to the teaching of geometric progression, which means that all theories harnessed in all Task Types are related to the teaching of Geometries and Functions. Course Part is the only section that is separated from the chapter about Sequence and Progression. However, the example given of perimeter and length of Kock curve refers to the teaching of GP.

Even though LD 1.3 contained Fractal Geometry in its Course Part – whereas LD 1.1 and 1.2 did not comprise this content –, we believe the textbook LD 1.3, in general, should have included more information on this theory, which would have provided students with better knowledge and a deeper exploration of fractals.

In view of what has been mentioned, we highlight that the exploration of Fractal Geometry teaching in this textbook is not limited to this content in itself, but also through the exploration of another mathematical content.

In general, we detected that LD 1.1 and LD 1.2 included the study of Fractal Geometry only in the section Proposed Activities, whereas LD 1.3 presented it in Proposed Activities, Course Part and Solved Activities.

In the next section, we will present our final comments on this investigation, in which we aim to highlight the main aspects we identified about the teaching proposals regarding Fractal Geometry over the analyzed publications.

## Conclusion

In order to answer the research question *Can Fractal Geometry be found in the high school textbooks adopted in Paraná state? If so, what are the teaching proposals for this content,* we focused on the most adopted textbook collections by the five most populous cities in the state of Paraná. The analysis carried out on these collections revealed significant aspects about the teaching proposals. Now we will present the characteristics we identified about the fractal geometry content.

We highlight that the Anthropological Theory of the Didactic, used in this research work both as a theory and methodology, was fundamental to the organization and understanding of the teaching proposal in question. The notion of mathematical organization and didactic organization provided support so that we could understand our research subject, which, in summary, consists in identifying and knowing the fractal geometry methodologies that are present in the textbooks analyzed and suggesting the praxeological quartets and didactic moments on the theme.

Based on the analyses we performed, there is reliable data to state that, in the selected publications, Fractal Geometry is seen as a theory used as an intermediary in the approach of other mathematical contents. The fact that this type of geometry is presented as a means, not as an end, allows us to understand that it is used as a tool in the aforementioned textbooks.

Although Base Nacional Comum Curricular (BNCC, National Curriculum Common Base, in free translation) does not suggest the study of non-Euclidean geometry content in the scope of basic education, our research allowed us to notice that other thematic units are approached in the proposals, such as algebra, functions and numbers. Therefore, there is a possibility to include skills contained in BNCC in association with proposals based on Fractal Geometry. It is important to highlight that Diretrizes Curriculares do Estado do Paraná (DCE - Paraná State Curriculum Guidelines, in free translation) (Paraná, 2008), according to what we previously mentioned in this paper, recommend approaching Fractal Geometry in the classroom. Along the same lines, Parâmetros Curriculares Nacionais (PCN – National Curriculum Parameters, in free translation) (Brazil, 1998) also recommended it during the development of the DCE.

Taking into account what is prescribed by the DCE (Paraná, 2008), we noticed that the official documents and the approaches contained in the analyzed textbooks follow the same lines. Furthermore, considering what the DCE recommend for Fractal Geometry teaching, we detected that the association between geometric knowledge and other mathematical contents, such as arithmetic and algebra, is appreciated.

In addition, for the teaching of this type of geometry, Paraná state curriculum guidelines prioritize the inclusion of definitions, statements and demonstrations of results, which are all considered inherent to geometric knowledge. In this sense, these aspects are presented concisely in the textbooks that comprised the subject in question.

The Tasks identified in the textbooks that approach Fractal Geometry knowledge were mostly presented in the chapter dedicated to Geometric Progression. It is important to emphasize that the analyzed Tasks are aimed at the school context only, involving the direct application of mathematical concepts, and that we could not find any Tasks aimed at the out-of-school context.

With regard to the level of the Fractal Geometry content that was encountered, it is safe to say that it is presented briefly. However, when we consider its presence in terms of praxeological organization, the textbooks follow and approach the requirements prescribed by the official guidelines – that is, the DCE –, but in a succinct way.

As expressed by the DCE (Paraná, 2008, p. 172), the identified Task Types presented explorations of fractals such as Kock curve and snowflake, Sierpinski triangle and carpet. Moreover, there is special attention to "definitions, inclusion of statements and demonstrations of results", as prescribed by the guidelines.

In view of the considerations expressed above, some questions emerge: will teachers find enough support in the textbook and in its characteristics, so that they can teach Fractal Geometry? Will they be able to overcome the insecurity related to developing this content if they do not have *a priori* knowledge of it?

These questions surely stimulate discussions, and we believe that changes may occur. One of them could be related to newly graduated teachers who might have studied subjects connected with non-Euclidean geometries in their curricula: a fact that could lead to an understanding that would support their classroom approach. As for teachers who have been working for a longer time, it is necessary to study the historical, mathematical and philosophical context of that geometry through the continuous development of their practice in order to make changes possible, since only the textbook itself does not provide enough material to support content approach in the classroom.

We understand that a careful study of this work could lead readers to reflect and raise questions to guide future investigations. For instance, here are some possibilities: how is Fractal

Geometry present in classroom teaching practice? What supporting materials do teachers use to approach this content besides the textbook?

We hope this discussion could contribute to studies and investigations related to the theme in question and that the visibility of the praxeological organization of Fractal Geometry teaching could be raised.

## Data Availability Statement

The data that support the results of this study are openly available at Figshare through the following DOI: https://doi.org/10.6084/m9.figshare.21719939.v1. The data were obtained from the following public domain resources: http://biblioteca.unespar.edu.br:8080/pergamumweb/vinculos/000096/00009660.pdf .

## Author Contribution Statement

This manuscript was written based on Ana Eliza Pescini's master's thesis, under the guidance of Mariana Moran. All the people meeting the authorship criteria are listed as authors, and all authors have contributed sufficiently to this work so that they take responsibility for its content, including participation in the conception, analysis, writing or revisions to the manuscript, as follows:

Study concept and design: Ana Eliza Pescini, Luan Padilha and Mariana Moran;

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Data collection and production: Ana Eliza Pescini and Luan Padilha;

Data analysis and interpretation: Ana Eliza Pescini, Luan Padilha and Mariana Moran;

Manuscript writing: Ana Eliza Pescini, Luan Padilha and Mariana Moran;

Manuscript intellectual review: Mariana Moran;

Final approval of the version submitted to the journal: Ana Eliza Pescini, Luan Padilha and Mariana Moran.

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