ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2023v25i2p356-393

An Analysis of Similarity Concept Presented in Textbooks in Brazil and the United States

Análisis del Concepto de Semejanza Presentado en Libros de Texto en Brasil y Estados Unidos

Une Analyse du Concept de Similitude Présenté dans les Manuels au Brésil et aux États-Unis

Uma Análise do Conceito de Semelhança Apresentado em Livros Didáticos no Brasil e nos Estados Unidos

Rúbia Barcelos Amaral [1]
São Paulo State University
Livre-Docente em Educação Matemática
https://orcid.org/0000-0003-4393-6127

Karen Hollebrands [2]
North Carolina State University
Ph.D., Curriculum & Instruction with emphasis in Mathematics Education
https://orcid.org/0000-0001-8266-1467

Abstract

This paper is a product of a research that investigates the presence of Geometry in textbooks, that is part of project of a Theorem – Reflection on Geometry and Mathematics Education research group. The focus here is to present results of the international study: how textbooks present the concept of similarity? For this, a qualitative research were developed and textbooks of three collections from Brazil and three from United States were examined. A synthesis of the literature related to the search for textbooks and the teaching and learning of the concept of similarity is presented, and the theoretical framework explores similarity as positioned at the crossroads of geometry and number to describe the ways in which different textbooks approach it. Sequence of topics and tasks presented in each textbook are described and then comparisons are made. There were theorems and problem types that were presented consistently across all textbooks, but differences in expectations related to proof and the use of coordinates and geometric transformations were identified. Only textbooks in the US included the use of transformations and coordinates and placed more emphasis on formal proof. Implications for the teaching and learning of similarity are provided.

[1] rubia.amaral@unesp.br
[2] kfholleb@ncsu.edu

Keywords: Textbook analysis, Mathematical tasks, International textbook comparison.

Resumen

Este artículo es producto de una investigación que investiga la presencia de la geometría en los libros de texto. El enfoque aquí es presentar los resultados del estudio: ¿cómo presentan los libros de texto el concepto de similitud? Para ello, se examinaron libros de texto de tres colecciones de Brasil y tres de Estados Unidos. Se presenta una síntesis de la literatura relacionada con la investigación de libros de texto y la enseñanza y aprendizaje del concepto de similitud, y el marco teórico explora la similitud posicionada en la encrucijada de la geometría y el número para describir las formas en que diferentes libros de texto la abordan. Se describe la secuencia de temas y tareas presentados en cada libro de texto y luego se hacen las comparaciones. Hubo teoremas y tipos de problemas que se presentaron de manera consistente en todos los libros de texto, pero se identificaron diferencias en las expectativas relacionadas con la demostración y el uso de coordenadas y transformaciones geométricas. Solo los libros de texto en los Estados Unidos incluyeron el uso de transformaciones y coordenadas y pusieron más énfasis en la prueba formal. Se proporcionan las implicaciones para la enseñanza y el aprendizaje de la similitud.

Palabras clave: Análisis de libros de texto, Similitud, Tareas matemáticas, Comparación internacional de libros de texto.

Résumé

Cet article est le résultat d'une recherche qui enquête sur la présence de la géométrie dans les manuels. Il s'agit ici de présenter les résultats de l'étude dont la question directrice est : comment les manuels présentent-ils le concept de similarité ? Pour cela, des manuels de trois collections au Brésil et trois aux États-Unis ont été examinés. Une synthèse de la littérature liée à la recherche sur les manuels et à l'enseignement et à l'apprentissage du concept de similarité est présentée, et le cadre théorique explore la similarité telle qu'elle se situe au carrefour de la géométrie et du nombre pour décrire la manière dont différents manuels l'abordent. La séquence des sujets et des tâches présents dans chaque livre est décrite, puis des comparaisons sont faites. On a trouvé des théorèmes et des types de problèmes présentés de manière cohérente dans tous les manuels, mais des différences dans les attentes liées à la preuve et à l'utilisation des coordonnées et des transformations géométriques ont été identifiées. Seuls les manuels aux États-Unis incluaient l'utilisation de transformations et de coordonnées et mettaient davantage l'accent sur la preuve formelle. Les implications pour l'enseignement et l'apprentissage de la similarité sont également présentées.

Mots-clés : Analyse de manuels, Similitude, Tâches mathématiques, Comparaison internationale de manuels

Resumo

Este artigo é fruto de uma pesquisa que investiga a presença da Geometria nos livros didáticos. O foco aqui é apresentar os resultados do estudo cuja pergunta diretriz é: como os livros didáticos apresentam o conceito de semelhança? Para isso, foram examinados livros didáticos de três coleções do Brasil e três dos Estados Unidos. Uma síntese da literatura relacionada à pesquisa de livros didáticos e ao ensino e à aprendizagem do conceito de semelhança é apresentada, e o referencial teórico explora semelhança como posicionada na encruzilhada da geometria e do número para descrever as maneiras pelas quais diferentes livros didáticos a abordam. A sequência de tópicos e tarefas presentes em cada livro são descritas e, em seguida, as comparações são feitas. Havia teoremas e tipos de problemas apresentados de maneira consistente em todos os livros didáticos, mas foram identificadas diferenças nas expectativas relacionadas à prova e ao uso de coordenadas e transformações geométricas. Apenas os livros didáticos nos Estados Unidos incluíam o uso de transformações e coordenadas e colocavam mais ênfase na prova formal. Ainda são apresentadas implicações para o ensino e a aprendizagem da semelhança.

Palavras-chave: Análise de livros didáticos, Semelhança, Tarefas matemáticas, Comparação internacional de livros didáticos.

An Analysis of Similarity Concept Presented in Textbooks In Brazil and the United States

Similarity is an important concept because it has numerous applications to the solution of real world problems and provides opportunities for students to connect what they have learned about ratio and proportions to geometric objects. Two objects can be considered similar numerically (e.g., if corresponding angles have equal measures and corresponding side lengths are proportional) or geometrically (e.g., if there exists a similarity transformation that maps one geometric object to another). Recent standards documents in the United States have recommended instructional approaches to the teaching and learning of similarity that are based on geometric transformations (CCSS-M, NCTM 2018). However, there is not much guidance about how transformations be used to introduce students to similarity. This transformation approach is different from traditional approaches to similarity that typically focus on the identification of corresponding angles that are congruent and side lengths that are proportional. We were curious to answer: how do textbooks in the United States and textbooks in Brazil present similarity to students? To investigate this question, we analyzed six mathematics textbooks from Brazil and the United States and focused on how the topic of similarity was structured and sequenced in each textbook. This is part of a big research that intends to analyze the presence of Geometry in textbook. Qualitative methods were used to identify tasks and chapters as the units of analysis. Cox's (2013) framework that characterizes similarity as positioned at the crossroads of geometry and number to describe the ways in which different textbooks presented similarity tasks.

We begin with an in-depth analysis of the approaches used in each textbook, starting with textbooks from Brazil, then the United States, and finally comparing and contrasting the approaches to similarity in both countries. Prior to that, we provide a synthesis of literature related to textbook research and the teaching and learning of the concept of similarity.

The Context of Research: Textbooks and the Concept of Similarity

Textbook Research

Textbooks are popular resources used by teachers to sustain and develop their teaching practice (Costa & Bittar, 2019; Guimarães et al., 2007; Lajolo, 1996; Lemos, 2006; Silva, 1996; Valverde et al., 2002). Research on textbooks and their use has gained international visibility in mathematics education research (Choppin, 2004). In particular, Choppin (2004) notes textbook research tends to address a broad array of issues. This contributes to a large number of research publications that makes it challenging to synthesize. However, it is important to distinguish the major categories of historical research:

> • those who, conceiving the textbook just as a historical document as any other, analyze the contents in a search for information extraneous to themselves [...] or those that are only interested in the content taught through the textbook;
> • those who, neglecting the contents of which the textbook is bearer, consider it as a physical object, that is, as a product manufactured, marketed, distributed, or even as a utensil designed for certain uses, consumed - and evaluated - in a given context (p.6).

The author highlights that in the first case, what the researcher writes is not actually an analysis of the textbook, but a description of the discipline presented through a particular medium; moreover, it is often the case that textbooks are only one of the sources to which the historian refers. In the second category, the historian directs his attention on the textbooks, contextualizing them in the environment in which they were conceived, produced, distributed, used and received. This analysis is independent of the contents of which textbooks are carriers. This distinction is surely schematic, since a researcher generally participates in work including both categories. Choppin (2004) further emphasizes that textbook authors are not simply spectators of their time; they claim another status, that of agent. Thus, the textbook is not a simple mirror. It modifies reality to educate new generations, providing a reformed, schematized, often-favored image. Actions contrary to morality are almost always punished exemplarily; social conflicts, criminal acts or everyday violence are systematically silenced. And historians are interested precisely in the analysis of this rupture between fiction and reality, that is, the intentions of the authors.

Choppin (2004) also draws attention to the fact that the development of textbooks (documentation, writing, pagination, etc.), material realization (composition, printing, bookbinding, etc.), marketing and distribution (whether public or private), involves the use of increasingly specialized techniques and work teams. Finally, textbook adoption, modes of use, and reception are able to mobilize numerous partners (teachers, parents, unions, associations, technicians, librarians, etc.). The selection and use of textbooks can be politicized and controversial. Research involving textbooks have considered and been influenced by these historical factors.

Freeman and Porter (1989) did an investigation "to (a) describe teachers' styles of textbook use and (b) examine the overlap between content taught and textbook content in elementary school mathematics" (p.403). They analyzed the overlap between textbook content and the content that was taught. Their findings contradicted the popular notion that elementary school teachers' content decisions are dictated by the mathematics textbooks they use. Their research showed important differences between the organization and content of the textbook, teachers' topic selection, content emphasis, and sequence of instruction.

Considering the importance of textbook research, especially research that analyzes their use in classrooms, Fan (2013) articulates and presents a new conceptual structure that encompasses the research objectives and methods present in the investigations that take the mathematical textbooks as a locus of inquiry. In this investigative proposal, textbooks, as research objects, can be seen as an intermediate variable in the context of education. And he adds that this new look at the educational role of books demands research that addresses these didactic materials not only as written documents, but those that also seek to elucidate their interrelations with other factors present in education.

Fan (2013) argues that researchers need to expand their questions and research methods on textbooks. When analyzing such methods, it has been noticed that, in general, there have been very few studies using empirical methods (observation of classes, interviews, etc.). Document analysis, he noted, was the method most used. Jones and Tarr (2007) assert "textbooks are present not only in classrooms, they are also frequently used by teachers and students, and influence the instructional decisions that teachers make on a daily basis" (p.5). They reported that some studies revealed.

> that most middle-grades […] mathematics teachers use most of the textbook most of the time […]. mathematics teachers of three fourths of the eighth-grade students […] reported using their textbook on a daily basis […] (and) two thirds of middle-grades mathematics teachers "cover" at least three fourths of the textbook each year. These findings tend to agree with results of research on students' use of mathematics textbooks as well, […] 72% of participating eighth graders stated that they did mathematics problems from a textbook every day (Jones & Tarr, 2007, p.5).

Fan (2013) suggests that methods for analyzing textbooks requires change. In 2011 he proposed a conceptual framework that conceptualizes textbooks as an intermediate variable in the context of education and defined mathematics textbook research as disciplined inquiry. He suggests relationships between mathematics textbooks and other factors in mathematics education be investigated.

Textbook analysis is a broad term primarily including (1) analysis of a single textbook or a series of textbooks, which often focuses on how a topic or topics are treated or how a particular idea or aspect of interest is reflected in the textbooks, and (2) analysis of different series of textbooks from the same country or, more frequently, different countries, often with focus on identifying their similarities and differences. The latter is also termed textbook comparison. Textbook comparison is based on an individual textbook or a series of textbooks. It can also be termed comparative textbook analysis. Johnsen (1993) generally defined textbook analysis as a product-oriented approach. He surveyed the literature on textbooks from Germany, France, UK, US and, Scandinavian countries. It revealed that textbook research using a product-oriented approach, which focused on subject matter knowledge, had dominated the field, while less research has been conducted on writing, developing and distributing textbooks (i.e. a process-oriented approach). Research about textbook use (i.e. a use-oriented approach) has started to receive slightly more attention.

Finally, as Fan (2013) notes, research on mathematics textbooks as a field of research is at an early stage, being considered recent in comparison to other fields of research in mathematics education. It is in this context that the results of the current research aims to contribute by conducting comparative textbook analysis focused on mathematical content (similarity) presented in textbooks in Brazil and the US.

Research on Similarity

We chose to examine similarity because it is easily related to real world situations. Lehrer, Strom & Confrey (2002), Cox (2013) and Lo, Cox & Mingus (2006) describe how the daily activities of children involve experiences that contribute to their ideas about scale, enlarging, or shrinking images. Cox (2013) adds similarity is a compelling context for research because the content has the potential to connect both geometric and numeric ways of reasoning.

To investigate how young children reason about similarity, Lehrer, Strom and Confrey (2002) conducted a classroom study with twenty-two third grade students with the goal "to support and document the emergence of multiple senses of mathematical similarity" (p. 359). Three children were identified as having special needs. Two sequences of tasks (first involving 2-dimensional forms and second 3-dimensional forms) were designed by the authors. Clinical interviews involving similarity content were also conducted with children. Their results indicated that "children’s explorations of similarity illustrate the continual interplay among tasks, tools, notations, and modes and means of argument in classrooms that promote understanding of mathematics" (p.392). The children’s understandings of the content was mediated by a distinct form of mathematical inscription, like ratio, algebraic “rule” and a line in a Cartesian system. Exploration of two dimensions helped students’ integration of multiple senses of similarity as ratio and as scale. During the year, students used their experiences learning about similarity as a resource for modeling phenomena in nature. This resulted in students experiencing epistemological dissonance between mathematics and science.

Cox (2013) discussed how similarity in middle school mathematics centers on the crossroad of geometry and number. We know that geometric and numeric approaches are different. For example, when determining whether two segments are proportional one could approach this question geometrically or numerically. Cox (2013) highlighted that "in a geometric context, the manipulation of quantities in a problem is not the same as manipulating the objects in the problem [...], the nature of the geometric context of similarity is significant and geometric proportional reasoning should be differentiated from numeric proportional reasoning". Her investigations were based in a set of clinical interviews of 21 students around 12-13 years old. She analyzed students' understanding of pre-similarity topics. She found that the "student use of visual reasoning did not indicate less developed conceptions of similarity" (p. 3). She affirms visual-based strategies helped students to think and that this resulted in the improvement of numeric strategies. She suggests offering tasks to students that require them to scale more complex geometric figures. She claims that this supports students' ability to quantify features of shape and the numeric relationships between them. Seven student strategies for approaching similarity tasks were identified: avoidance, additive, visual, blending, pattern building, unitizing and functional scaling. They are describing in the table below. Cox (2013) also highlighted how tasks influence strategies students selected and used.

Table 1.

Strategies Used by Students to Scale Geometric Figures (Cox, 2013, p.13)

Characteristic Type	Strategy	Description Two figures are similar/nonsimilar if…
Nonspecific (6,98%)	None indicated	(A student's response was not specific enough to ascertain what characteristics of the shape the student perceived, or how the student arrived at a conclusion about similarity.)
Appearance (26,65%)	Cosmetic	… they are visually alike/different.
	Shape type	… they are different sizes of the same/different shape type.
	Relative position	… the relative positions of the sub-shapes do not/do change.
Angle (8,77%)	Angle	… corresponding angles are of equal/different measures.
Length (35,24%)	Corresponding length comparison	… one length is longer or the same as another.
	Constant gap	… there exists/does not a consistency of a gap length within two figures.
Relationship (22,36%)	Constant difference	… there is/is not a constant difference relationship between pairs of corresponding lengths.
	Constant ratio	… there exists/does not exist constant ratio between corresponding lengths.
	Qualitative relationship	… there is/is not a constant qualitative relationship in the components of two figures.
	Dilation	… one can be transformed into the other through some act of expansion or dilation.
	Tiling	… one can be transformed into the other using tessellation or "tiling".

Cox and Lo (2014) present a deeper analysis of students' numeric and non-numeric approaches to solving similarity tasks. To do this, they conducted clinical interviews with middle school students and looked at the strategies and reasoning used by students to differentiate similar figures from non-similar figures. They determined which factors influenced students’ strategy selection. Two factors were more influential: 1) "the complexity of the figures being compared"; and 2) "the type of distortion present in non-similar pairings" (p. 1). Cox and Lo (2014) concluded that students use the distortion's presence or absence to visually decide if two figures are similar. They claim visual reasoning supports students to improve their numeric reasoning strategies. They also stated that distortion is a visual activity that makes students reflect upon and evaluate the validity and accuracy of the objects during the process of differentiating figures and during the process of perceiving ratios among the lengths of corresponding sides. A summary of student's strategies is presented in Table 2.

Table 2.

Strategies and Non- Strategies Students Used Cox & Lo (2014, p.11)

Characteristic Type	Strategy	Description Two figures are similar/nonsimilar if…
Nonspecific (6,98%)	None indicated	(A student’s response was not specific enough to ascertain what characteristics of the shape the student perceived, or how the student arrived at a conclusion about similarity.)
Appearance (26,65%)	Cosmetic	… they are visually alike/different.
	Shape type	… they are different sizes of the same/different shape type.
	Relative position	… the relative positions of the sub-shapes do not/do change.
Angle (8,77%)	Angle	… corresponding angles are of equal/different measures.
Length (35,24%)	Corresponding length comparison	… one length is longer or the same as another.
	Constant gap	… there exists/does not a consistency of a gap length within two figures.
Relationship (22,36%)	Constant difference	… there is/is not a constant difference relationship between pairs of corresponding lengths.
	Constant ratio	… there exists/does not exist constant ratio between corresponding lengths.
	Qualitative relationship	… there is/is not a constant qualitative relationship in the components of two figures.
	Dilation	… one can be transformed into the other through some act of expansion or dilation.
	Tiling	… one can be transformed into the other using tessellation or “tiling”.

In another study, Lo, Cox, and Mingus (2006) conducted a conceptual-based curricular analysis of the concept of similarity. In this study they focused on three middle grade textbooks series, considering the concept definitions and concept images presented in the books. They highlight the function of the textbooks arguing the nature and the sequence of the tasks in this material "have an impact on the development of concept images that support students as they make sense of the terms 'similar figures' or 'scalar drawings' and the properties they hold" (p. 221). Lo, Cox, and Mingus (2006) observe all textbooks they analyzed have the same goal: to help students to elaborate a conceptual understanding of similarity and to be prepared to use these properties to solve contextual problems even though the textbooks have different numbers of lessons.

The authors claim two textbooks[3] analyzed introduce the similarity conception based on the context of scaling and the third textbook[4] presents two different definitions of similarity. One explores the sufficient condition for two objects to be similar and the other provides a necessary condition. Looking at concept images, Lo, Cox and Mingus (2006) assert that all curricula studied have two goals in common: to assist students to develop notions of similarity; and to solve problems by applying properties of similarity. They identified three major types of tasks to explore the notions of similarity: differentiating, measuring, and constructing. Differentiating tasks require students to decide if two figures have the same shape or to recognize in a set of figures which ones have the same shape. For these activities, students should consider the intuitive notion of same shape or properties of similarity. In measuring activities, students use direct or estimated measurements of objects to analyze patterns and relationships. In a constructing activity, students are invited to use tools like grid paper, ruler, and protractor to develop a solution to the task. The authors observe materials or space can limit student solution strategies.

Considering the similarity concept, Lo, Cox & Mingus (2006) claim similarity is "closely related to the concept of ratio and proportion" (p. 223). They add proportional reasoning is a subject extensively studied. For example, Lamon (1993) investigated students' abilities to solve problems which were classified within four different semantic types. She found the stretching/shrinking problems which required multiplicative reasoning were most difficult. Kaput & West (1994) examined students' use of informal reasoning strategies when solving missing value proportion problems and considered how their approach was influenced by task variables. Hart (1984) and Lamon (1993) analyzed student's errors and difficulties when exploring proportional tasks. Finally, Chazan (1987) identified three difficulties in learning about similarity: 1) definition of similarity, ratio and proportion; 2) dimensional growth ("students are surprised to see that area does not grow in the same ratio as sides do" (p.4)); and 3) the meaning of proportional relationships.

Students often begin formal study of similarity with extensive concept images (Tall & Vinner, 1981) related to similarity based on their experiences playing with miniature toys,

3 "Study together" and "MathScape".
4 "Connected Mathematics 2".

enlarging or shrinking images (Lehrer et al., 2002), resizing typographic fonts (Cox et al. 2007), and viewing posters or illustrations of everyday objects (van den Brink & Streefland, 1979). Research studies have documented that children as young as eight years old can employ visual perception and use pre-proportional strategies to solve similarity problems (Lehrer et al., 2002; Swoboda & Tocki, 2002; van den Brink & Streefland, 1979). This demonstrates that visual perception and unconscious conceptions of invariance play a key role in understanding geometric ratio and proportion. This ability to visually perceive the relational size of objects and extrapolate the size of such objects in a new context is largely unexplained. Freudenthal, quoted by his colleagues, says, "I go even as far as saying that congruencies and similarities are built into the part of our central nervous system that processes our visual perceptions". The mechanism by which it is built, and the method by which we perceive is to him "an enigma" (van den Brink & Streefland 1979, p. 408).

Our study builds on prior research that has analyzed similarity topics in high school textbooks in the United States (grades 9-12). We focus on analyzing similarity topics in textbooks from the United States and Brazil. In the United States, similarity is typically studied by students in grade 9 or 10. Since 2010, new standards in the US have called for a different approach to teaching similarity – one that involves geometric transformations (dilations). In Brazil, national standards also recommend an approach to similarity using dilations. It is presented in 9° grade also, but it is the last grade of middle school. National Brazilian Standards includes:

> It should also be emphasized the tasks that analyze and produce transformations and dilation of flat geometric figures, identifying their variant and invariant elements, in order to develop the concepts of congruence and similarity. These concepts should be highlighted in the Elementary Education, so that students are able to recognize the necessary and sufficient conditions to obtain congruent or similar triangles and to apply this knowledge to simple demonstrations, contributing to the formation of a type of important reasoning for Mathematics, hypothetical-deductive reasoning (Brazil, 2017, p.270).

One may describe this transformation approach as visual – an approach to similarity that has been studied and recommended by prior research (Cox & Lo, 2014). We were curious to see how mathematics textbooks in the US and Brazil were approaching similarity and how the approaches compared in textbooks in the same country and between countries.

Methods and Findings

This study is part of a bigger project that investigates Geometry in textbooks. The focus of the project reported here is to examine the approach to similarity in six textbooks, three from Brazil and three from the U.S. Our choice of textbooks was determined by several factors. With recent initiatives, Brazilian textbooks also include digital learning objects (DLO) as well as printed material (usually by CDs or websites accessed by password printed in the book). Analysis of this technology innovation is part of the larger project. In the first year of this initiative, only three of ten middle school textbooks evaluated and approved by the Government of Brazil included DLOs and they were chosen to be analyzed in the current project.

At the high school level, textbooks in the US tend to present mathematics in two different ways. Some textbooks have a single subject focus. For example, in 9th, 10th, and 11th grade students might use Algebra I, Geometry, and Algebra II textbooks. Similarity would be included in the Geometry textbook and the main focus of the text would be on Geometry topics. Other textbooks in the US present topics in an integrated manner, similar to Brazil and other countries. Students in 9th, 10th, and 11th grade might use textbooks that are titled Integrated Mathematics I, Integrated Mathematics II, and Integrated Mathematics III. We selected textbooks from the US that presented mathematical topics in an integrated manner. In these textbooks, similarity was included in an Integrated Mathematics II or III textbook. Students would typically use this textbook during the 9th or 10th grade when they are 14 or 15 years old. The US books were chosen based on their wide distribution in the United States. Two "traditional" textbooks were selected that are commonly used across the US. Another, "reform" oriented textbook was selected that is also widely used across the US. Similarity is introduced in middle school and studied in more depth at the high school level. Because the books selected from Brazil approached the content in an integrated manner, integrated high school textbooks were selected.

Table 3 shows the number of pages and percent of each book devoted to similarity. The way each textbook organizes the content is described in the Table 4. To analyze the approach each text used to address similarity we considered the explanation given by the textbook authors and the tasks proposed to the students.

Table 3.

Number of pages and percent of each book devoted to similarity

	Total number of pages in the textbook	Number of pages focused on similarity	Percentage of the book focused on similarity	Total of tasks[5] of similarity
Book 1-BR	272	21	7.7%	80
Book 2-BR	270	28	10.3%	140
Book 3-BR	240	25	10.4%	70
Book 1-USA	1322	104	7.8%	360
Book 2-USA	941	76	8.1%	538
Book 3-USA	642	71	11.0%	247

We describe sections of each textbook to illustrate the geometric structure of each, and to better understand how similarity is presented.

Table 4.

Similarity topics in each textbook

<table>
<tr><td colspan="2">BRAZIL</td></tr>
<tr><td rowspan="4">Book 1</td><td>CHAPTER 1 - Similarity
The idea of similarity
Ratio of similarity
Application of similarity</td></tr>
<tr><td>CHAPTER 2 - Similarity of triangles
A special case
The special case on practice
The case AA</td></tr>
<tr><td>CHAPTER 3 - Using the similarity of triangles</td></tr>
<tr><td>CHAPTER 4 - Side splitter theorem
Using the theorem</td></tr>
<tr><td>Book 2</td><td>UNIT 5
1. similar figures
2. similar polygons
3. similar triangles
Thales and the application of similar triangles
4. Side splitter theorem
Proof of side splitter theorem</td></tr>
<tr><td>Book 3</td><td>CHAPTER 6 - Similarity
Thumbnails
Proportional segments
Side splitter theorem
Side splitter theorem and the triangles
Similarity of figures
Dilation (homotetia)
Similar triangles</td></tr>
<tr><td colspan="2">UNITED STATES</td></tr>
</table>

5 We considered each item as a task (for example, a task with three items was considered three tasks).

	UNIT 7 - SIMILARITY AND RIGHT TRIANGLES
Book 1	MODULE 16 - Similarity and transformations 16.1 Dilations Investigating properties of dilation Dilating a line segment Applying properties of dilations Determining the center and scale of dilation 16.2 Proving figures are similar using transformations Confirming similarity Determining if figures are similar Finding a sequence of similarity transformation Proving all circles are similar 16.3 Corresponding parts of similar figures Corresponding parts of similar figures Justifying properties of similar figures using transformations Applying properties of similar figures 16.4 AA similarity of triangles Exploring Angle-Angle similarity for triangles Proving Angle-Angle triangle similarity Applying Angle-Angle similarity
	MODULE 17 - Using similar triangles 17.1 Triangle proportionality theorem Constructing similar triangles Proving the triangle proportionality theorem Applying the triangle proportionality theorem Proving the converse of triangle proportionality theorem Applying the converse of the triangle proportionality theorem 17.2 Subdividing a segment in a given ratio Partitioning a segment in a one-dimensional coordinate system Partitioning a segment in a two-dimensional coordinate system Constructing a partition of segment 17.3 Using proportional relationships Exploring indirect measurement Finding an unknown height Finding an unknown distance 17.4 Similarity in right triangles Similarity in right triangles Finding geometric means of pairs of numbers Proving the geometric means theorems Using the geometric means theorems Proving the Pythagorean theorem using similarity
	CHAPTER 9 - Proportions and Similarity 9.1 Ratios and proportions Write and use ratios Use properties of proportions 9.2 Similar polygons Identify similar polygons Use similar figures 9.3 Similar triangles Identify similar triangles Use similar triangles

Book 2	9.4 Parallel lines and proportional parts Proportional parts within triangles Proportional parts with parallel lines 9.5 Parts of similar triangles Special segments of similar triangles Triangle angle bisector theorem 9.6 Similarity transformation Identify similarity transformations Verify similarity 9.7 Scale drawing and models Scale models Use scale factors
Book 3	Book 2 (10th grade) UNIT 3: Coordinate Methods Lesson 1: A Coordinate Model of a Plane 1. Representing Geometric Ideas with Coordinates 2. Reasoning with Slopes and Lengths 3. Representing and Reasoning with Circles Lesson 2: Coordinate Models of Transformations 1. Modeling Rigid Transformations 2. Modeling Size Transformations 3. Combining Transformations Lesson 3: Transformations, Matrices, and Animation 1. Building and Using Rotation Matrices 2. Building and Using Size Transformation Matrices Lesson 4: Looking Back Book 3 (11th grade) Unit 3 Lesson 1: Reasoning about Similar Figures 1. When are Two Polygons Similar? 2. Sufficient Conditions for Similarity of Triangles 3. Reasoning with Similarity Conditions Lesson 2: Reasoning about Congruent Figures 1. Congruence of Triangles Revisited 2. Congruence in Triangles 3. Congruence in Quadrilaterals 4. Congruence and Similarity: A Transformation Approach Lesson 3: Looking Back

Different Approaches

In this section we discuss the textbook approaches. We start with Brazilian textbooks, exploring similar and different aspects. After this, we describe the U.S. textbooks. We finish comparing the two countries perspectives.

Brazil x Brazil. Brazil has a National Program to evaluate textbooks. Editors can submit textbooks and the Government, after evaluating them, creates a list of approved books. All textbooks have descriptions that are compiled in a "Textbook Guide" that is sent to all Brazilian public schools. Teachers at each school can choose the textbook that will be used. These books are then used by students for three years. It is a cycle of three years because each year is focused on a different level of evaluation: elementary, middle and high school. Chosen textbooks are sent to all public school students in Brazil free of charge. It is a big program and an expensive government investment. In Brazil, similarity is addressed in 9th grade, middle school. The approaches to similarity in the textbooks have some differences, but we observed commonalities that we will describe.

Book 1 has the following content sequence: similarity (the concept), ratios and proportions, similarity of triangles (the AA case) and the side splitter theorem (In a triangle, a segment parallel to one side of a triangle divides the other two sides of the triangle proportionally).

The chapter starts exploring the concept of similarity as "the word similarity meaning look like. But in Geometry this word has a specific meaning[...]. In Geometry the word similarity is connected with the idea of same shape" (Centurión & Jakubovic, 2012, p.10). A formal definition for similar polygons is presented based on congruent angles and proportional sides. As an example, the authors use two maps (with 4 points, associating the quadrilateral polygon), and some simple figures with three, four, and more sides. One point we would like to highlight is the fact the authors use the proportional with bigger/smaller numbers and the tasks suggest as answer smaller/bigger (Figure 1). It can be confusing to the teacher and, especially, to the students if the teacher doesn't explain this difference.

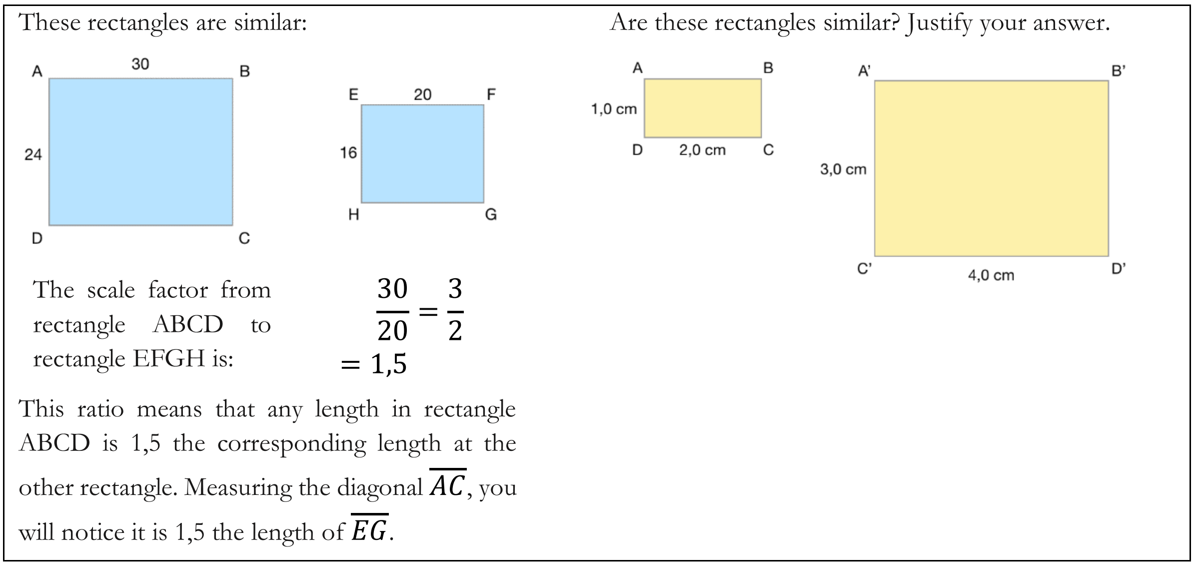


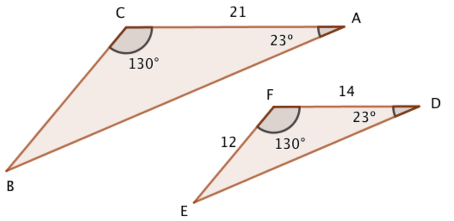
Figure 1.

Based on Centurión & Jakubovic (2012, p.12)

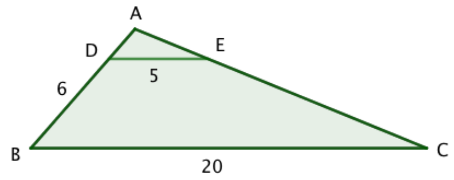
A few tasks ask students to determine if two figures are similar by examining the lengths of corresponding sides or the measures of corresponding angles. Tasks also ask students to find the ratio or proportion between to similar figures.

In chapter 2, the authors explore the importance of the angles for the similarity of two polygons. It is written that congruent angles are important for polygons to be similar, but it is not enough in general. It is enough only for triangles, and the Angle-Angle proof is presented. Tasks in this chapter are based on this property and most of them ask students to check if the two given triangles are similar, or if it is assumed they are, to find a missing measure (Figure 2).

1. Find the length of $\overline{BC}$ of the triangle ABC, after comparing it to the triangle .



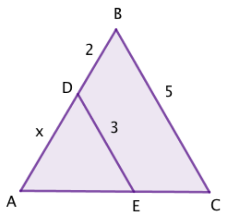
2. In this figure, $DE/\quad BC$. Find the length of $\overline{AD}$.



Hint: Compare ΔADE and ΔABC.

3. In these triangles, $DE \parallel BC$. Find the value of x

a)



b)

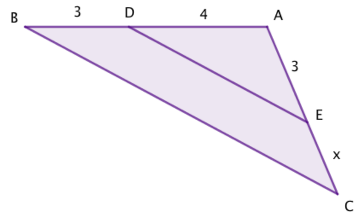
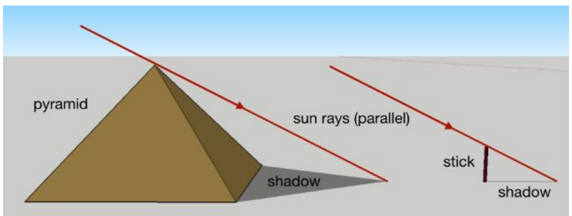


Figure 2.

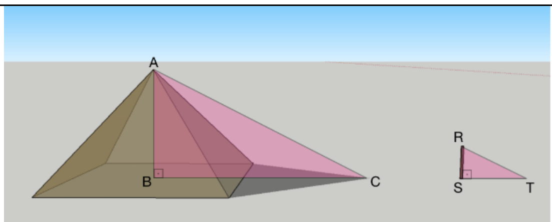
Based on Centurión & Jakubovic (2012, p.18)

More about triangle similarity is presented in chapter 2 including a problem solved by Thales in Egypt during the 6th Century (Figure 3).



Thales considered these two imaginaries triangles:

Look at the "landscape" and determine the height of the tree.



Observe these triangles:
$\hat{B} \equiv \hat{S}$ (right angles)
$\hat{C} \equiv \hat{T}$ (angles of inclination of the sun's rays)
So by AA case: $\Delta ABC \sim \Delta RST$

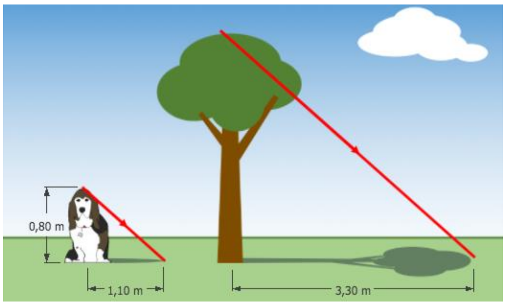


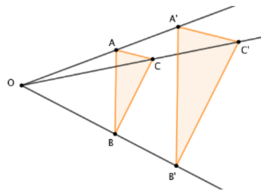
Figure 3.

Based on Centurión & Jakubovic (2012, p.22)

The last chapter of similarity content focuses on the side splitter theorem. The task asks students to practice the theorem and some street maps are proposed as context problems.

Dilations are mentioned on the last page of the chapter, in a "challenge task" (Figure 4).

5. Triangle A'B'C' is an enlargement of triangle ABC, by a dilation centered at O with a scale factor of 1,6.



Which elements preserves the same measures after a dilation (enlargement or reduction)?

Figure 4.

Based on Centurión & Jakubovic (2012, p.30)

Like Book 1, Book 2 presents similarity in the following sequence: similar figures, similar polygons, similar triangles and side splitter theorem. The chapter begins with the claim that "In everyday life, we call similar such things or ideas that have the same or very similar elements" (Leonardo, 2010, p.82). As an example, the authors show a photo in two different sizes, as enlargement and reduction, saying "in both situations remains the original form, what changes is only the size" (p.82). After this, the author states figures are similar "when the measurements of the corresponding segments are proportional and measures of the corresponding angles are equal" (p.83).

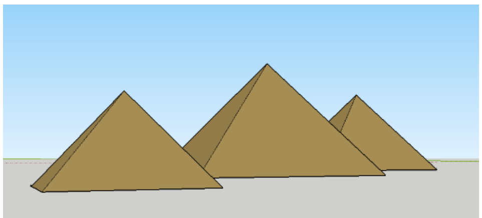
After that, author presents three cases of similarity of triangles and based at the same story about pyramid and Egypt to describe the Thales application of triangles (Figure 5).

Thales and a similar triangle application

Little is known about Thales of Miletus (624-548 BC), mathematician, philosopher and Greek scientist. However, several accounts of eras later than the one in which the philosopher lived record that he made several calculations on the height of a pyramid observing its shadow.

Do you know the pyramids of Egypt? They are very old constructions made of large blocks of stone to serve as a tomb for the pharaohs, that is, they were built to house the mummified body of a ruler, as well as his most valuable objects.

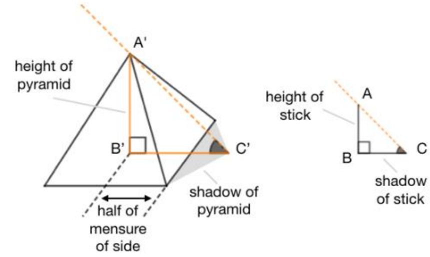
It is said that, around 600 BC, Thales, on one of his trips to Egypt, was challenged by the pharaoh to calculate the height measure of the Khufu pyramid.

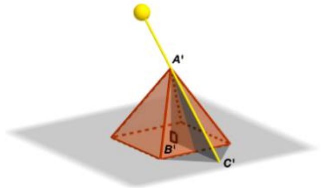


On a sunny day, Thales noted that the square-shaped pyramid formed a shadow, like is shown in the diagram at the side. At that moment, Thales planted a stick on the ground and realized that it also cast a shadow.

He waited until the height and shadow of the stick had the same measure and asked one of his helpers to immediately measure the length of the shadow and measure the side of the pyramid.

Analysing the measures found, and based on the fact that the solar rays are parallel, the philosopher had the idea to make a similar scheme to this one:





Triangles $A'B'C'$ and ABC are similar, because they have one right angle and the sun rays strike the objects with the same angle.

Since the triangles are similar, the measurements of the corresponding sides are proportional.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC}$$

Therefore, Thales concluded as the measure of the height of the stick and the length of its shadow were equal, the height of the pyramid was equal to the length of its shadow plus half of the length of its side.

Figure 5.

Based on Leonardo (2010, p.90-91)

The side splitter theorem and its proof is also presented, and a task with a street map. A set of 14 contextual tasks are present to finish the chapter.

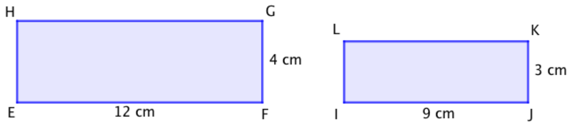
The content sequence in Book 3 is: thumbnails, proportional segments, side splitter theorem, similarity of figures, dilation, similar triangles (the three cases of similarity). This book is different from the others. It starts with the idea of thumbnails and proportional segments, and then explains what a proportion is. As we commented in book 1, here we could note authors present the proportion most of the time as bigger/smaller numbers and the tasks explore more the use of smaller/bigger (Figure 6).

In this chapter, we will study the scale factor between line segments. To do this, consider the following line segments.

The ratio between segments AB and BC is given by dividing the AB measure by the CD measure. $\frac{AB}{CD} = \frac{8}{2} = 4$

Therefore, the ratio between AB and CD segments is 4, that is, $\frac{AB}{CD} = 4$.

Now consider the rectangles EFGH and IJKL.



Let's calculate the scale factor of the line segments EF and FG, and of IJ and JK.

$\frac{EF}{FG} = \frac{12}{4} = 3$ $\frac{IJ}{JK} = \frac{9}{3} = 3$

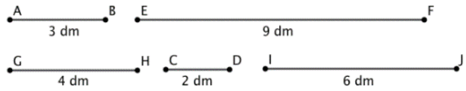
So, $\frac{EF}{FG} = \frac{IJ}{JK}$, that is, the scale factor of the line segments are equal. In this case, we say the line segments EF and FG are proportional to the line segments IJ and JK.

The scale factor between two segments is given by the division by dividing the measures of the line segments, considering the same unit of measurement.

The scale factor of $\overline{AB} e \overline{BC}$ é $2,5, pois \frac{AB}{BC} = \frac{5}{2} = 2,5$

Tasks:

1. According to the line segments, check which equalities are true.



a) $\frac{AB}{IJ} = \frac{GH}{EF}$ e) $\frac{AB}{EF} = \frac{CD}{GH}$

b) $\frac{IJ}{GH} = \frac{AB}{CD}$ f) $\frac{IJ}{CD} = \frac{CD}{GH}$

c) $\frac{IJ}{AB} = \frac{GH}{CD}$ g) $\frac{GH}{IJ} = \frac{AB}{CD}$

d) $\frac{EF}{CD} = \frac{IJ}{GH}$ h) $\frac{EF}{IJ} = \frac{AB}{CD}$

2. Determine the scale factor between line segments.

a) AB = 5cm e BC = 15cm
b) CD = 4cm e CE = 10cm
c) AC = 20cm e CD = 4cm
d) EF = 8m e FG =16m
e) EG = 24m e CD = 4m
f) FG = 16cm e AC = 20cm

Figure 6.

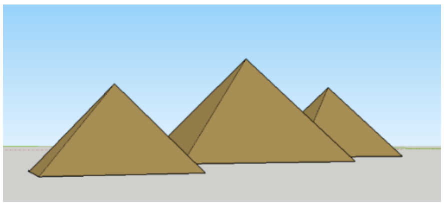
Based on Souza & Pataro (2013, p. 124-125)

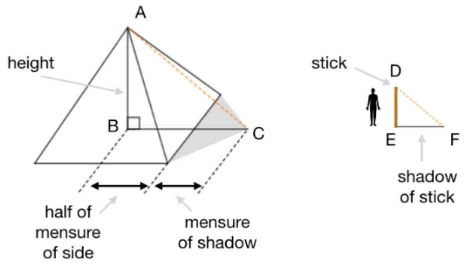
After this content, the book explores the side splitter theorem, showing a particular proof (that works just in a specific case with some particular conditions), and proposes tasks, with the map of street as the only contextual task in 33 tasks related to this subject. In this section there is a task that explains how to divide a segment in equal parts, using the set-square.

Following this, the book introduces the concept of similarity of polygons, based in the congruent sides and proportional angles, like the other textbooks. A few pages are devoted to dilation, using the compass (4 tasks) and the computer (2 examples). Cases of similarity are presented, with no proof, and the problem of Thales in Egypt is presented, inside a task using the same solution (Figure 7).

According to reports, Thales of Miletus lived in Egypt, for a while, where he aroused admiration when calculating the height of a pyramid.

Representing this situation using a mathematical model, we obtain triangles ABC and DEF.





To accomplish this feat, Thales grounded a stick of known size on the ground, and at the time the stick shadow size was the same size as the stick, he also calculated that the height of the pyramid would be equal to the length of its shadow plus half of the length of its base.

A. Are these triangles similar? If so, what is the case of similarity?
B. According to these triangles, what equality can we write to get the height of the pyramid?

Figure 7.

Based on Souza & Pataro (2013, p. 141)

The contextual problem with a shadow (Figure 8), as the other textbooks, is presented in one task. All the chapters have in the end a set of questions called "reflecting about the chapter" that pose interesting questions to discuss with students during class (eg. "What is the difference of congruence and similarity?", "why do you do the dilation transformation?", "Based on contents of this chapter, write some questions about them. Sit with your colleagues, exchange your questions and discuss the solutions").

At a certain time of day, a 1.65m person projects a 1.16m shadow. At that moment, a tree projects a shadow of 13.92m. How high is the tree?

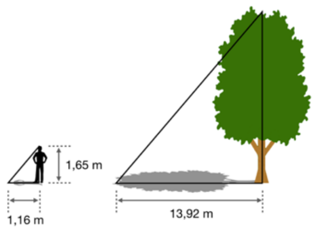


Figure 8.

Based on Souza & Pataro (2013, p.142)

Looking at Brazilian textbooks we can see they consider the same definition of similarity (based on the same measure of angles and proportional measure of sides) and they have almost the same sequence to explore the subject. The study of similarity is based on the idea of similar polygons, taking some special attention to triangles and the cases of similarity, Thales and his problem at Egypt and the side splitter theorem.

Dilation is explored only in Book 3, with few tasks. Book 1 mentions it in one task, but superficially. Most of the tasks ask students to check if the figures are similar or state they are similar and ask students to find missing measurements. All of them have few tasks with contextual problems which little more than 50% are high level tasks (author 1 & author 2, 2017).

USA x USA. The first textbook has two chapters devoted to similarity and proportion. The sequence of topics in this textbook is: dilations, proving figures are similar using transformations, connecting angles and sides of figures, the AA (Angle-Angle) case of similarity, the SAS (Side-Angle-Side) and SSS (Side-Side-Side) cases of similarity, the triangle proportionality theorem, subdividing a segment in a given ratio, proportional relationships, similarity in right triangles. The first unit begins with a chapter entitled "similarity and transformation." The question used to motivate the chapter is "how can you use similarity and transformation to solve real-world problems?" (p. 825). Transformations are described using rules that are applied to coordinates of geometric figures (Figure 9).

Properties of transformations

Example 1: Stretch ΔABC with points A(1,2), B(3,2), and C(3,-1) horizontally and vertically by factor of 4.

(x,y) → (4x, 4y) Write the transformation rule
A’(4,8), (B’(12,8), C’(12,-4) Use the transformetion to write each transfermed point

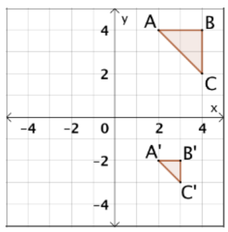
Describe the transformation.

1. Stretch ΔDEF with points D(-2,1), E(-1,-1), and F(-2,-2) horizontally and vertically by factor of 3.
2. Is the stretch a rigid motion?
3. Is it true that ΔDEF □ ΔD’E’F’?

Similar figures

Example 2: Transform ΔABC with points A(3,4), B(-1,6), and C(0,1) by shifiting it 2 units to the right and 1 unit up.

(x,y) → (x+2, y+1) Write the transformation rule
A’(5,5), (B’(1,7), C’(2,2) Write the transformed point



Describe the transformation shown in the graph.

4. Write the rule used to transform ΔABC.
5. Describe in words the transformation shown in the figure. Answers may vary. Sample: Compress ΔABC vertically and Horizontally by a factor 2, and then shift it to the right 1 unit and down 4 units.

Figure 9.

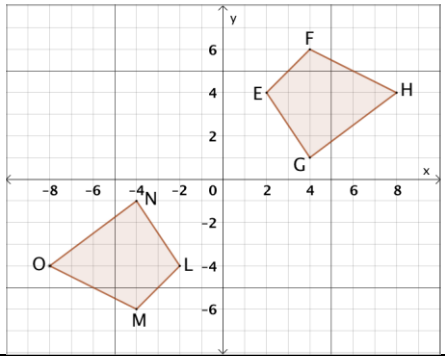
Based on Kanold et al. (2015, p. 826)

The first topic explored is dilation. "A dilation is a transformation that can change the size of a polygon but leaves the shape unchanged" (p. 827). Directions are given about dilation properties, how dilating a line segment, how to apply properties of dilations, how to determine the center and the scale factor. Similarity is defined to include congruence.

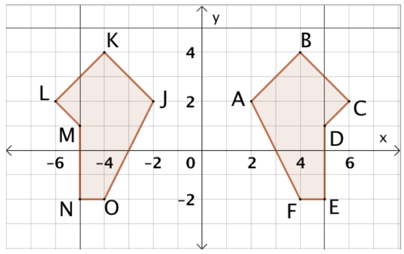
> a similar transformation is a transformation in which an image has the same shape as its image. Similarity transformations include reflections, translations, rotations, and dilations. Two plane figures are similar if and only if one figure can be mapped to the other through one or more similarity transformation (p. 837).

Some examples (Figure 10) are given to show how similarity is explored using different polygons on the coordinate plane.

A – Trace patio EFHG. Turn the paper so that patio EFHG is mapped onto patio LMON. Describe the transformation. What does this confirm about the patios?



B- Trace statues ABCDEF and JKLMNO. Fold the paper so that statue ABCDEF is mapped onto statue JKLMNO. Describe the transformation. What does this confirm about statues?



C- Describe the transformation you can use to map vertices of garden RST to correspondingvertices of garden DEF. What does this confirm about the gardens?

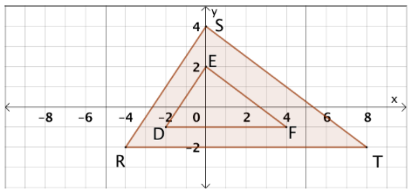


Figure 10.

Based on Kanold et al. (2015, p. 837-838)

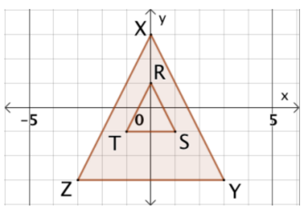
The use of dilations and algebraic are present to students to determine if two figures are similar (Figure 11).

Explain 1: Determining if figures are similar

You can represent dilations using the coordinate notion (x, y) → (kx, ky), where k is the scale factor and the center of dilation is the origin. If 0 < k < 1, the dilation is a redution. If k > 1, the dilation is an enlargement.

Example 1: Determine whether the two figures are similar using similarity transformation. Explain.

A- ΔRST and ΔXYZ



To map ΔRST onto ΔXYZ, there must be some factor that dilates ΔRST.

Pre-image	**Image**
R(0,1)	X(0,3)
S(1,-1)	Y(3,-3)
T(-1,-1)	Z(-3,-3)

Figure 11.

Based on Kanold et al. (2015, p. 838)

Explain 2 states that sometimes it is necessary to use more than one transformation to describe the mapping of one figure to the other.

In the third section, similarity is described in terms of angles and sides. They state that figures are similar if the sides lengths are proportional and the angles measures are equal. This definition is explored in most of the tasks and connected with algebraic descriptions of transformations (Figure 12).

14. Which transformations will not produce similar figures? Select all the apply and explain your choices.
A. (x, y) → (x–4, y) → (-x, -y) → (8x, 8y)
B. (x, y) → (x+1, y+1) → (3x,2y) → (-x, -y)
C. (x, y) → (5x, 5y) → (x, -y) → (x+3, y-3)
D. (x, y) → (x, 2y) → (x+6, y-2) → (2x, y)
E. (x, y) → (x, 3y) → (2x, y) → (x-3, y-2)

Consider this model of a train locomotive when answering the next two questions.
21. If the model is 18 inches long and the actual locomotive is 72 feet long, what is the similarity transformation to map from the model to the actual locomotive? Express the answer using the notations x → ax, where *x* is a measurement on the model and *ax* is the corresponding measurement on the actual locomotive.

22. If the diameter of the front wheels on the locomotive is 4 feet, what is the diameter of the front wheels on the model? Express the answer in inches.

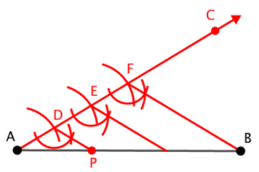
Figure 12.

Based on Kanold et al. (2015, p. 856-858)

The next section explores the AA triangle similarity theorem. Based on it, the SSS similarity theorem and the SAS similarity theorem are introduced. Tasks are proposed before the section that engages students in the triangle proportionality theorem. Contextual problems ask to apply the theorem in maps. Subdividing a segment in a given ratio is the focus of the next section (Figure 13). It is studied in one- and in two-dimensional coordinate system. Proportional relationships are explored and examples and tasks examine shadows, statues, and other objects (eg. Figure 14). The last section addresses similarity in right triangles.

Given the directed line segment from A to B, construct the point P that divides the segment in the given ratio from A to B.

9. 1 to 2



10. 3 to 2

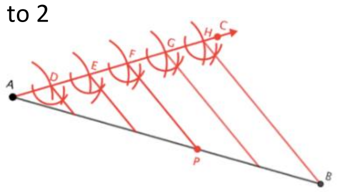
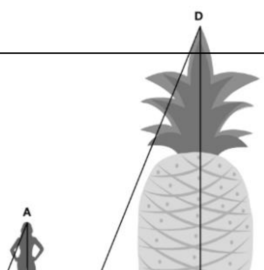


Figure 13.

Based on Kanold et al. (2015, p.895)

12. A student wanted to find the height h of a statue of a pineapple in Nambour, Australia. She measured the pineapple's shadow and her



own shadow. The student's height is 5 feet 4 inches. What is the height of the pineapple? ΔABC ≈ ΔDEP by the AA Similarity Criterion. AC = 5ft 4in. = 64in. BC = 2ft = 24in. EF = 8ft 9in. = 105in. $\frac{AC}{DF} = \frac{BC}{EF\prime}; \frac{64}{h} = \frac{24}{105}; \frac{h}{64} = \frac{105}{24}; 64\left(\frac{105}{24}\right) = 280$ The height h of the pineapple is 280 inches or 23 feet 4 inches.	

Figure 14.

Based on Kanold et al. (2015, p.908)

Book 2 follows the following sequence of similarity topics: ratio and proportion, similar polygons, similar triangles, fractals, similarity transformation, scale drawings and models.

This textbook starts by exploring ratios and proportions. The textbook states that "ratio is a comparison of two quantities using division" (p. 543). Some contextual and non-contextual examples are presented. The proportion is introduced as "an equation stating that two ratios are equal" (p. 544) followed by additional examples.

After this, a second section is presented to explore similar polygons, introducing the idea that "similar polygons have the same shape but not necessarily the same size" (p. 551). It is suggested to use scale factors and proportions in tasks involving similarity. The Figure 15 is an example.

Example 3. Use similar figures to find missing measures
In the diagram, ACDF ~ VWYZ.
a. Find x.
Use the corresponding side lengths to write a proportion.

$\frac{CD}{WY} = \frac{DF}{YZ}$ Similar proportion

$\frac{9}{6} = \frac{x}{10}$ CD = 9, WY = 6, DF = x, YZ = 10

9(10) = 6(x) Cross products property

90 = 6x Multiply

15 = x Divide each side by 6

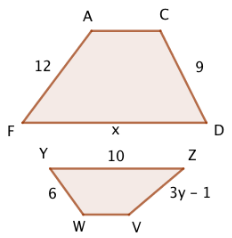


Figure 15.

Based on Carter et al. (2012, p.553)

The next section focuses on similar triangles, starting with the claim "two triangles are similar if two pairs of corresponding angles are congruent" (p. 560). The three cases of similarity are introduced. Two other theorems are set up: Triangle Proportionality Theorem and Triangle Midsegment Theorem. The side splitter theorem explored in Brazilian textbooks is introduced too, but it is done by a Corollary 1 about proportional parts of parallel lines: "if three or more parallel lines intersect two transversals, then they cut off the transversals proportionally" (p.574), and the Corollary 2: "if three or more parallel lines cut off congruent segments on one transversal, then cut off congruent segments on every transversal" (p.575), as shows the Figure 16.

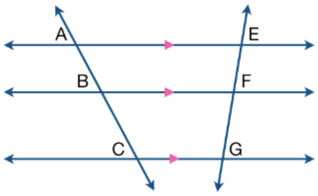
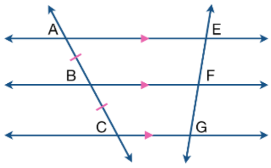
Corollary 9.1 Proportional parts of parallel lines. Is there or more parallel lines intersect two transversals, then they cut the transversals proportionally. Example If $\overline{AE}/$ $\overline{BF}/$ $\overline{CG}$, then $\frac{AB}{BC} = \frac{EF}{FG}$	Corollary 9.2 Congruent parts of parallel lines. If three or more parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal. Example If $\overline{AE}/$ $\overline{BF}/$ $\overline{CG}$, and $\overline{AB} \cong \overline{BC}$, then $\overline{EF} \cong \overline{FG}$
	

Figure 16.

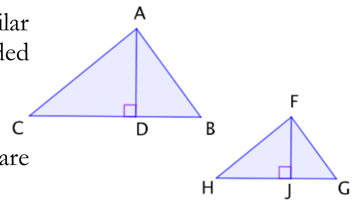
Based on Carter et al. (2012, p.574-575)

The use of formalism is very present in this textbook. After the Corollary above, more theorems about special segments of similar triangles and triangle angle bisector are presented (Figure 17 and 18) with a proof later:

Special segments of similar triangles.
You learned in Lesson 9-2 that the corresponding side lengths of similar polygons, such as triangles, are proportional. This concept can be extended to other segments in triangles.
Theorems. Special segments of similar triangles

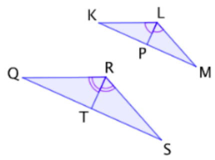
9.8 If two triangles are similar, the lengths of corresponding altitudes are proportional to the lengths of corresponding sides.

Abbreviation ~Δs have corr. altitudes proportional to corr. sides.
Example If ΔABC ~ ΔFGH, then $\frac{AD}{FJ} = \frac{AB}{FG}$.



9.9 If two triangles are similar, the lengths of corresponding angle bisectors are proportional to the lengths of corresponding sides.

Abbreviation ~ Δs have corr. ∠ biserctors proportional to corr. sides.
Example If ΔKLM ~ ΔQRS, then $\frac{LP}{RT} = \frac{LM}{RS}$.



9.10 If two triangles are similar, the lengths of corresponding medians are proportional to the lengths of corresponding sides.

Abbreviation ~ Δs have corr. medians proportional to corr. sides.
Example If ΔABC ~ WXY, then $\frac{CD}{YZ} = \frac{AB}{WX}$.

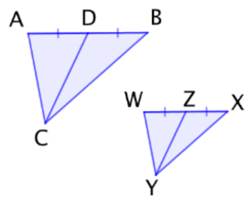


Figure 17.

Based on Carter et al. (2012, p.583)

Theorem 9.11 Triangle angle bisector
An angle bisector in triangle separates the opposite side into two segments thar are proportional to the lengths of the other two sides.

Example If $\overline{JM}$ is an angle bisector of ΔJKL, then $\frac{KM}{LM} = \frac{KJ}{LJ}$.

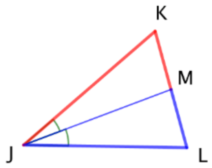


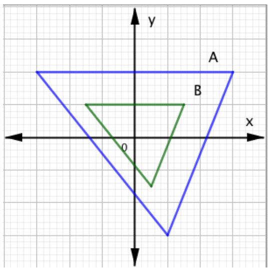
Figure 18.

Based on Carter et al. (2012, p.586)

The next section is dedicated to fractals. After that, a new section connects similarity and transformations. Dilations are introduced as a type of similarity transformation. The types of dilations, enlargement and reduction are defined in a sequence and examples and tasks explore some math or context problems about them. Most of the tasks are similar to these examples (Figure 19):

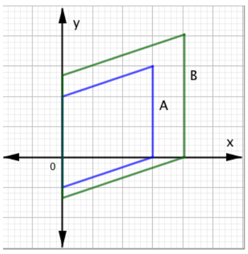
Example 1. Identify a dilation and find its scale factor
Determine whether the dilation from A to B is an enlargement or a reduction. Then find the scale factor of the dilation.

a.



B is smaller than A, so the dilation is a reduction.
The distance between the vertices at (-3, 2) and (3, 2) for A is 6 and from the vertices at (-1.5, 1) and (1.5, 1) for B is 3. So the scale factor is 3/6 or 1/2.

b.



B is larger than A, so the dilation is an enlargement.
The distance between the vertices at (3, 3) and (3, 0) for A is 3 and between the vertices at (4, 4) and (4, 0) for B is 4. So the scale factor is 4/3.

Figure 19a.

Based on Carter et al. (2012, p.594)

Practice and Problem Solving
Example 1. Determine whether the dilation from A to B is an enlargement or a reduction.
To find the scale factor on the dilation.

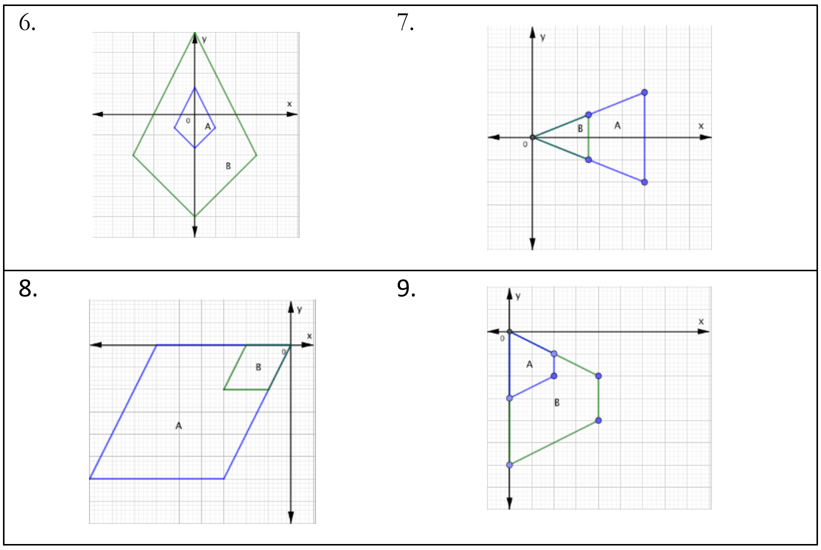


Figure 19b.

Based on Carter et al. (2012, p.596)

3. Graph the original figure and its dilated image. Then verify that the dilation of a similarity transformation.

14. M(1, 4), P(2, 2), Q(5, 5); S(-3, 6), T(0, 0), U(9, 9)

15. A(1, 3), B(-1, 2), C(1, 1); D(-7, -1), E(1, -5)

16. V(-3, 4), W(-5, 0), X(1, 2); Y(-6, -2), Z(3, 1)

17. J(-6, 8), K(6, 6), L(-2, 4); D(-12, 16), G(12, 12), H(-4, 8)

Figure 19c.

Based on Carter et al. (2012, p.597)

Example 3. Verify similarity after dilation

Graph the original figure and its dilated image. Then verify that the dilation is a similarity transformation.

a. **Original**: A(-6, -3), B(3, 3), C(3, -3); image: X(-4, -2), Y(2, 2), Z(2, -2)

Graph each figure. Since ∠C and ∠Z are both right angles, ∠C ≅ ∠Z. Show that the lengths of the sides that include ∠C and ∠Z are proportional.

Use the coordinate grid to find the side lengths.

$\frac{XZ}{AC} = \frac{6}{9}$ or $\frac{2}{3}$, and $\frac{YZ}{BC} = \frac{4}{6}$ or $\frac{2}{3}$, so $\frac{XZ}{AC} = \frac{YZ}{BC}$.

Since the lengths of the sides that include ∠C and ∠Z are proportional, ΔXYZ ~ ΔABC by SAS Similarity.

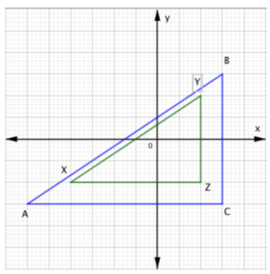


Figure 19d.

Based on Carter et al. (2012, p.598)

Scale drawings and models are the focus of the last section. Miniatures, statues and maps are explored in contextual problems.

In the third textbook, similarity is first presented in Book 2 (10^{th} grade) in a unit focused on animation, transformations, and the coordinate plane. Students use coordinates to describe translations, reflections, rotations, and size transformations centered at the origin in the plane. Students explore compositions of transformations using coordinates in the plane. The size transformation is later named a similarity transformation and the following description is provided:

The composition of a rigid and a size transformation is called a similarity transformation.

i. Corresponding segment lengths of the image are changed (magnified or reduced) by the scale factor.
ii. Corresponding angle measures are the same for the preimage and image shape.
iii. The area of the image is the scale factor squared times the preimage area (p. 217).

The book then introduces matrix representations of transformations, including similarity transformations.

In Book 3 of this series (11^{th} grade) the sequence of topics are: similar polygons; conditions for similarity of triangles; reasoning with similarity conditions; applications, connections, reflections and extensions; congruence and similarity - a transformation approach.

The first lesson of unit 3 of the textbook explores "reasoning about similarity". It starts with Escher figures, relating different triangles with same shape and different sizes (Figures 20 and 21). In a sequence, a figure shows a projector with the enlargement of an image and it is aimed that its factor from the original image depends essentially on the distance from the projector to the screen (Figure 22). Two questions are proposed: "how can you test whether two polygons are similar? How can you create a polygon similar to a given polygon" (p.164). To study these questions tasks (part of them related with the real world) are given to make students investigate about similar polygons. In one task it is defined that "two polygons with the same number of sides are similar provided their corresponding angles have the same measure and the ratios of lengths of corresponding sides is a constant" (p.165).

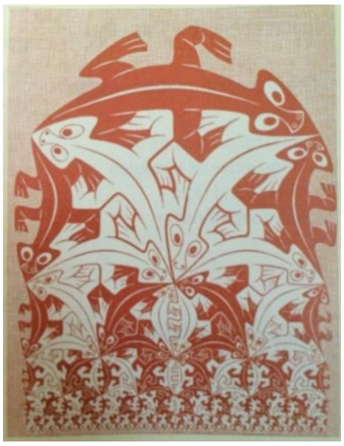


Figure 20.

Hirsch et al. (2015, p.162)

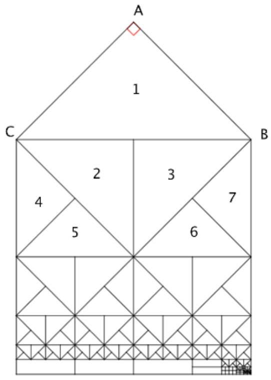


Figure 21

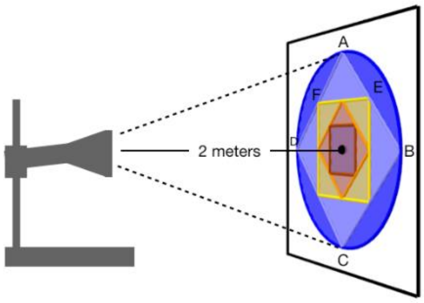


Figure 22

Based on Hirsch et al. (2015, p.162-163)

In the sequence, sufficient conditions for similarity of triangles is explored. After a suggestion of the use of an app to observe what happens to the angles and side measures of the triangles with dragging, students are invited to examine more explicitly the transformation-based test to check if two triangles are similar, analyzing the minimal conditions that will ensure that they are similar. Again, a set of tasks are proposed to investigate one preview question: "what combination of side or angle measures are sufficient to determine that two triangles are similar?" (p.169). During the tasks some theorems are explored (eg. Figure 23) that summarized in the end of the lesson. The cases of similarity of triangles are studied on this approach.

4. Suppose you know that in ΔABC and ΔDEF, $m\angle C = m\angle F$, $d = k \cdot a$, and $e = k \cdot b$.
   a. Could you prove that $\Delta ABC \sim \Delta DEF$? If so, explain how. If not, explain why not.
   b. Complete the following statement.
      If an angle of one triangle has the same measure as an angle of a second triangle, and if the lengths of the corresponding sides including these angles are multiplied by the same scale factor k, then…
   c. Compare your statement in Part b with your classmates.

The conclusion you reached in Part c of Problem 4 is called the Side-Angle-Side (SAS) Similarity Theorem. This theorem gives at least a partial answer to the question of finding minimal conditions that will guarantee two triangles are similar. In the next several problems, you will explore other sets of sufficient conditions.

Figure 23.

Based on Hirsch et al (2015, p.170)

After sufficient conditions are considered, similarity and proportionality is presented. Resources are suggested as the used of pantograph. Transformations are included, with the use of technology suggested (Figure 24). Another set of tasks are proposed with the goal to develop connections between similarity and other contents, as perimeters and areas in similar figures for example (Figure 25).

Size transformation principle. To size transform a set of points, you find the size-transformation image of each point in the set.

7. Now conduct each of the following experiments using interactive geometry software or paper, pencil, and a ruler.

a. Draw a ΔPQR as show at the right. Mark and label any two points on $\overline{PQ}$. Repeat for sides $\overline{QR}$ and $\overline{PR}$. Choose a point C in the exterior of ΔPQR as the center of a size transformation with magnitude 2.5. Find and label the image of ΔPQR and the six points on its sides.

i. Does collinearity of points appear to be preserved by this size transformation? Does betweenness of points appear to be preserved? Compare your finding with those of others and agree on written if-then summary statements.

ii. What kind of figures are the images of $\overline{PQ}$, $\overline{QR}$, and $\overline{PR}$? How are the lengths of the images related to the lengths of their preimages?

iii. What kind of figures are the images of ∠PQR, ∠QRP, and ∠RPQ? How are the measures of images and preimages related?

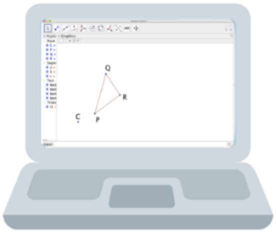


Figure 24.

Based on Hirsch et al (2015, p.177)

17. There are important connections between the perimeters of similar figures and between the areas of similar figures.

a. Suppose ΔP'Q'R' is the image of ΔPQR under a size transformation center C and magnitude k.

i. Write an argument to prove that perimeter ΔP'Q'R'= k (perimeter ΔPQR).

ii. Use a result from Connections Task 13 to help prove that area ΔP'Q'R'= k² (area ΔPQR)

b. Suppose one polygon is the image of another polygon under a size transformation with center C and magnitude k.

Figure 25.

Hirsch et al (2015, p.178)

Some tasks study properties of fractals. Extensions are also included. Most of them based in tasks exploring algebraic problems and relating tasks ("in connection task ...") as the example below.

14. Look back at your work for Applications Task 5. Since ΔABD ~ΔBCD, it follows that $\frac{AD}{BD} = \frac{BD}{CD}$.

Note that BD appears twice in the proportion. The length of $\overline{BD}$ is the *geometric mean* of the lengths of $\overline{AD}$ and $\overline{CD}$.

a. Using the language of geometric mean, state a theorem about the altitude to the hypotenuse of any right triangle.

b. The **geometric mean** of two positive numbers a and b is the positive number x such hypotenuse of any right triangle.

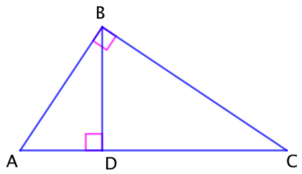


Figure 26.

Based on Hirsch et al (2015, p.186)

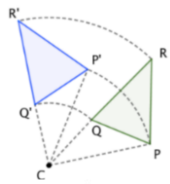
27. In Connections Task 14, you conjectured and the provided a geometric proof that if a and b are two positive numbers, their arithmetic mean is greater than or equal to their geometric mean. Use the facts that $(a + b)^2 \geq 0$ and $(a - b)^2 \geq 0$ along with algebraic reasoning to prove the **arithmetic-geometric mean inequality**, $\frac{a+b}{2} \geq \sqrt{ab}$

Figure 27.

Based on Hirsch et al (2015, p.191)

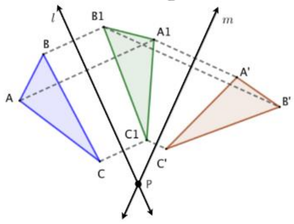
The next lesson is focused on congruence. One of the sections is dedicated to "congruence and similarity: a transformation approach" (p. 208). More questions are proposed to explore the content, for example "How can you use the composite of a size transformation and rigid transformation to prove sufficient conditions for similarity of triangles?" (p.208). Some connections between reflection, rotation and translation are investigated in the tasks.

The Reflection-Rotation Connection. Recall that a rotation is a transformation that "turns" all points in the plane about a fixed center point through a specified angle. Thais is, if points P' and Q' are the images of points P and Q under a counterclockwise rotation about point C, then CP = CP', CQ = CQ', CR = CR', and m∠PCP' = m∠QCQ' = m∠RCR'.



In Problems 3 and 4, you will investigate connections between line reflections and rotations.

3. The diagram below shows ΔA'B'C' the image of ΔABC under a composition of two reflections across intersecting lines, first a reflection across l and then a reflection across m.
   a. How does ΔA'B'C'appear to be related to ΔABC by position?



b. A portion of the diagram at the bottom of the previous page is reproduced here.
   i. Explain why PA = PA' and PB = PB'
   ii. Explain why m∠A'P'B' = m∠APB
   iii. Explain why m∠APA' = m∠BPB'.

c. Suppose X is a point on l (X ≠ P) and X' is the image of X under the composition of a reflection across line *l* followed by a reflection across line *m*.
   i. Draw a diagram.
   ii. Explain why it is also the case that m∠XPX' = m∠APA'.

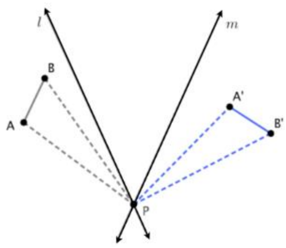


Figure 28.

Based on Hirsch et al, 2015, p.210 and 211)

When we look across the three textbooks from the United States we notice similarities and differences.

In one US textbook, students are first introduced to similarity by considering corresponding sides and angles and their lengths. The other two textbooks introduce similarity using dilations. All three books use coordinates to represent polygons in the plane and students examine the effect dilations have on the coordinates. Only one of the US textbooks introduces students to matrix representations of polygons and describes how multiplying a matrix by a scalar is one way to represent how a size transformation is applied to a polygon. While all three textbooks describe properties of similar figures, the second book is the most formal with the presentation of theorems and proofs. All three books also make use of context in the application of similarity to the real world.

Brazil x USA. As we observed, Brazilian and U.S. textbooks have approach similarity differently. The Brazilian textbooks chose have almost the same approach, based the same definition specially that is static, as define Cox and Lo (2014). The transformations are not explored, they are just mentioned in few tasks, but not in a clear connection with similarity. In U.S. textbooks the transformations are explored, but in a different way, as we discussed before.

The biggest difference we can observe is "static x dynamic" approach. When we consider two figures are similar when they have same angles and proportional sides, we are looking to the static properties of the figures. When we consider two figures are similar if one figure can map the other, we are exploring the movement, the possibility to drag the figure. It is based on the transformation, especially on the concept of dilation, to find the same shape and size. This dynamic approach is present in U.S. textbook while the static bases the Brazilian textbooks (even static is presented at U.S. textbooks too).

Considering this specific difference, other differences are consequence. The coordinate system is not mentioned in Brazilian textbook connected with similarity. If the translation is not studied, this system is not necessary. In the U.S. textbooks, it is not explored in the same approach. As we said before, textbook 2 and 3 just use the idea of dilation, but not the other transformation (which this system is essential).

Another difference is the number of pages of U.S. and Brazilian textbooks. This fact impacts the number of tasks to explore the content (290 in Brazil and 1300 in the U.S.). It can be a good point of discussion in a teacher training, reflecting about the practicing of the teacher preparing the class (eg. Is it better to have a lot of tasks? Is it good to have more options or it is one more effort to the teacher choosing the tasks?)

Because both Brazilian and U.S. textbooks explore the "static definition" of similarity we can identify commonalities, in particular the ways in which problems are posed within contexts. Some of them are based in the same context as shadow and maps. Tasks exploring the goal of identify if two figures are similar or assuming they are the students have to find the missing measure is either present in both textbooks.

Discussion

The purpose of this study was to examine three textbooks from Brazil and three textbooks from the United States to answer: how do textbooks in the United States and textbooks in Brazil present similarity to students? While there were commonalities in the theorems that were presented across all six textbooks (e.g., Side Splitter Theorem) there were also differences in the way the content was sequenced, the types of tasks that were presented, the emphasis on proof, use of coordinates and geometric transformations, and the presentation of similarity using numeric or visual approaches.

The sequence of topics varied in terms of whether the topic was first introduced formally, through theorems, or informally through the use of contextual based problems and real-world situations. There appears to be more consistency in Brazil in the sequence of the topics. All three books begin by discussing the concept of similarity in general and then apply it to similar triangles with the side-splitter theorem appearing later in the sequence. There is a focus on the lengths of corresponding sides and measures of corresponding angles which could be described as numeric. In the US, all three books begin by exploring similar figures before similar triangles. However, there are differences in how proof is approached. In one book the AA similarity theorem is introduced early in the sequence and later in a different book. A third book begins with the use of coordinates.

All six books seem to approach the concept of similarity numerically (Cox, 2013). This numeric approach may be in the form of ratios and proportions (common in Brazil). As Lehrer, Strom and Confrey (2002) and Lo, Cox and Mingus (2006) highlight textbooks contribute to students' integration of multiple conceptions of similarity that can include thinking about ratio and scale. US books connected similarity to dilations in the plane with the use of coordinates. Although they attends more to transformations, which could be described as visual (that Lehrer, Strom and Confrey (2002) noted that helps students to understand the ratio and proportion concept, and later the similarity concept), the way they are implemented focuses on numeric values associated with lengths of sides and angles, which is numeric. As Cox (2013) discussed similarity can connect geometry and number, and visual-based strategies can facilitate students' learning.

Silva (2005) notes that in the 1960s Brazil tried to give Geometry an axiomatic treatment, with a focus on algebraic structures, set theory, and geometric transformations. In practice, the author observed that teachers were not implementing geometric transformations and matrices. Rather they perpetuated the traditional teaching of Euclidean geometry. We note that the contemporary treatment of similarity in Brazil has a focus on the numeric approaches to similarity. A similar movement, "the new math" occurred in the US in the 1960s. There was a focus on set theory and formal algebra in curricula. Teachers were unprepared to implement the more formal approach and it was replaced with "back to basics" curriculum in the 80s. The treatment of geometry during this era focused on axioms and proofs. Since then, standards called for a de-emphasis on two-column proof and a focus on reasoning and justification. Recent standards also call for a transformation approach to congruence and similarity. There is still evidence of formal proof in some textbooks and the ways which transformations are treated vary from a coordinate/numeric approach.

Although all six textbooks presented students to the same concept, similarity, the way in which it was presented varied. How these varied presentations of the concepts influence students' learning and conceptualization of similarity would be an interesting study to pursue. Lo, Cox and Mingus (2006) mentioned three major types of tasks to explore the notions of similarity: differentiating, measuring and constructing. As with their research, it is possible to find all three of these kinds of tasks in the six textbooks analyzed, with a focus first two types.

The results show how diverse the treatment of the presentation of the content, the tasks, the and formal reasoning expected of students. We can summarize highlighting different ways similarity is introduced in the studied textbooks, noting it is related to different ideas of contextual life situations. Lo, Cox and Mingus (2006) noted all textbooks have the same goal that is to elaborate a conceptual understanding of similarity and to prepare students to use these properties to solve contextual problems. So, to use connections between these ideas in a contextual situation is fundamental. For this, tasks need to receive teacher attention to connect them with real life in contextual problems; to reflect about the way the concept is presented and proposed of the tasks (eg. authors use the proportional with bigger/smaller numbers and the tasks propose as answer smaller/bigger). The formal approach is considered in a different level by the authors, how explored more or less the perspective of based on theorems and proof to present the concept.

The concept involves most of the reflections about similarity. We could identify how close the conceptualization of similarity in Brazil, and that it is presented in a different way in U.S. Obviously it influences the fact Brazilian and U.S. study of similarity at textbooks are different. Silva (2005) notes that historically in Brazil in the 1960s it happened that tried to give Geometry an axiomatic treatment, with resources to algebraic structures and set theory, including the theme of geometric transformations. Some pilot experiments began to introduce new concepts of geometry in the curriculum, such as geometric transformations, isometry and dilation. In practice the author observes that effectively in the schools there was no place for the geometric transformations and for matrices, having perpetuated the traditional teaching of Euclidean geometry. With this scenario, we note that Brazil has a focus on the static perspective of similarity, while U.S. expand it to the movement, with the study of transformation, even this work is different on the three analyzed textbooks. Other point of concept to branch is the relation between congruence and similarity, that received little attention from the authors (and can keep not noted by the teachers).

All of these points can be expanded to other Geometry contents, and we share the challenge to compose the mosaic of Geometry concept analysis in textbooks in order to better understand the presence of Geometry in this very used teacher resource.

Acknowledgments

We are grateful to the CAPES - *Coordination for the Improvement of Higher Education* for funding (Process N. YYY).

Identification of each author's contribution

First author is responsible for the Conceptualization, Methodology, Investigation, Resources and Writing - Original Draft.

The secondary author is responsible for the Writing - Review & Editing, Visualization and Supervision.

References

Amaral, Rúbia Barcelos; Hollebrands, Karen. An analysis of context-based similarity tasks in textbooks from Brazil and the United States. *International Journal of Mathematical Education in Science and Technology*, Londres, v. 48, n. 8, p. 1166-1184.

Brazil (2017). *Base Nacional Comum Curricular*. *Educação é a base*. Ministério da Educação. <http://basenacionalcomum.mec.gov.br/images/BNCC_EI_EF_110518_versaofinal_site.pdf>. Access in: 17/08/2021.

Carter, John A.; Cuevas, Gilbert J.; Day, Roger; Malloy, Carol E. (2012). *Integrated Math. Vol. 2*. Columbus, OH: McGraw Hill.

Centurión, Marilia; Jakubovic, José (2012). *Matemática: teoria e contexto*. São Paulo: Saraiva.

Chazan, Daniel. (1987). Similarity: Unraveling a conceptual knot with the aid of technology. In *Proceedings of the Eleventh International Conference of the Psychology of Mathematics Education* (p. 3–9). Montreal: PME.

Choppin, A. (2004). Educação e Pesquisa. São Paulo. n.3, v.30. https://files.eric.ed.gov/fulltext/ED383532.pdf>.

Choppin, Alain. (2004). História dos livros e das edições didáticas: sobre o estado da arte. *Educação e Pesquisa*, v.30, n.3, p. 549-566, set./dez. <https://doi.org/10.1590/S1517-97022004000300012>.

Costa, Jéssica Serra Corrêa & Bittar, Marilena (2019). A teoria antropológica do didático no estudo do cálculo mental. *Educação Matemática Pesquisa*. n.5 v.21. <https://doi.org/10.23925/1983-3156.2019v21i5p445-454>.

Cox, Dana. (2013). Similarity in middle school mathematics: at the crossroads of geometry and number. *Mathematical Thinking and Learning*, 15(1), p. 3–23. <https://doi.org/10.1080/10986065.2013.738377>

Cox, Dana; Lo, Jane-Jane. (2014). Detecting distortion: bridging visual and quantitative reasoning on similarity tasks. *Mathematics Education Research Journal*. *26*, p.1-23. <https://link.springer.com/article/10.1007%2Fs13394-013-0108-1>.

Fan, Lianghuo. (2013). Textbook research as scientific research: Towards a common ground on issues and methods of research on mathematics textbooks. *ZDM – Int J Math Educ*. 45, 765–777. <https://link.springer.com/article/10.1007%2Fs11858-013-0530-6>.

Freeman, Donald J.; Porter, Andrew C. (1989). Do textbooks dictate the content of Mathematics instruction in Elementary Schools? *American Educational Research Journal, 3*(26). Fall. <https://doi.org/10.3102/00028312026003403>

Guimarães, Gilda; Gitirana, Veronica; Cavalcanti, Milka; Marques, Mabel. (2007). Livros didáticos de matemática nas séries iniciais: análise das atividades sobre gráficos e tabelas. In: *Encontro Nacional de Educação Matemática*. *Anais*. Belo Horizonte. https://www.researchgate.net/profile/GildaGuimaraes/publication/305116276_Livros_Didaticos_de_Matematica_nas_Series_Iniciais_analise_das_atividades_sobre_graficos_e_tabelas/links/578cda5408ae59aa668149ce/Livros-Didaticos-de-Matematica-nas-Series-Iniciais-analise-das-atividades-sobre-graficos-e-tabelas.pdf>.

Hart, Kim M. (1984). Ratio: children's strategies and errors. A report of the strategies and errors in secondary mathematics project. London: NFER-Nelson.

Hirsch, Christian; Fey, James; Hart, Eric; Schoen, Harold; Watkins, Ann. (2015). *Core-Plus Mathematics Contemporary Mathematics in Context*. Columbus, OH: McGraw Hill.

Johnsen, Egil B. (1993). *Textbooks in the kaleidoscope: A critical survey of literature and research on educational texts*. Oslo: Scandinavian University Press.

Jones, Dustin; Tarr, James. (2007). An examination of the levels of cognitive demand required by probability tasks in middle grades mathematics textbooks. *Statistics Education Research Journal*. 6. p. 4-27. <https://www.academia.edu/26156030/An_Examination_of_the_Levels_of_Cogitive_Demand_Required_by_Probability_Tasks_in_Middle_Grades_Mathematics_TEXTBOOKS2?bulkDownload=thisPaper-topRelated-sameAuthor-citingThis-citedByThis-secondOrderCitations&from=cover_page>.

Kanold, Timothy D.; Burger, Edward B.; Dixon, Juli K.; Larson, Matthew, R.; Leinwand, Steven J. (2015). *Integrated mathematics*. Orlando, FL: Houghton Mifflin Harcourt.

Kaput, James J.; West, Maria M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In: *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany: State University of New York Press. <https://psycnet.apa.org/record/1994-98042-005>.

Lajolo, Marisa. (1996). Livro didático e qualidade de ensino. In: *Em aberto*. Ministério da Educação e Desporto SEDIAE/INEP, v.16, n.69. <https://doi.org/10.24109/2176-6673.emaberto.16i69.2061>.

Lamon, Susan J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education,* 24, p. 41-61. <https://doi.org/10.5951/jresematheduc.24.1.0041>.

Lemos, Maria Patrícia F. (2006). O estudo do tratamento da informação nos livros didáticos das séries iniciais do ensino fundamental. *Ciência e Educação*, v.12, n.2, p.171-184. <https://doi.org/10.1590/S1516-73132006000200005>.

Lehrer, Richard; Strom, Dolores; Confrey, Jere. (2002). Grounding metaphors and inscriptional resonance: Children's emerging understanding of mathematical similarity. *Cognition and Instruction*, 20 (3), 359-398. <https://doi.org/10.1207/S1532690XCI2003_3>.

Leonardo, Fabio.M. (2010). *Projeto Araribá: Matemática*. São Paulo: Moderna.

Lo, Jane-Jane; Cox, Dana; Mingus, Tabitha. (2006). A conceptual-based curricular analysis of the concept of similarity. In *Proceedings of the Eleventh International Conference of the Psychology of Mathematics Education* (pp. 3–9). v.2, p.222-228.

Silva, Ezequiel T. (1996). Livro didático: do ritual de passagem à ultrapassagem. In: *Em aberto*. Ministério da Educação e Desporto SEDIAE/INEP, v.16, n.69. <https://doi.org/10.24109/2176-6673.emaberto.16i69.%25p>.

Silva, Maria Célia L. (2005). A geometria escolar ontem e hoje: algumas reflexões sobre livros didáticos de Matemática. *Revista Iberoamericana de Educación Matemática*. n.3, Septiembre. p. 73-85. <http://funes.uniandes.edu.co/14570/>.

Souza, Joamir R; Pataro, Patrícia R.M. (2012). Vontade de saber. São Paulo: FTD.

Swoboda Ewa; Tocki, Jerzy. (2002). How to prepare prospective teachers to teach mathematics: some remarks. Paper presented at the Second International Conference on the Teaching of Mathematics (at the undergraduate level), Crete, Greece. <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.469.9549&rep=rep1&type=pdf>.

Tall, David. O.; Vinner, Shlomo. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169. <https://link.springer.com/article/10.1007/BF00305619>.

Valverde, Gilbert A.; Bianchi, Leonard. J; Wolfe, Richard G.; Schimdt, William H.; Houang, Richard T. (2002). According to the book: using TIMSS to investigate the translation of policy into practice through the world of textbooks. Dordrecht: Kluwer.

Van den Brink, Jan; Streefland, Leen. (1979). Young children (6–8) - Ratio and proportion. *Educational Studies in Mathematics*, 10(4), p. 403–420. <https://www.jstor.org/stable/3481826?casa_token=TqqAtIfRYWIAAAAA%3AjRNKhMhXcv59XUtg5Nfo5wjS4BpLdgAdYggfNT0QMVshvHeabAnhNJOHh0IoexFuFWP_UhcPLEwrI7Vml0ORAyuAtHe1KSdcI31alVTpiSOYmo_OOE&seq=1#metadata_info_tab_contents>.