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Linear diophantine equations through problem solving: possibilities for mathematics degree courses

Ecuaciones lineales diofánticas a través de la resolución de problemas: posibilidades para carreras de matemáticas

Équations diophantiennes linéaires par la résolution de problèmes : possibilités pour les cours de mathématiques

Equações diofantinas lineares por meio da resolução de problemas: possibilidades para cursos de licenciatura em matemática

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Abstract

The study of Diophantine equations began from the propositions of Diophantus of Alexandria, currently supporting problem solving in several areas and working with integer numbers in several contexts. Thus, this topic has been addressed in undergraduate mathematics courses, especially in the Number Theory subject. When considering these aspects, we aim to present the possibility of working with linear Diophantine equations through the mathematics teaching-learning-assessment methodology through problem solving (MEAAMaRP), adopted by the Grupo de Trabalho e Estudos em Resolução de Problemas [Working Group and Studies in Problem Solving] (GTERP), for undergraduate mathematics courses, based on an experience carried out in a pilot class of a graduate program. To this end, we followed the qualitative

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research paradigm and carried out an intervention in a matter of a graduate program in mathematics education with characteristics of a teaching experiment. We followed the ten steps proposed to solve a problem according to the MEAAMaRP. We used content analysis to organize and treat the data, mainly its three main phases: pre-analysis, exploration of the material and treatment of the results and interpretation. We could see that the participants appropriated the concept of linear Diophantine equations, made inferences from the dialogue, and attempted to solve the generating problems. Thus, we indicate the work with problem solving in mathematics degree courses since it can enable the student to develop autonomy and group work, among other results.

Keywords: Mathematics education, Number theory, Teaching-learning-assessment of mathematics through problem solving.

Resumen

El estudio de las ecuaciones diofánticas se inició con las proposiciones de Diofanto de Alejandría, pudiendo actualmente apoyar la resolución de problemas en diferentes áreas y trabajar con números enteros en diferentes contextos. Así, este tema ha sido abordado en los Cursos de Licenciatura en Matemática, asociados a la disciplina de Teoría de Números. Al considerar estos aspectos, pretendemos presentar una posibilidad de trabajar con ecuaciones diofánticas lineales a través de la Metodología de Enseñanza-Aprendizaje-Evaluación de las Matemáticas a través de la Resolución de Problemas (MEAMaRP) adoptada por el Grupo de Trabajo y Estudios en Resolución de Problemas (GTERP), para la Licenciatura Cursos en Matemáticas, a partir de una experiencia realizada en una clase piloto en un programa de posgrado. Para ello, seguimos el paradigma de la investigación cualitativa y realizamos una intervención en una disciplina de un Programa de Posgrado en Educación Matemática con características de experimento didáctico. Se siguieron los 10 pasos propuestos para resolver un problema según MEAAMaRP y se utilizaron técnicas de análisis de contenido para la organización y tratamiento de los datos, principalmente sus tres fases principales: preanálisis, exploración de materiales y tratamiento de resultados e interpretación. Pudimos ver que los participantes se apropiaron del concepto de ecuaciones diofánticas lineales e hicieron inferencias a partir del diálogo e intentaron resolver los problemas generadores. Así, indicamos el trabajo con Resolución de Problemas en los Cursos de Licenciatura en Matemáticas, ya que es capaz de posibilitar que el alumno desarrolle aspectos como la autonomía y el trabajo en grupo.

Palabras clave: Educación matemática, Teoría de los números, Enseñanza-aprendizaje-evaluación de las matemáticas a través de la resolución de problemas.

Résumé

L'étude des équations diophantiennes a commencé avec les propositions de Diophante d'Alexandrie, actuellement en mesure de soutenir la résolution de problèmes dans différents domaines et de travailler avec des nombres entiers dans différents contextes. Ainsi, ce sujet a été abordé dans les Cours de Licence de Mathématiques, associés à la discipline de la Théorie des Nombres. Lors de l'examen de ces aspects, nous visons à présenter une possibilité de travailler avec des équations diophantiennes linéaires à travers la méthodologie d'enseignement-apprentissage-évaluation des mathématiques par la résolution de problèmes (MEAAMaRP) adoptée par le groupe de travail et d'études en résolution de problèmes (GTERP), pour le diplôme Cours de Mathématiques, basés sur une expérience réalisée dans une classe pilote d'un programme d'études supérieures. Pour ce faire, nous avons suivi le paradigme de la recherche qualitative et réalisé une intervention dans une discipline d'un programme d'études supérieures en didactique des mathématiques présentant les caractéristiques d'une expérience d'enseignement. Les 10 étapes proposées pour résoudre un problème selon MEAAMaRP ont été suivies et des techniques d'analyse de contenu ont été utilisées pour l'organisation et le traitement des données, principalement ses trois phases principales : pré-analyse, exploration matérielle et traitement des résultats et interprétation. Nous avons pu voir que les participants se sont appropriés le concept d'équations diophantiennes linéaires et ont fait des inférences à partir du dialogue et des tentatives de résolution des problèmes générateurs. Ainsi, nous indiquons le travail avec la résolution de problèmes dans les cours de licence en mathématiques, car il est capable de permettre à l'étudiant de développer des aspects tels que l'autonomie et le travail de groupe.

Mots-clés : Enseignement des mathématiques, La théorie du nombre, Enseignement-apprentissage-évaluation des mathématiques par la résolution de problèmes.

Resumo

O estudo das equações diofantinas teve início a partir das proposições de Diofanto de Alexandria, atualmente sendo capaz de subsidiar a resolução de problemas em diversas áreas e o trabalho com números inteiros em diversos contextos. Assim, este tópico tem sido abordado em cursos de licenciatura em matemática, especialmente na disciplina Teoria dos Números. Ao considerarmos estes aspectos, objetivamos apresentar uma possibilidade de trabalho com equações diofantinas lineares por meio da metodologia de ensino-aprendizagem-avaliação de matemática através da resolução de problemas (MEAAMaRP), adotada pelo Grupo de Trabalho e Estudos em Resolução de Problemas (GTERP), para cursos de licenciatura em matemática, a partir de uma experiência realizada em uma aula-piloto de um programa de pós-graduação. Para tanto, seguimos o paradigma qualitativo de pesquisa e realizamos uma intervenção em uma disciplina de um programa de pós-graduação em educação matemática com características de um experimento de ensino. Seguimos as dez etapas propostas para resolver um problema segundo a MEAAMaRP e utilizamos a análise de conteúdo para a organização e tratamento dos dados, principalmente suas três principais fases: pré-análise, exploração do material e tratamento dos resultados e interpretação. Pudemos constatar que os participantes se apropriaram do conceito de equações diofantinas lineares e realizaram inferências a partir do diálogo e tentativas de resolução dos problemas geradores. Assim, indicamos o trabalho com a resolução de problemas em cursos de licenciatura em matemática, já que é ele capaz de possibilitar ao estudante o desenvolvimento da autonomia e o trabalho em grupo, entre outros resultados.

Palavras-chave: Educação matemática, Teoria dos números, Ensino-aprendizagem-avaliação de matemática através da resolução de problemas.

According to Boyer (1996), the most famous work of Diophantus of Alexandria, a pioneer in Diophantine equations, is the *Arithmetica*, composed of thirteen books characterized by a high degree of mathematical ability. Diophantus strongly influenced the development of Diophantine equations, as he had a different way of thinking than the Greeks in terms of methodology and had his own recording style.

According to Eves (2004, p. 207), Diophantus of Alexandria is deemed one of the pioneers and most important among the significant mathematicians studying number theory because

> he had enormous importance for the development of algebra and strongly influenced Europeans who later dedicated themselves to number theory. As in the case of Heron, no one knows for sure Diophantus' nationality and when he lived exactly. Although there is some tenuous evidence that he may have been Heron's contemporary, most historians tend to place him in the third century of our era. Apart from the fact that his career flourished in Alexandria, nothing more is clear about him, although one finds in the Greek anthology an epigram that seems to give some details of his life.

Campell and Zazkis (2002) corroborate this statement by arguing that with Diophantine equations, we can solve problem situations applied in different areas, work with integer numbers in different contexts to develop strategies and expand the repertoire of problem solving without necessarily resorting to algorithms, which favors reading, interpreting statements, and developing conjectures. They explain that teaching and learning require a trial and error strategy, verifying numerical calculations, and using properties to enable the student to understand the object studied.

Following Campell and Zazkis (2002), in the mathematics degree course of the university campus where this study was carried out, the Number Theory subject considers the study of Diophantine equations. One of the objectives of the subject states: "The knowledge of arithmetic of whole numbers is fundamental, both for the future mathematics teacher, in basic education, and the future researcher in the different areas associated with mathematics", and that "the objective of this subject is to familiarize students with relevant content and with typical methods of approaching problems in this context" (Programa de Ensino, 2018). In addition, the syllabus also includes the study of divisibility, the fundamental theorem of arithmetic, congruences, quadratic remains, Diophantine equations, and Fermat's theorem.

In this context, Resende and Machado (2012) present that, in teacher education, the elementary theory of numbers must consider aspects such as its topics present in basic education[5], its relations with natural and integer numbers, and its ideas present in school mathematics, in addition to being an excellent opportunity for mathematical research. Thus, among the aspects highlighted in the teaching methodology foreseen for the mentioned subject, which converges with the notes of Resende and Machado (2012), some classes involve problem solving, develop content around number theory, and provide students with a reflection on their teaching and learning in basic education.

Given the above, we were mobilized by the following motivating question of this thematic edition: How should the elementary theory of numbers be worked on in the initial education of mathematics or pedagogy teachers so that it can effectively support the teaching work in basic education? Moreover, we aim to present the possibility of working on linear Diophantine equations through the mathematics teaching-learning-assessment methodology through problem solving (MEAAMaRP) for mathematics teaching degree courses, based on an experiment carried out in a pilot class in a graduate program.

Problem solving as a line of research, according to the authors Onuchic and Allevato (2011, p. 77), is pioneered by Polya[6]. In his studies, Polya was concerned with "figuring out how to solve problems and how to teach strategies that would lead to seeing ways to solve problems." That is, the focus of this approach turns to ways of developing the ability to think about procedures, techniques or methods so that the individual can explain the meaning that this makes for him.

However, according to the authors, for such development, the teacher must be the mediator, inserting provocations in the middle of the activities, promoting discussions and students' involvement, and giving students greater responsibility for the learning they intend to achieve. When arguing about the teacher's role, Polya (1995, p. v) corroborates this statement:

> If they fill up their time by drilling students into routine operations, they annihilate the interest and stunt students' intellectual development, thus wasting their opportunities. But if they challenge the students' curiosity, presenting them with problems compatible with their knowledge and helping them through stimulating questions, they can instill in students a taste for independent reasoning and provide them with certain means to achieve this goal.

Thus, students must assume the role of active participants, have responsibility for their teaching and learning process, and analyze their own methods and the solutions obtained to the

[5] T.N.: Basic education comprises primary education (grades 1 to 9 of elementary education) and secondary education (grades 1 to 3 of high school)

[6] George Polya (1887-1985) was a Hungarian-American mathematician. He published the book entitled *How to solve it*, which received in Brazil the curious title *The art of solving problems*.

problems, always aiming at constructing knowledge. In teaching and learning through problem solving, the Base Nacional Comum Curricular [National Common Curricular Base] (BNCC) (Brasil, 2018) highlights how important it is to work with the methodology throughout the school journey. The BNCC brings as a backdrop for mathematics teaching the development of mathematical subjects through problem solving because:

> It is advisable to reiterate the justification for the use in the BNCC of "Solving and Creating Problems" instead of "Solving Problems". This option expands and deepens the meaning given to problem solving: the creation assumes that students investigate other problems involving the concepts treated; its purpose is also to promote reflection and questioning about what would happen if some data were changed or some conditions were added or removed. (Brasil, 2018, p. 536)

Therefore, we understand that problem solving can be dealt with in different segments since it enables contextualization and interdisciplinarity, promoting the intertwining between experienced situations and issues related to mathematical tools, providing conditions for students to formulate critical-reflective thinking by building the solution of a problem situation. In the specific case of teacher education, working with problem solving and other methodologies can make teachers better prepared for their future professional reality (Onuchic & Morais, 2013).

In this context, this work is the result of the studies of the subject *Tópicos Especiais em Educação Matemática: Resolução de Problemas - Ensino, Aprendizagem e Avaliação* [Special Topics in Mathematics Education: Problem Solving - Teaching, Learning and Evaluation] offered in 2022 in a graduate program in mathematics education in the southeastern region of Brazil, which followed the MEAAMaRP adopted by the Grupo de Trabalho e Estudos em Resolução de Problemas (GTERP). This text is organized into three main sections in addition to this one, which involve the methodological aspects, the presentation and discussion of the data and the final considerations.

Methodological Route

When considering the objective of this article –stated in the previous section– we adopted the qualitative research approach (Lüdke & André, 1986). This factor is justified since our main concern is the process of constituting possibilities to teach topics of number theory]from problem solving and how this manifests itself in the activities applied, that is, in the steps developed throughout the experience carried out.

To this end, we conducted an intervention in a subject of a graduate program in mathematics education with characteristics of a teaching experiment since "it is a research methodology that seeks to explore and explain students' mathematical activities" (Borba, Almeida, & Gracias, 2019, p. 46). We understand this experience as a possibility to evaluate an innovative approach to a number theory mathematical topic. Thus, we developed a lesson plan on linear Diophantine equations, a topic addressed in the Number Theory subject[7] in the teaching and research degrees in mathematics of the university campus to which the graduate program is linked. In Table 1, we present a summary of this lesson plan.

Table 1.

Lesson Plan (Prepared by the authors, 2022)

Subject	Number Theory – teaching and research degrees in mathematics
Lecturers/Teachers	The authors of the article
Hours/class	3 c/h
Semester/Year	2022/2
Content	Linear Diophantine equations
General objective	Present the concept, theorems and procedures for solving linear Diophantine equations through the methodology of mathematics teaching-learning-assessment through problem-solving in GTERP.
Specific objectives	Review the procedure for solving the greatest common divisor; review the Euclid division algorithm; understand the definition of linear Diophantine equation; understand the condition of existence of a linear Diophantine equation; know how to use the theorems and results in different situations and problems.
Auxiliary resources	Printed sulfite sheets with examples and exercises.
Strategy	Use the mathematics methodology of teaching-learning-assessment through problem solving in GTERP to work on the concept of linear Diophantine equations with higher education students of the mathematics degree course.

Since the environment in which the intervention was carried out involved discussions and studies on problem-solving methodology, we decided to draw up the lesson plan using the MEAAMaRP. According to Allevato and Onuchic (2021, p. 47), this methodology can propel students' learning using the problem as a "starting point and guidance for learning new concepts and new mathematical content".

Thus, the authors suggest developing the activities in ten stages, as shown in Figure 1. In this process, it is mainly up to the teacher to propose a generating problem or accept one the

[7]This subject has already been taught by the first author of the article.

students proposed (step 1) and, throughout the resolution steps, to assist the groups in understanding it and observe and encourage them. To Allevato and Onuchic (2021), the function of the generating problem is to make the student able to build a new content or concept, in our case, that of linear Diophantine equations.

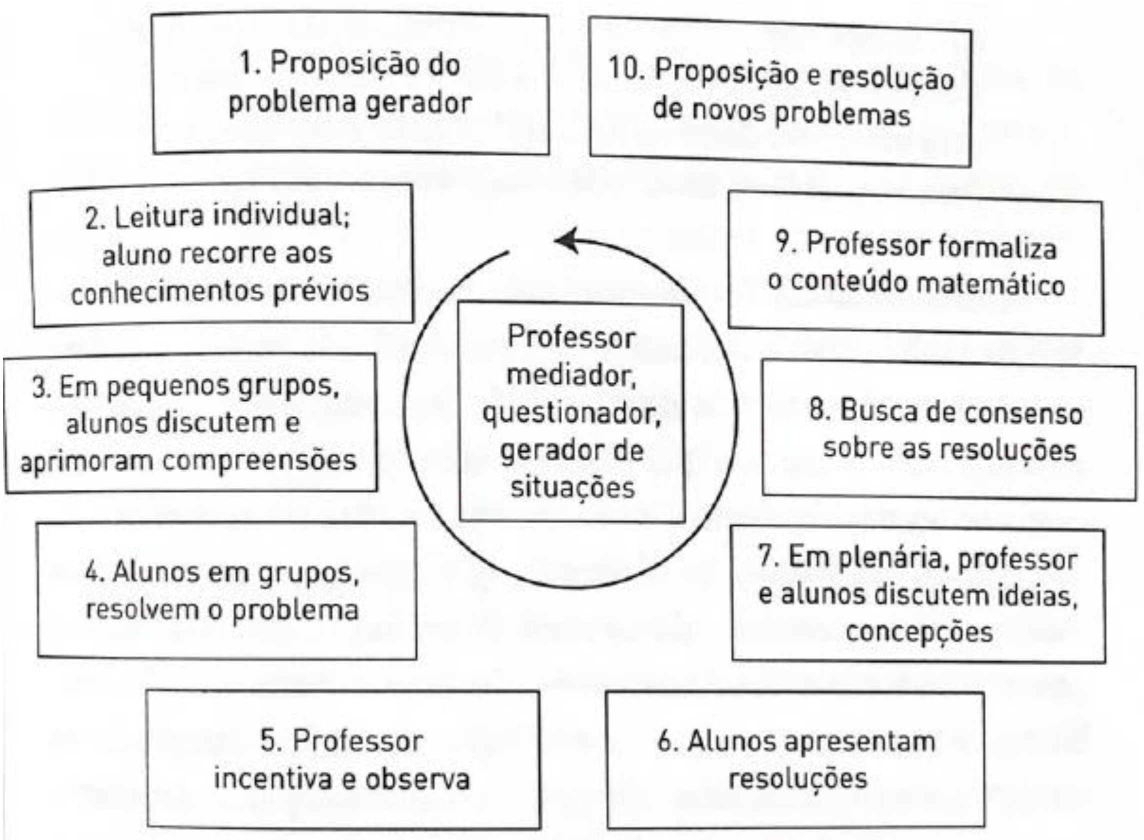

Figure 1.

MEAAMaRP scheme (Allevato & Onuchic, 2021, p. 51).

The second step is performed after each student has received the problem and sought to understand it. Then, students gather in groups to carry out the third and fourth steps of the methodology. At this moment, the teacher helps the groups to understand the problem and to seek strategies to solve it based on their previous knowledge, without providing ready answers (step 5).

Continuing in the sixth stage, each group chooses representatives to present their resolutions on the blackboard to seek a consensus on the correct result later (steps 7 and 8). “This is when there is a great improvement in mathematical reading and writing and relevant construction of knowledge about the content” (Allevato & Onuchic, 2021, p. 50).

In the ninth stage, the teacher must formally present, using appropriate mathematical language, the concepts and procedures necessary to solve the generating problem. Finally, in the last stage, the teacher proposes new problems to analyze whether the students understood the mathematical content. In this step (10) of the methodology adopted by the GTERP, Allevato and Onuchic (2021) suggest that students can propose the problems. BNCC (Brasil, 2018) indicates that basic education students should be able to propose problems at this level.

In addition to these aspects, we highlight that the methodological path adopted throughout this research considered some techniques of content analysis (Bardin, 2016), especially its three main phases: pre-analysis, exploration of the material, and treatment of results and interpretation. In this context, the pre-analysis involves the study of the MEAAMaRP and the review of mathematical contents addressed in number theory subjects, culminating in the preparation of the lesson plan and work proposal.

In turn, the exploration of the material is characterized by the intervention carried out, through which we develop the activities foreseen in the lesson plan with graduate students simulating a higher education class. Finally, the treatment stage of the results and interpretation considers the discussion of the proposal through the report of the experience and its analysis, a fact explained in the next section.

Data Presentation and Analysis

The class began with the presentation of the proposal to be worked on based on the MEAAMaRP for introducing a specific mathematical topic, not mentioning what it would be. The five participants in this teaching experience received a printed sheet with the ten steps of the MEAAMaRP, described in Figure 1, as guidance on the procedures adopted in class. In addition, they also received another sheet with the statement of three problems, presented in Table 2, which would be solved by the groups formed, which had been selected by the proponents of this article (step 1 of the methodology).

Table 2.

Problems. Prepared by the authors, adapted from Braga & Carmo (2018)

1 – The cinema of the Mall of Rio Claro offers, on Wednesdays, tickets with values of R$10.00 in the morning and R$12.00 in the afternoon. How many ways can tickets be sold so that you have a balance of R$200.00 at the end of the day?
2 – Edson is a music fan; he sets aside R$125.00 a month to buy CDs and DVDs. On average, a CD costs R$10.00 and a DVD, R$16.00. What are the possibilities of purchasing the items, spending exactly R$125.00?
3 – A farmer wants to buy duck and chicken pups, spending R$1770.00. A duck puppy costs R$31.00, and a chicken puppy costs R$21.00. How many of each of the two types will the farmer be able to buy?

The problems were directed to them with the instruction that they were read individually to use the previous knowledge of mathematics present (step 2 of the methodology). Then, the participants met in groups A and B to discuss the hypotheses and strategies they should use to solve the three proposed problems and solve them, corresponding to steps 3 and 4 of the methodology (Figure 2). At this moment, teachers circulated in the groups, observing and encouraging the participants in the discussions and seeking the solution of the problems (step 5 of the methodology).

Figure 2.

Participants solving the problems. Research data (2022)

After solving the problems in groups, the participants shared their resolutions on the blackboard, describing the strategies adopted and the solutions obtained (step 6 of the methodology). At that moment, we could observe and understand diverse ways of solving the same problem using their previous knowledge. When sharing the solutions to the problems on the blackboard, there was also a search for consensus on the resolutions presented, in addition to discussing mathematical ideas and conceptions on Diophantine equations (steps 7 and 8 of the methodology). Next, we present and report on some of the participants' strategies.

Problem 1

Figures 3, 4 and 5 illustrate solutions presented by the groups to problem 1, stated in Table 2.

Group A solutions

This group presented two solutions to this problem (Figures 3 and 4). The first solution used the trial-and-error strategy to find numerical values that would satisfy the problem. According to the proponents, the four numerical values presented all solve the problem. They deduced that there were no more solutions from the number of tickets for the afternoon shift, since this number would have to be a multiple of 5.

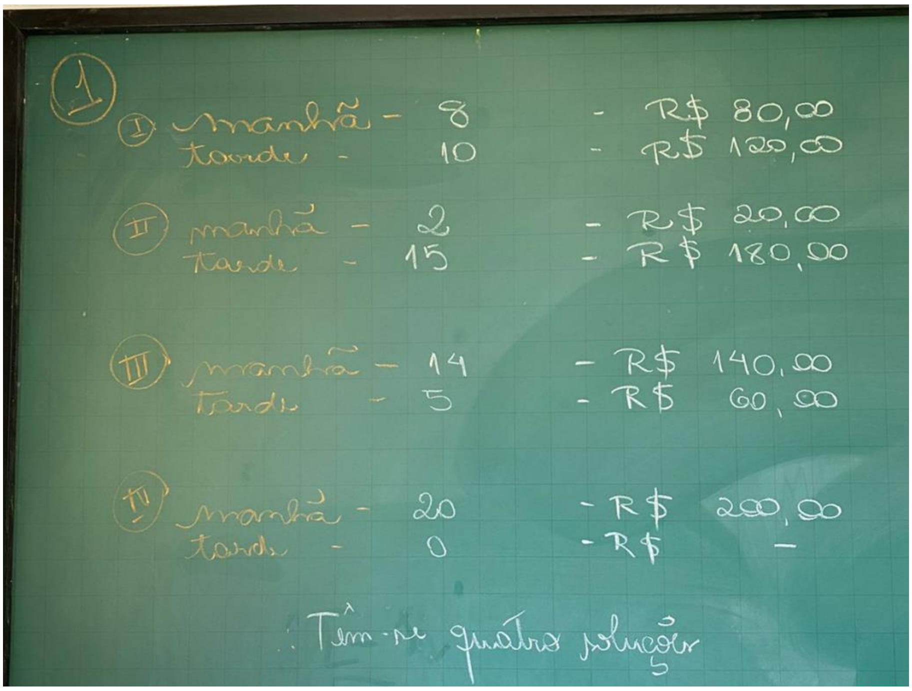

Figure 3.

A solution to problem 1 given by group A. Research data (2022)

In the other solution, presented in Figure 4, there is the generalization of the problem through the equation $10x + 12y = 200$, being x the number of morning tickets and y the number of afternoon tickets. They isolated the value of x as a function of the variable y and concluded that the value of y must be a multiple of 5. However, they presented only a numerical solution to the problem, found by inspection, and questioned how to find the other solutions using this equation.

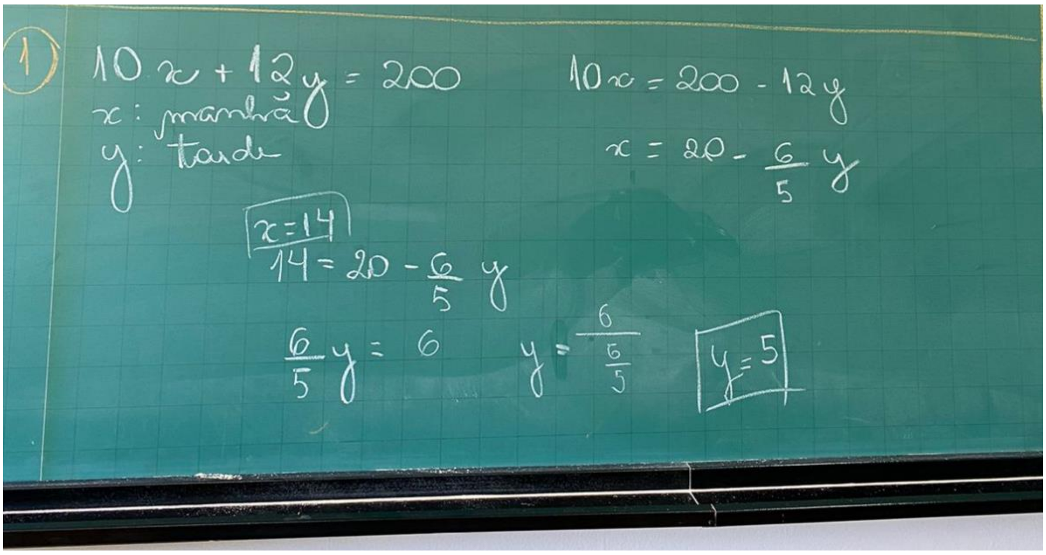

Figure 4.

Other solution to problem 1 given by group A. Research data (2022)

Regarding the solution illustrated in Figure 3, we observed that there was progress towards giving a generalization for the solution of the problem using an equation to model it.

Group B solution

In the strategy developed to solve the problem, the members of group B noted that the number of tickets for the afternoon shift would have to be a multiple of 5. Thus, through mental calculation and trial and error, they found four possible integer solutions, illustrated in Figure 5.

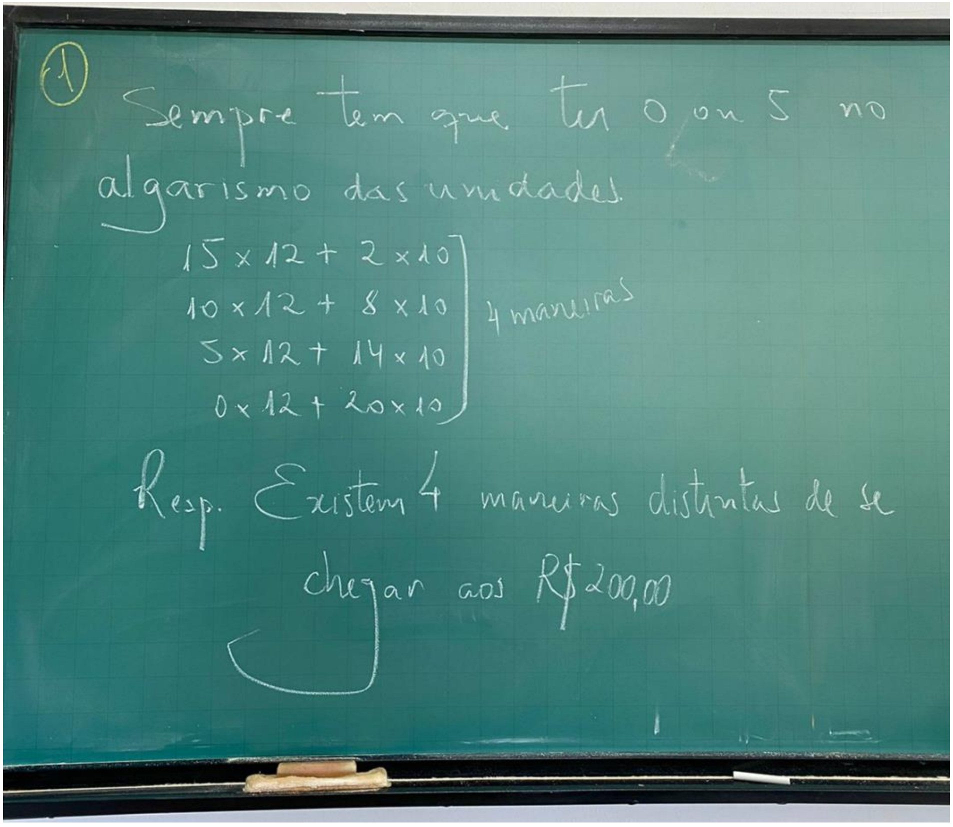

Figure 5.

A solution to problem 1 given by group B. Research data (2022)

After exposing the solutions to problem 1 and discussions carried out with the teachers' measurement, some conclusions were produced: the equation $10x + 12y = 200$, which x represents the number of morning tickets and y represents the number of afternoon tickets, models this problem, making it possible to isolate one of the variables as a function of the other; the maximum common divisor (mdc) between 12 and 10 is 2, that it 2 divides 200. They conjectured that the divisibility of the coefficients of the equation could be a condition for the existence of a solution.

Problem 2

Figures 6 and 7 present the solutions given by groups A and B for problem 2 stated in Table 2.

Group A solution

In the solution presented by group A, the proponents wrote the general equation that models the problem given by $10C + 16D = 125$, where C represents the number of CDs and D represents the number of DVDs. Through trial and error, they found no solutions to this problem.

They concluded that, for this equation to have an integer and positive solution, the coefficients that accompany C and D should be multiples of 5, which does not occur. Therefore, this problem has no solution.

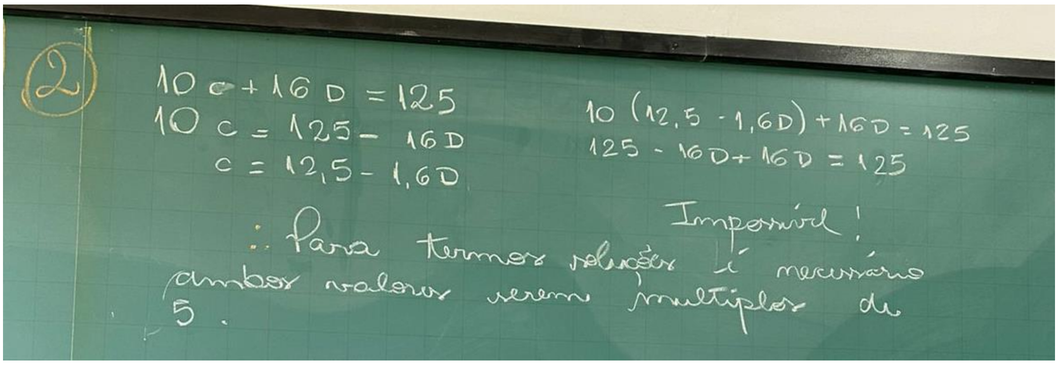

Figure 6.

A solution to problem 2 given by group A. Research data (2022)

Group B solution

The solution presented by this group (Figure 7) is similar to that of group A. The proponents wrote the general equation that models the problem, $10x + 16y = 125$, where x represents the number of CDs and y represents the number of DVDs, and isolated the variable x as a function of y. They also observed that the value of y would have to be a multiple of 5, not 2, for x to be a positive integer number, which does not happen in this case.

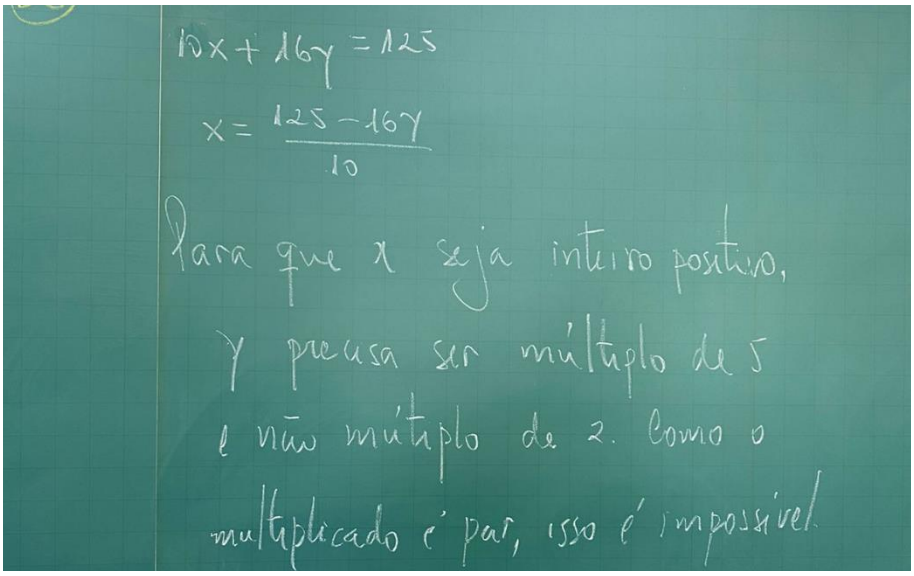

Figure 7.

Solution to problem 2 given by group B. Research data (2022)

In the discussions and search for consensus on the topic based on the data produced, the participants deduced that $mdc(10,16) = 2$ and that 2 does not divide 125. They compared this data with that obtained in problem 1, in which $mdc(12{,}10) = 2$ e 2 divides 200. They conjectured that an equation of this nature has integer solutions if the mdc of the coefficients of the equation divides its independent term.

Next, we move on to the presentation and discussion of the solutions to problem 3, set out in Table 2.

Problem 3

Group A solution

The solution to this problem given by group A and illustrated in Figure 8 presents the equation that models it and only two solutions found by trial and error. In the explanation, the group members observed that if the number of ducks increased, for example, the number of chickens would have to decrease and vice versa. They reported that they no longer investigated solutions to the problem because the values were too high to perform multiplications and sums.

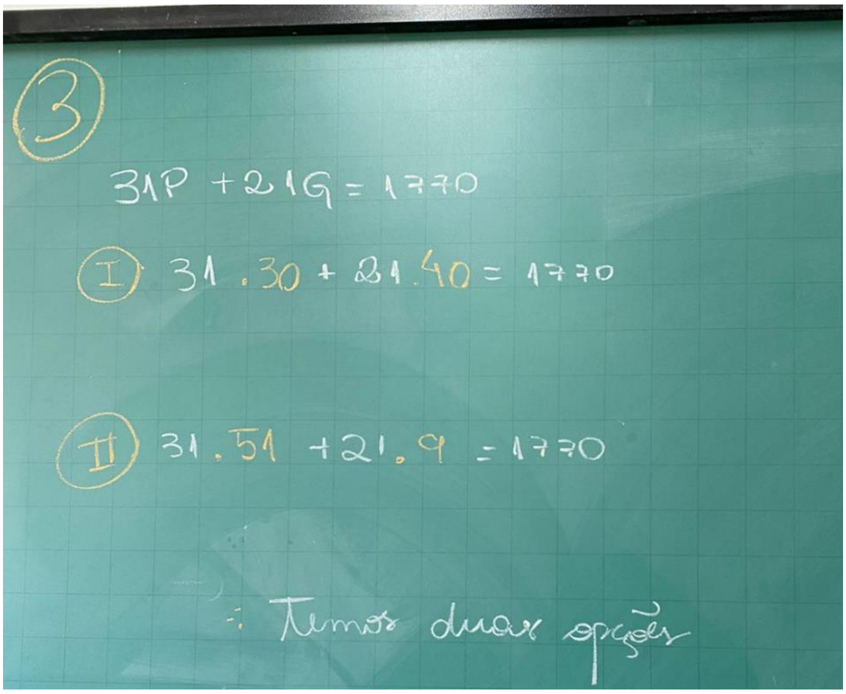

Figure 8.

A solution to problem 3 given by group A. Research data (2022)

Group B solution

In the solution presented by group B, described in Figure 9, the proponents presented the equation that models the problem, given by $31x + 21y = 1770$, being x the number of duck and y chicken pups, and isolated the variable y as a function of x. From the analysis of conditions for these variables, the group presented only one solution to the problem.

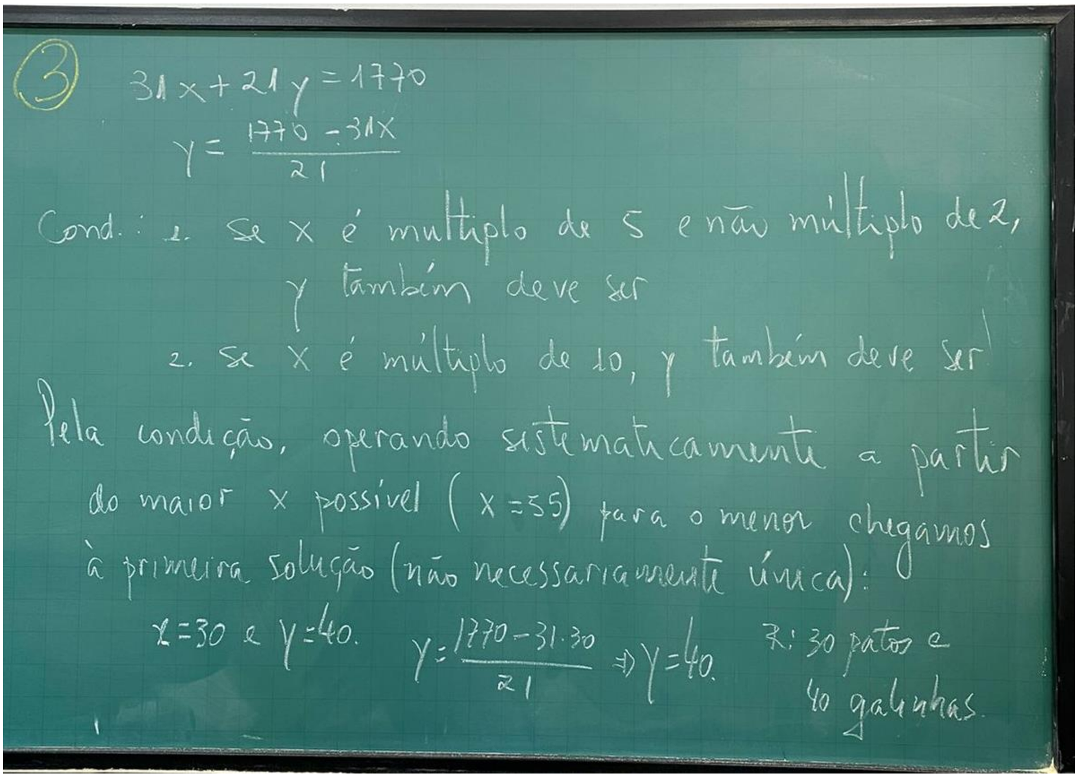

Figure 9.

Solution to problem 3 given by group B. Research data (2022)

In the discussions about the solutions to this problem, which relied on the teachers' mediation, the participants observed, in this case, that $mdc(31{,}21) = 1$ and 1 divides 1770, and this would be a condition for the existence of integer solutions (positive) to the problem.

Almost all the solutions presented for the three problems were incomplete, and we agreed with the participants that the correct solutions would be available. So, the solution of problem 3 was presented and discussed after the content was systematized.

From the strategies adopted – resolution, socialization and discussion – some consensus was produced: (i) This type of problem involves equations with two variables, with the solution in the set of integers; (ii) A condition for the existence of solutions for this type of equation is the divisibility of the maximum common divisor of the coefficients of the equation by the independent term of this equation.

Also, during the discussion, some questions arose about the solutions to the proposed problems, such as (a) How to find the number of solutions for this type of equation? Is this number finite or infinite? (b) Is there any method to find all solutions?

Next, we formalized the concept of Diophantine equations, starting with a resumption of the Euclid division algorithm, corresponding to step 9 of the GTERP methodology.

Formalization of mathematical contents

In this step, we present the main concepts about linear Diophantine equations of two variables, based on references indicated in the basic bibliography available in the Number Theory subject (Programa de Ensino, 2018): Martinez et al. (2013); Graham, Knuth, and Patashnik (1995); Niven and Zuckerman (1991); and Vinogradov (1977).

Lemma - Euclid's division algorithm

Let a and b be two positive integer numbers and $a = bq + r$, with $0 \leq r < b$, with r integer. Then $mdc(a, b) = mdc(b, r)$

A demonstration of this motto is found in Martinez et al. (2013, p. 19).

Definition - linear Diophantine equation

A linear Diophantine equation in two variables is an expression of the form $ax + by = c$, in which $a, b, c \in Z$, with $a \neq 0$ or $b \neq 0$, and the pair $(x, y) \in Z^2$, being Z the set of integer numbers. We call the solution of a Diophantine equation every pair $(x_0, y_0) \in Z^2$ that satisfies the equation $ax_0 + by_0 = c$.

Example 1: Solve the Diophantine equation $8x + 7y = 9$.

$(2, -1)$ is a solution because $8 \,.\, 2 + 7 \,.\, -1 = 16 - 7 = 9$.

Theorem - the condition of existence of a solution

The linear Diophantine equation $ax + by = c$ has an integer solution if, and only if, $mdc(a, b)$ divides c, that is, $mdc(a, b)|c$.

Demonstration:

Let x_0 and y_0 be particular solutions of the equation $ax + by = c$. Since $mdc(a, b)$ divides a and b, it also divides $ax_0 + by_0$, and, since (x_0, y_0) is a solution of this equation, then $ax_0 + by_0 = c$ and $mdc(a, b)$ divides c.

On the other hand, suppose $mdc(a, b)$ divides c, then $c = mdc(a, b). k$, into some k integer. In addition, there are integer numbers x_0 and y_0, such that $mdc(a, b) = ax_0 + by_0$. Multiplying this equation by k, we obtain $mdc(a, b). k = akx_0 + bky_0$, that is, $akx_0 + bky_0 = c$ and so exists (kx_0, ky_0) in Z^2, which is the solution of the equation $ax + by = c$.

Thus, the Diophantine equation $ax + by = c$ has infinite solutions c if it is a multiple of the greatest common divisor of a and b. Otherwise, it has no solution.

Theorem - solution of the equation $ax + by = c$

Be (x_0, y_0) a particular solution of $ax + by = c$ where a and b are non-zero integer numbers. Then, any integer solution of this equation is given by:

$$x = x_0 + bdk \text{ and } y = y_0 - adk,$$

where $mdc(a, b) = d$ and k is any integer.

A demonstration of this motto can be found in Braga and Carmo (2018, pp. 24-25).

Example 2: Finding a solution to the Diophantine equation $32x + 9y = 7$.

Like $mdc(a, b) = (32, 9) = 1$ and $1 \mid 7$, the equation has an integer solution. Using Euclid's algorithm, we have the following possibilities:

$$32 = 9 . 3 + 5;\ 9 = 5 . 1 + 4;\ 5 = 4 . 1 + 1;\ 4 = 1 . 4 + 0.$$

Isolating the mdc and writing it as a function of $a = 32$ and $b = 9$, from the tests of the Euclidean divisions, we obtain:

$$1 = 5 – 4 . 1;$$

$$1 = 5 – (9 – 5 . 1) . 1;$$

$$1 = 5 – 9 . 1 + 5 . 1;$$

$$1 = 5 . 2 – 9 . 1;$$

$$1 = (32 – 9 . 3) . 2 – 9 . 1;$$

$$1 = 32 . 2 – 9 . 7.$$

Thus, we find a linear combination for 32 and 9 such that

$$32 . 2 + 9 . (-7) = 1.$$

Multiplying this particular solution by 7 , we get $32 . 14 + 9 . (-49) = 7$.

Therefore, (14, -49) is a trivial solution to the given equation. The general solution is as follows:

$$x = 14 + 9k \text{and } y = -49 - 32k, \text{ where } k \in Z.$$

After formalizing these contents, we discuss the solution of problem 3 using the technique presented in Example 2. Briefly, the solution to this problem is given by: $31x + 21y = 1$, which is the Diophantine equation that models the problem, with $mdc(31, 21) = 1$ and 1 divides 1770. So, the problem has a solution, one of which is given by $x_0 = 30$ and $y_0 = 40$. The other solutions are of the form $x = 30 + 21k$ and $y = 40 - 31k$, with k integer. In this case, as x and y must be positive integers, then the possibilities are $k = -1, 0, 1$, resulting in the respective pairs of solutions $(9, 71), (30, 40)$ and $(51, 9)$.

We consider it pertinent to emphasize that, although at first the three problems proposed and described in Table 2 present similarities, each involves different resolution strategies. The solutions to problem 1 can be found by trial and error, which hardly happens in problem 3, which we observed throughout the presentation of the resolutions (step 6) and plenary (step 7). In addition, the problem 2 proposed has no solution.

Finally, we gave participants a list of exercises to pin down the concepts studied. In other words, the proposition and resolution of new problems, the last step of the GTERP methodology.

We could observe similarities and divergences between the experience carried out and the study by Mendes, Pereira, and Proença (2020). These authors indicated four weaknesses of the work with solving problems in the training of mathematics teachers: (i) difficulties regarding undergrads' lack of knowledge about the mathematical content of basic education; (ii) undergrads' difficulties in communicating their ideas about the mathematical content worked on; (iii) students' difficulties understanding the problems; and, (iv) difficulties related to the time for the undergraduates to carry out the activities and develop the teaching approach.

By observing these four weaknesses and the experience we carried out, we indicate that the graduate students in mathematics education participating in the study, and most of them graduated in mathematics, presented knowledge about the contents of basic education and could communicate their ideas in the plenary. However, we highlight the third and fourth points of Mendes, Pereira, and Proença (2020), cases exemplified by students' initial difficulty in observing differences between the three proposed problems (see Table 2) and the actual performance of the activities.

Final Considerations

In this article, we aimed to present a possibility of working on linear Diophantine equations through MEAAMaRP in mathematics degree courses, based on an experiment carried out in a pilot class in a graduate program. Therefore, we consider an intervention in a subject of a graduate program in mathematics education with characteristics of a teaching experiment.

We consider this experiment relevant because teacher education, as Tardif (2018) pointed out, includes the sum of disciplinary knowledge, professional knowledge, curricular knowledge and experiential knowledge. Disciplinary, professional, and curricular knowledge is produced in the academic sphere, and experiential knowledge is produced by teachers in their own pedagogical practice in schools. In this sense, research and experiences carried out in the academic environment presenting and discussing other methodologies for teaching and learning mathematics, such as problem solving, can support teaching work in basic education. In addition, the experience simulated a classroom environment, allowing us to think about experiential knowledge in teacher training.

Thus, in the study carried out, the meetings were permeated by discussions and learning, focusing on *how to*[8] develop strategies and approach mathematical subjects through problem solving with students from different segments (middle school final years, high school and undergraduate level), using the GTERP methodology. The discussions during the classes provided an environment of dialogue, practice and reflection because it was possible to remember how we studied throughout the academic journey and how to *be a teacher*[9] in the classroom. In addition, we understand how powerful this line of research is and plays a significant role in students' teaching and learning process.

Although most of the participants in the intervention had degrees in mathematics and, at some point in their training, had studied linear Diophantine equations, they did not remember the mathematical topic covered. We verified that they could appropriate the concept and, throughout steps 6 to 8 of this methodology, made inferences from the observations and attempts to solve the generating problems.

Thus, we emphasize that it is crucial to work on mathematical subjects based on the problem-solving methodology since it enables the students to develop autonomy and expand the repertoire to express their opinions and strategies; expand their ability to analyze, verify, and interpret the results obtained; understand that mistakes in the course of solving are part of the teaching and learning process; and develop skills to work in groups. Thus, we hope that the experience proposed and presented throughout this text can inspire teachers working in mathematics degree courses to use the teaching-learning-assessment of mathematics through problem solving in teaching topics of number theory, particularly linear Diophantine equations.

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[8] We understand that this *as* indicates distinct possibilities of developing the same problem.
[9] *Being a teacher* refers to the ways of acting in the classroom.

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