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Transição do aritmético para o algébrico à luz de ideias de Yves Chevallard

Transition from Arithmetic to Algebra in Light of Yves Chevallard's Ideas

Transición de la aritmética al álgebra a la luz de las ideas de Yves Chevallard

Le passage de l'arithmétique à l'algèbre à la lumière des idées d'Yves Chevallard

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Resumo

Tem-se como objetivo desse artigo refletir sobre a transição da aritmética para a álgebra a partir de publicações de Yves Chevallard. Para isso, selecionamos quatro artigos desse autor que tratam dessa abordagem, no âmbito do sistema de ensino francês, mas com implicações no ensino de álgebra no Brasil. Os artigos foram selecionados por meio de leituras prévias e pelas relevâncias das ideias contidas nos mesmos. A partir das reflexões busca-se uma possível resposta à questão: Quais aspectos epistemológicos da transição do aritmético para o algébrico são revelados em artigos de Yves Chevallard? Os traços metodológicos assumidos são da pesquisa bibliográfica e da análise de conteúdo. As conclusões indicam que os aspectos epistemológicos do aritmético e do algébrico seguem uma modelização matemática algébrico/numérico, intermediada pelo processo de transposição didática.

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Palavras-chave: Aritmético e algébrico, Modelização matemática, Transposição didática**.**

Abstract

The aim of this article is to reflect on the transition from arithmetic to algebra based on Yves Chevallard's publications. For this, we selected four of Chevallard's articles focused on that approach within the French education system, but with implications for algebra teaching in Brazil. The articles were selected through previous readings and relevance of the ideas contained in them. From the reflections, we sought to answer the following question: What epistemological aspects of the transition from arithmetic to algebra are revealed in Yves Chevallard's articles? The methodological features assumed are bibliographic research and content analysis. The conclusions indicate that the epistemological aspects of arithmetic and algebra follow an algebraic/numerical mathematical modeling, intermediated by the didactic transposition process.

Keywords: Arithmetic and algebra, Mathematical modeling, Didactic transposition.

Resumen

El objetivo de este artículo es reflexionar sobre la transición de la aritmética al álgebra a partir de las publicaciones de Yves Chevallard. Para ello, seleccionamos cuatro artículos de este autor que abordan este enfoque en el contexto del sistema educativo francés, pero con implicaciones para la enseñanza del álgebra en Brasil. Los artículos fueron seleccionados a través de lecturas previas y relevancia de las ideas contenidas en ellos. A partir de las reflexiones buscamos una posible respuesta a la pregunta: ¿Qué aspectos epistemológicos de la transición de la aritmética al álgebra se revelan en los artículos de Yves Chevallard? Las características metodológicas asumidas son de investigación bibliográfica y análisis de contenido. Las conclusiones indican que los aspectos epistemológicos de la aritmética y del álgebra siguen un modelado matemático algebraico/numérico, intermediado por el proceso de transposición didáctica.

Palabras clave: Aritmética y álgebra, Modelado matemática, Transposición didáctica.

Résumé

L'objectif de cet article est de réfléchir sur le passage de l'arithmétique à l'algèbre à partir des publications d'Yves Chevallard. Pour cela, nous avons sélectionné quatre articles de cet auteur qui traitent de cette approche, dans le contexte du système éducatif français, mais avec des implications pour l'enseignement de l'algèbre au Brésil. Les articles ont été sélectionnés à partir des lectures antérieures et par la pertinence des idées qu'ils contiennent. À partir des réflexions, nous cherchons une réponse possible à la question : Quels aspects épistémologiques de l'arithmétique à l'algèbre sont révélés dans les articles d'Yves Chevallard ? Les caractéristiques méthodologiques adoptées sont la recherche bibliographique et l'analyse de contenu. Les conclusions indiquent que les aspects épistémologiques de l'arithmétique et de l'algèbre suivent une modélisation mathématique algébrique/numérique, intermédiée par le processus de transposition didactique.

Mots-clés : Arithmétique et algèbre, Modélisation mathématique, Transposition didactique.

The Transition from Arithmetic to Algebra in the Light of Yves Chevallard's Ideas

Mathematics teaching in France and other countries, such as Brazil, Spain, and Germany, has been marked by problems that have affected curricular proposals long before and after the modern mathematics reform (Chevallard, 1984, 1989, 1990, 1994). In those curriculum reframing, the mathematics corpus began to be questioned, raising discussions about the epistemologies of mathematical knowledge related to arithmetic, geometry, and algebra. We clarify that we do not intend to compare the French educational system with the Brazilian or other countries' system, nor do we intend to expose issues related to teacher education before and after modern mathematics was introduced in schools in Brazil. Instead, we aim to uncover studies that deal with the epistemologies of the objects of school mathematics[5], mainly the mathematical objects that structure school arithmetic and algebra[6].

The problem around arithmetic and algebra dates back to the 15th century, with arithmetic linked to the practices of merchants and traders (Chevallard, 1984). In the transition from arithmetic to algebraic thinking, algebra was paramount in advancing knowledge and leveling up society (Chevallard, 1984), but such expansion opposed both ways of thinking. In addition, the modern mathematics reform in France from the late 1960s caused another opposition in the curriculum structure, putting geometry and numerical structures teaching on irreconcilable sides (Chevallard, 1994). Catalán (2003) and Gascón (2011) revoiced this clash. However, rather than a broader analysis, this study is centered on four works by Chevallard (1984, 1989, 1990, 1994) that discuss the opposition between arithmetic and algebra thinking, dealing with epistemological discussions on the knowledge of arithmetic and elementary algebra that make up school mathematics.

The methodological traits of our approaches are qualitative, using bibliographic research (Severino, 2007) and content analysis. For such, we chose the documents (Chevallard's articles), explored this source, translated excerpts, and concluded through arguments and interpretations (Bardin, 2011).

This article explores the transition from arithmetic to algebraic thinking by drawing on insights from four of Yves Chevallard's publications. From a content analysis of his articles, we aim to provide a possible answer to the question: What epistemological aspects of arithmetic and algebra are revealed in Yves Chevallard's 1984, 1989, 1990, and 1994 studies?

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[5] [...] The history of school mathematics translates into the history of the didactic transpositions from mathematics to the teaching of mathematics (Valente, 2005, p. 21).

[6] Several studies understand them as generalized arithmetic (Usiskin, 1995; Catalán, 2003; Gascón, 2011).

Revisiting ideas for new notes

We begin our analysis from Yves Chevallard's texts published under the title: *Le passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au college* (The transition from arithmetic to algebra in elementary education mathematics teaching in Brazil). It is a series of three articles, divided into three parts, carrying the subtitles: *L'évolution de la transposition didactique* (The evolution of didactic transposition) (1984); *Perspectives Curriculaires: la notion de modélisation* (Curricular perspectives: the notion of modeling) (1989) and *Voies d'attaque et problèmes didactiques* (Ways of coping and didactic problems) (1990). These works shed light on the problems with arithmetic and algebra teaching in French schools.

In these writings, Chevallard cites and analyzes fragments of several authors to demonstrate how the predominance of arithmetic occurred in different periods in the context of social and cultural practices. Chevallard wrote his text based on his analysis of those fragments, which revealed epistemological elements of the mathematical knowledge of arithmetic and algebra. Those elements later served as a theory for the author to reveal the clash of ideas between the importance of teaching arithmetic and algebra in the French educational system. The body of the text exposes Chevallard's (1984) intent to initiate a discussion that he did not exhaust in this first article but advanced in others, which, indeed, materialized with the publication of two more articles.

In the introduction of the 1984 article, the author announces that the reform in the corpus of mathematics through modern mathematics (which occurred in the late 1960s) called attention to the problem of the curriculum long established and legitimized in the French educational system. To explain this problem, Chevallard (1984) resorted to several works that structure mathematical knowledge in the arithmetic and algebraic fields: Jacques Pelletier du Mans (1554), François Viète (1591)[7], Euler (1774)[8], Clairaut (1760)[9], M. Terquem (1827), Newton (1802), A. Lentin and J. Rivaud (1961)[10], Smith (1953)[11], Chevallard and Johsua (1982)[12].

In the sections, Chevallard revealed the position of some of the authors above that advocated arithmetic knowledge in the French school curriculum. According to their

[7] Ditto François Viète (1630).
[8] Ditto Euler (1795).
[9] Ditto Clairaut (1746).
[10] Ditto A. Lentin and J. Rivaud (1967).
[11] Ditto Smith (1958).
[12] Ditto Chevallard and Johsua (1991).

understanding, the epistemologies of the objects of arithmetic were structured by different civilizations throughout human culture and constituted the school curriculum in many centuries. However, other authors deemed algebra the structure of a mathematical corpus that could overcome the limitations of arithmetic practices, which had begun to be questioned more incisively at the end of the 16th century (Chevallard, 1984).

> This undoubtedly necessary reminder of history allows us to point the finger at a crucial fact from which we will draw a few more consequences: the disappearance of a secular way of organizing the mathematical corpus of teaching in recent years. Until then, the opposition between arithmetic and algebra was indeed traditional. Ancient tradition, affirmed from the beginning by Viète himself at the end of the 16th century – , and, in any case, well established in usage: spanning the entire 19th century, was only extinguished in the early 1970s (Chevallard, 1984, p. 52, our translation).

The strength of Viète's ideas comes from the fact that he proposes symbolism for algebra. They shed new light on the arithmetic teaching objects culturally established in different social practices. Likewise, Chevallard (1984) saw that "this tradition – equivalent to a simultaneously epistemological and didactic conception that produces an unaltered teaching text for a long time, or at least with a slow evolution – is opposed in two phases [...]" (Chevallard, 1984, p. 52, our translation). In the first phase, we have the arithmetic corpus, essential to the future learning of mathematics; in the second phase, the ideas of an algebra corpus that wants to occupy its space as cultural and school knowledge. However, Chevallard (1984, p. 52, our translation) inferred that: "[...] Arithmetic provides the set of requirements on which, in a second phase, the authors base the algebra course [...]". The mathematician exemplified this second phase by citing Euller's book *Elements of Algebra*, translated into French in 1774.

In the following sections of the article, Chevallard (1984)[13] discussed the topis: *Une frontière oubliée* (p. 52) (A Forgotten Frontier), *Le passage de l'arithmétique a l'algèbre* (p. 53) (The Transition from Arithmetic to Algebra), *Le devenir de l'arithmétique dans la* reforme (p. 58) (The Future of Arithmetic in the Reform), *Une algèbre introuvable?* (p. 61) (An Algebra not Found?), *L'algèbre sans algèbre?* (p. 64) (An Algebra Without Algebra?), *La dialectique numérique/algébrique* (p. 72) (The Numeric/Algebraic Dialectic) and *Une conception empiriste du réel mathématique* (p. 76) (An Empiricist Conception of the Mathematical Real) in an interconnected way, revealing arithmetic epistemologies intertwined

[13]We recommend the original article as further reading to broaden understandings of these sections.

with algebraic ones, but announced that the of arithmetic practice *habitus*[14] would coexist with the algebraic practice *habitus*.

> The elimination of the arithmetic/algebraic opposition, in effect, alters the connection conditions in relation to arithmetic and algebra. The former relationship between the working tool and the object being worked on seems to have been lost. Both domains – the numeric, the literal – will coexist in a simple juxtaposition, existing ones that find their own justification in themselves. The relationships, usual among those two orders of mathematical reality, now seem to be abolished. Or rather, they open spaces for new inverted relationships: it is no longer the algebraic that allows studying the numerical; it is the numerical that "justifies" and "allows understanding" the algebraic [...] (Chevallard, 1984, pp. 76-77, our translation).

Chevallard argued the excerpt was extracted from a manual of the *quatrième du collège*[15], from the 1970s, as shown in Table 1.

Table 1.

The algebraic as the essence of the numeric (Chevallard, 1984, p. 77, our translation)

III - DIFFERENCE OF TWO DECIMALS

$x \in \mathbb{D}$, $y \in \mathbb{D}$, $z \in \mathbb{D}$, $z = x - y$ means that $z + y = x$.

How is *z* called in place of *x* and *y* in this order?

$z = 13 - (-7)$	$z = x - y$
$z + (-7) = 13$	$z + y = x$
$[z + (-7)] + 7 = 13 + 7$	$(z + y) + (-y) = x + (-y)$
$z + [(-7) + 7] = 13 + 7$	$z + [y + (-y)] = x + (-y)$
$z + 0 = 13 + 7$	$z + 0 = x + (-y)$
$z = 13 + 7$	$z = x + (-y)$

For all *x* in $\mathbb{D}$, for all *y* in $\mathbb{D}$, $x - y$ is a decimal and $x - y = x + (-y)$.

Examples: $8 - (-7) = 8 + 7 = 15$; $9 - 14 = 9 + (-14) = -5$.

The operation that for each pair $(x; y)$, $x \in \mathbb{D}$, $y \in \mathbb{D}$ makes the decimal $x - y$ correspond is the subtraction in $\mathbb{D}$.

The decimals Chevallard (1984) mentioned must be understood as[16] relative integer decimals, taught in the Brazilian educational system as relative rational numbers (symbolized in textbooks by $\mathbb{Q}$, such that the relative integers ($\mathbb{Z}$) and positive and negative fractional

[14] [...] systems of durable and transposable *provisions*, structured structures predisposed to function as structuring structures, that is, as generating and organizing principles of practices and representations [...] (Bourdieu, 2013, p. 87).

[15] Equivalent to the eighth grade of elementary school in Brazil

[16] We call decimal number any rational number *x* for which there is $a \in \mathbb{Z}$, and $n \in \mathbb{N}$ such that $x = \frac{a}{10^n}$. In other words, a rational *x* is a decimal number when there is an $n \in \mathbb{N}$ such that $x \cdot 10^n$ be integer (Dumas, 2005, p. 36, our translation).

numbers, are contained in this set). The ideas in Table 1 made up the 1971 program for the last year of middle school (grade 8), the *quatrième*.

Chevallard concluded this first part on the transition from arithmetic to algebra by emphasizing that:

> Such a conception of knowledge misses the real of which it would be a question precisely of producing knowledge, because it lacks its constitution as an object of knowledge. The algebraic is no longer used to know the numeric. From now on, it is just an essentializing shorthand that describes, summarizes, and separates the essence of the accident. The "modern" surgency of the dichotomy of "observation" and "theory" – which current books revive at will – corresponds to a dissolution of the object of knowledge in favor of the real object, now shown in an abstraction that is considered immediate and easy. The didactic order will be organized around this imaginary epistemology (Chevallard, 1984, p. 81, our translation).

The quotation was an implicit statement that the author would continue the discussions of the first part of *The Transition from Arithmetic to Algebra in Middle School Mathematics Teaching*. In this second part, published in 1989, Chevallard addressed curriculum perspectives and the notion of modeling: *Le passage de l'arithmetique a l'algebrique dans l'enseignement des mathematiques au college – Deuxieme Partie – Perspectives Curriculaires: La Notion de Modelisation* (The Transition from Arithmetic to Algebra in Middle School Mathematics Teaching – Part Two – Curriculum Perspective: The Notion of Modeling). In the introduction, Chevallard (1989) reopened the discussion he had begun in the first article of the series, more precisely, in *Une conception empiriste du réel mathématique*. However, in this section, he exposed *La réforme Chevènement et le triomphe empiriste* (Chevallard, 1989, p. 43) (The Chevènement Reform and the Empiricist Triumph).

The Chevènement reform, occurred in the late 1960s, proposed to rescue the numbers (arithmetic) to the detriment of algebra. This was a disruptive perspective for the French education system curriculum (Chevallard, 1989). This reform underscored the numbers as practical and reality-driven, reducing the emphasis on abstract concepts required in algebra. The Chevènement reform relegated algebra aspects to the background without excluding them, and using letters was considered a previous generalization of the studies of numerical calculations (Chevallard, 1989).

In the introductory section of the second article, Chevallard also dealt with three subsections: *Du calcul formel au calcul fonctionnel*; *Du collège au lycée et au-delà*; *Un problème d'ingénierie curriculaire* (From Formal to Functional Calculation; From Middle to High School and Beyond; A Curriculum Engineering Problem) (Chevallard, 1989, pp. 46-49).

Those subsections reveal problems in arithmetic and algebra teaching in the French education system involving the didactic transposition process established in the official high school program.

> This, in fact, is the essential contradiction. The didactic transposition, which modifies the functioning of the objects of knowledge, gives a certain specificity to the official program that the prodigal teaching proposes to the student. This official program engenders in the student a personal program that, as it is in the official program, will enjoy limited suitability as the referred object of knowledge, which will no longer be a pure didactic bet, it will be just a tool of the student's didactic-mathematical activity: for example, the factorization of an algebraic expression can cease to be the objective of its activity, becoming the means to solve a third degree equation, knowing one of its roots (Chevallard, 1989, p. 47, our translation).

The objects of knowledge Chevallard cited (algebraic expression and equation of the third degree) transit from middle to high school and get to university education (in Brazil, from elementary to high school and higher education). Those objects present problems in the school curriculum, or rather, in school algebra teaching. They are in the official curriculum, and they are taught; therefore, they have been one of the problems of curriculum engineering for a long time. Thus, those objects of knowledge are part of the problem announced by Chevallard (1989):

> The general didactic problem to which we are led can be formulated as follows: is it possible to define and implement a *state of the education system* (i.e., a *curriculum*) that determines an official program for the algebra that is most *appropriate* to the tasks in which it will be used, mainly in high school? (Chevallard, 1989, p. 49, our translation, emphasis added).

Although the problem was identified in the 1980s in France, it remains relevant in mathematics teaching in Brazil today. To propose an answer, first, we must understand the epistemology of the algebraic objects in school mathematics (Valente, 2005). For that, we can start with the section Chevallard (1989, p. 49) called *Calcul algébrique et systèmes de nombres* (Algebraic Calculus and Numbering Systems). This section is unique because it leads back to reflections on the mathematical knowledge in transition in the French education system but somehow connected with the Brazilian education system. There is no denying that the breadth of both conceptions is close, even though decades have passed, and today, we have our thoughts on future technologies. Subsection *Une incontournable dialectic* (An Unavoidable Dialectic) (Chevallard, 1989, p. 50) emphasizes the historical importance of the numbering systems ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{D}$, $\mathbb{Q}$, and $\mathbb{R}$), which transit from elementary to high school and reach university.

We see in numbering systems underlying epistemologies that need to be understood by mathematics educators (teachers or others), mainly in basic education. As previously stated, those epistemologies, which are access bridges to the number and algebraic knowledge, coexist and do not stand alone. There is a habitus of teaching practices that guide the stages (years, grades, cycles, etc.) of the education system as a whole, which is manifest in the words of Chevallard (1989):

> The first domains of calculations found – in history as in school – are constituted by the different *number systems*, successively introduced and studied from primary school to high school: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{D}$, $\mathbb{Q}$, and $\mathbb{R}$. Although those number systems do not have their calculation domains only for those levels - which leads us to think here of vector calculus [...] (Chevallard, 1989, p. 50, our translation, emphasis added).

Formally, Chevallard (1989) explains that the notion of a number system is defined as follows:

> ∗ *Addition* (denoted by +), associative, commutative binary operation, having a neutral element (denoted by 0);
> ∗ *Multiplication*, associative, commutative binary operation, having a neutral element (denoted by 1), and distributivity in relation to addition.
> The effectively targeted number systems also have
> ∗ *(Total) order relation* compatible with addition and multiplication (Chevallard, 1989, pp. 50-51, our translation, emphasis added).

This formal way of defining number systems has epistemological implications for mathematical knowledge, which is, in most cases, restricted to specific groups (mathematicians and professors who teach mathematics at universities). Basic education mathematics teachers study this as part of their initial or continuing education, however, mostly dissociated from the transposition process of school algebra objects, which allows [...] *an explicit didactic choice*, inserting the new notion into the text of the knowledge taught, *satisfying the dialectic of old and new* [...] (Chevallard & Johsua, 1991, p. 171, our translation).

Chevallard (1989), the notion of a number system, and the order relation compatible with the binary operations of addition and multiplication are fundamental for the verification of the simplification rule. For example, let us have two polynomials of the first degree with their coefficients in the number system (SN), i.e. P(x) = ax + b and Q(x) = cx + d (a, b, c and d in SN). We will call it the "equation of the first degree over SN" when we have equality of the type P(x) = Q(x). Therefore, *any first degree equation over SN that is not identically verifiable in that SN has more than one solution* (Chevallard, 1989, p. 51). The simplification rule can be visualized as follows:

Let us have P(x) = ax + b and Q(x) = cx + d. If P(x) = Q(x), then ax + b = cx + d. Simplifying the equality, the solution to the equation is:

$$ax + b = cx + d \Leftrightarrow ax + b + (-cx) + (-b) = cx + d + (-cx) + (-b) \Leftrightarrow$$

$$ax + (-cx) = d + (-b) \Leftrightarrow x \cdot [a + (-c)] = [d + (-b)] \Leftrightarrow$$

$$x \cdot [a + (-c)] \cdot \frac{1}{[a+(-c)]} = [d + (-b)] \cdot \frac{1}{[a+(-c)]} \Rightarrow x = \frac{[d+(-b)]}{[a+(-c)]}, \text{ com } [a + (-c)] \neq 0.$$

Understanding the simplification rule leads to what Chevallard (1989, p. 51) calls *Un problème fondamental* (A Fundamental Problem). This fundamental problem goes back to the study of the properties of number systems.

> We will finally add one last property to the definition of number systems. This property is motivated by the recurrent and fundamental problem that sustains the number systems studied in middle school: such systems, in fact, *do not contain enough numbers*. The need for its repeated extension (from ℕ to ℤ, from ℤ to ℚ etc.) stems from this insufficiency, for which we can find a double origin (Chevallard, 1989, p. 51, our translation, emphasis added).

The double origin for numerical insufficiency stems, according to Chevallard (1989), from *measuring quantities and the existence of numbers resulting from an acceptable algebraic calculation*. Consequently, extending the natural numbers (ℕ) to relative integers (ℤ), from relative integers to rationals (ℚ), from rationals to real numbers (ℝ) and, later, from real to complex numbers (ℂ) becomes necessary to embrace algebraic calculations, such as equation-solving processes. The effect of extending the number systems is in *La maîtrise formelle du calcul fonctionnel* (The Formal Mastery of Functional Calculus) (Chevellard, 1989, p. 52). For Chevallard, this mastery is based on two objectives.

> The *first objective* must ensure the teaching of a satisfactory *formal* manipulation of the algebraic calculus, or, in the most developed version, of the calculation in the R(x) body of rational fractions – an objective that is especially important for students who intend to study beyond high school.
> The dialectical mastery between formal manipulation of the algebraic calculation (or better: of algebraic calculations) and knowledge of number systems thus constitutes a *second goal* of algebra instruction in high school. This goal derives from a double observation: it cannot mean mastering the *functional* algebraic calculus without

> employing the algebraic calculus correctly; one cannot employ algebraic calculation without establishing a dialectic between the number and the algebraic [...] (Chevallard, 1989, pp. 52-53, our translation, emphasis added).

Both objectives –and more intensely the second, led Chevallard (1989, p. 53) to deal with the *La modélisation mathématique* (Mathematical Modeling). This section is divided into eight subsections: *De l'extramathématique à l'intramathématique, Systèmes et modèles, Le cas du pendule simple, Mathématique et mathématisé, La production de connaissances, Réversibilité de la relation de modélisation, Récurrence du processus de modélisation e Modèles locaux, modèles régionaux* (From Extramathematical to Intramathematical, Systems and Models, The Case of the Simple Pendulum, Mathematics and Mathematized, The Production of Knowledge, Reversibility of the Modeling Relation, Recurrence of the Modeling Process, and Local Models, Regional Models) (Chevallard, 1989, pp. 53-58).

In these eight subsections, Chevallard presented epistemological aspects that reveal knowledge of arithmetic and algebraic objects in mathematical modeling. It is important to note that mathematical modeling is not based solely on a methodology for teaching mathematics. As Chevallard (1989) stated, it is a part of mathematical activity, and enables the study of mathematical objects and the creation of other mathematical objects from existing ones. From this perspective, the author revealed the epistemological implications of algebraic knowledge in mathematics.

In subsection *De l'extramathématique à l'intramathématique*, Chevallard (1989) indicated something fundamental: "The question of the functionality of algebraic calculus [...]" (p. 53, our translation). This functionality "[...] must still be analyzed in its general principles as well as in its concrete modalities [...]" (Chevallard, 1989, p. 53, our translation). For the author, intramathematics studies mathematical objects, such as number systems. The extramathematical refers to the use of studies of mathematical objects in physical, biological, and social systems, among others. Chevallard clarified that "[...] it is usual in the mathematical study of such non-mathematical systems for which the name of *mathematical modeling* is employed" (Chevallard, 1989, p. 53, our translation, emphasis added). According to the author, mathematical modeling must be well understood, so he explained systems and models as follows:

> We will first introduce a simplified scheme, which essentially assumes two identity registers: a mathematical or non-mathematical system and a (mathematical) *model* of that system. The modeling process comprises, schematically, three stages.
>
> **1.** The system to be studied is defined by the specification of the *pertinent* "aspects" of what we want to make of that system or the set of *variables* we decompose into the domains of reality where they appear. Letters x, y, z, a, b, c etc. will represent

those variables, reserving the right to return to the – main – question that raises the use of those variables.

2. Then, the model is built to appropriately establish a certain number of relations, R, R', R" etc., between the variables taken into account in the first stage, being *the set of those relationships* the model of the system to be studied.

3. The model thus obtained is handled to produce *knowledge* about the studied systems, a knowledge that takes the form of new relations between the system variables.

Phase 3 is always properly mathematical, while the previous ones originate from the domain of reality, which is supposed to reveal the system – the mathematics that acts on the mathematical object, etc. (Chevallard, 1989, p. 53, our translation, emphasis added).

As Chevallard (1989) pointed out, systems and models serve as a connection for us to visualize the mathematical work involved in a modeling process. So much so that he demonstrated that through the model obtained for the case of the simple pendulum: "An elementary mathematical work on this "raw" model leads to the fundamental relation $T = f(A)\sqrt{L/g}$, which allows, in turn, to produce knowledge about the studied system [...]" (Chevallard, 1989, p. 53, our translation).

The subsection *La production de connaissances* (The Production of Knowledge) reveals the complexity of a large-scale mathematical model, the Pythagorean theorem, which, in one of its algebraic forms, can assume the following representation: $z^2 = x^2 + y^2$. However, this model is more evident in the study of right triangles, whereby the measures of the sides of a right triangle are in the Pythagorean relation: *the length of the hypotenuse squared is equal to the sum of the squares of the lengths of the two legs*. So far, nothing new, but Chevallard (1989, pp. 55-56, our translation, emphasis added) extended the discussion:

A model is usually interesting when it allows us to produce knowledge that otherwise would not be easily given. Let us consider the Pythagorean theorem: it provides a characteristic relation of right triangles, which constitutes a *right triangle model* (a model whose variables are the measures a, b, and c of the sides): $c^2 = a^2 + b^2$. This equality has a classical interpretation in the system register: the area of the square built on the hypotenuse is equal to the sum of the areas built on the two sides of the right angle; moreover, as we know, by demonstrating (by geometric considerations of equalities of areas) this last equality we can establish the Pythagorean algebraic relation. But this looks productive from relations that we get not directly from a geometric point of view: multiplying it by $\pi/8$; getting the equality $\pi c^2/8 = \pi a^2/8 + \pi b^2/8$, whose immediate geometric interpretation is: the area of the semicircle having the hypotenuse in diameter is equal to the sum of the areas of the semicircles constructed on both sides of the right angle. And multiplying the Pythagorean equality by a suitable numerical coefficient ($kc^2 = ka^2 + kb^2$), the same could be said about equilateral triangles ($k = \sqrt{3}/4$, or any other similar figures built on the sides of the triangle.

The modeling Chevallard indicated as the study of the model of right triangles generates knowledge and promotes a more comprehensive mathematical work. In the model, we recognize epistemological elements typical of school algebra but justified by arithmetic reasoning, i.e., we can realize the algebraic/numerical dialectic.

The other sections of the second article solidify the notion of modeling: *Mathematics and Modeling, The Tools of Mathematization, The Closed World and the Infinite Universe*. However, we will not discuss them here, but we leave it up to the reader to check the full text of Chevallard (1989).

Our attention now turns to the last article in the series on *The Transition from Arithmetic to Algebra in Middle School Mathematics Teaching* (Chevallard, 1990). With this article, Chevallard closed the trilogy of the discussion that began in the 1980s. The third article in the series resumes the ideas of the second, but from new perspectives, already perceived in the subtitle: *Voies d'attaque et problèmes didactiques* (Ways of Coping and Didactic Problems) (Chevallard, 1990, p. 5). This article is divided into five sections: *La modélisation comme concept, Constructions croisées: le numérique et l* 'algébrique, *Les entiers naturels comme objet d'étude, Premiers repères d'un programme de recherche* e *Problématique du programme de recherche* (Modelling as a Concept, Interconnected Constructions: The Numerical and the Algebraic, Natural Integers as an Object of Study, First References of a Research Program, and Problems of the Research Program) (Chevallard, 1990, p. 5-37). Each of these sections has subsections. Next, we will focus on the notions contained in some of these subsections.

Starting with the second section, the first subsection, *Un outil fondamental: les entiers naturels* (A Fundamental Tool: The Natural Integers) (Chevallard, 1990, p. 13), shows the importance of studying the natural numbers. Historically and didactically, the natural integers are so constituted as to be considered "[...] the first tool of mathematical modeling" (Chevallard, 1990, p. 13, our translation). This number system is a reference for teaching other number systems, which ensures that it receives special attention from the French education system curriculum (Chevallard, 1990). This idea can be extended to the curriculum of the Brazilian education system.

Natural integers deserve attention because, as a study tool, they intervene "[...] in two levels in the process of algebraic modeling [...]" (Chevallard, 1990, p. 14, our translation). Both levels are understood as follows: "[...] on the one hand, the variables that define the studied system can be SN values. On the other hand, the formulation of the relations that govern the system that can use algebraic expressions with coefficients in SN" (Chevallard, 1990, p. 14, our translation). Although natural integers receive significant attention, they are insufficient, as previously discussed, highlighting the need to extend them from the set of natural numbers ℕ to the set of the relative integers ℤ, implying what Chevallard (1990) considered the *L'insuffisance des entiers naturels comme outil* (The Insufficiency of Natural Integers as a Tool).

The extension of ℕ to ℤ is crucial for the epistemology of algebra because before, the *relative* numbers were called *algebraic* numbers (Chevallard, 1990). This information leads us to what the author called "artificial" numbers. Chevallard (1990, pp. 17-18, our translation) alludes to the "artificial" numbers:

> Negative numbers are therefore introduced into mathematical practice, not as "natural" integers, i.e., as numbers used in counting or "measuring" finite sets, but as a calculation tool, an *artifice* used to calculate – just like an automobile can be considered a "transport artifice". For this reason, they were long ago categorized – including fractions – in the category of artificial numbers (Smith, 1925, Chapter IV), allowing them to be used flexibly, thus more pleasantly, as a powerful mathematical tool, the algebraic calculus.

The negative numbers boosted mathematical knowledge in the algebraic field. This powerful mathematical tool gave a new dimension to algebraic calculation, mainly to the "rule of signs" game, known long before those numbers were invented (Chevallard, 1990). In Table 2, we have more explanations about the "artificial" numbers.

Table 2.

Fragments of the history of "artificial" numbers (Chevallard, 1990, p. 18, our translation)

The "artificial" numbers – here, the negative ones – are not, therefore, primarily motivated by the study of systems with variables that take positive and negative integer values, as we stubbornly want to believe, presenting rare systems of this type – altitude, lift, losses, and earnings etc. They are born *of requirements internal to mathematical work* (exactly: algebraic). Undoubtedly, its introduction, which extends the numerical domain, will raise many historical questions that necessarily find an echo in the middle school curriculum. It is known that Diophante, in the third century A.D., rejected as absurd the equation $4x + 20 = 4$, which would give $x = -4$. In the 16th century, Cardano called negative numbers false numbers, and Descartes, in 1637, called *false roots* the negative roots of an equation. Considering negative numbers as "true" numbers required us to risk writing equalities such as $(+15) + (-20) = -5$ (Bombelli, 1572), which was eventually accepted in the seventeenth century. Above all, let us say that in the progressive normalization of the status of the negatives, trivializing their use, *letters will play an essential unifying role*. Using a single letter, *unaffected by a sign*, to designate, indifferently, a positive or negative number seems to have appeared around 1659 in Hudde (Smith, 1925, p. 259).

Table 2 contains fragments of the epistemology of the relative integers. This allows us to think about how the extension of ℕ to ℤ was structured. We will see the main ideas of this extension in Table 3.

Table 3.

The transition from ℕ to ℤ (Chevallard, 1990, pp. 18-19, our translation)

The transition from ℕ to ℤ, as indicated, then starts from the following problem. For all a > 0, it is necessary to introduce a "number", denoted by –a such that, for any natural integers b and c, it turns out that b > ac, and we can write: b – ac = b + (–a)c. Then, let there be a number such that, according to the definition (in ℕ) from b – ac, we have: b = ac + b + (–a)c.

1. Let us take b = a and c = 1. The number –a must therefore check the equality a = a + a + (–a), or even a + (–a) = a – a = 0. In other terms, –a is the *solution to the equation x + a = 0*. If the set we want to obtain from ℕ by adding "numbers" –a (*a* natural integer > 0), is effectively a *number system* ℤ (as defined above); so, this equation has on ℤ only one solution, and thus the equation x + a = 0 *characterizes* –a *completely*.

2. Assuming that ℤ is a good system of numbers, from equality a + (–a) = 0, we first deduce (by multiplying by c) the equality ac + (–a)c = 0, and hence (by adding b), the equality ac + b + (–a)c = b. If the natural integers a, b, and c are such that b > ac, then we have: b – ac = b + (–a)c. *Thus, to form ℤ from ℕ, it is necessary and sufficient to associate, for every integer a > 0, a number –a such that a + (–a) = 0.*

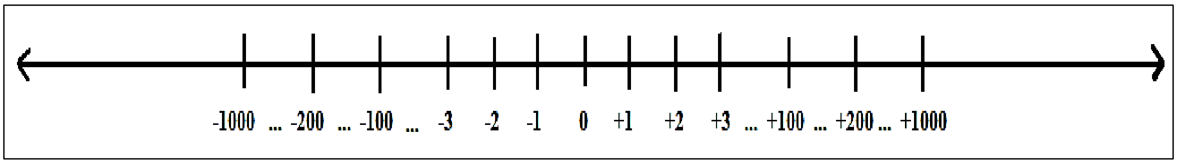
The hypothesis that there may be such a number system is solid. This implies particularly that, on ℤ, any equation of the form x + a = 0 has a solution: if, in fact, *a* belongs to ℕ, then –a belongs to ℤ and is the solution to the equation; if *a* belongs to ℤ – ℕ, there exists b in N such that a = –b, and the considered equation is then satisfied for x = b. We observe, in general, that –a is the (unique) solution of the equation x + a = 0, for *a* belonging to ℤ. If a = –b, with b in ℕ, then we have: b = –a = –(–b). But generally, for all *a* in ℤ, we have: –(–a) = a.

The hypothesis that ℤ is a number system allows us to see how convenient it is *to calculate* in ℤ. So be numbers –a and –b, with a and b in ℕ. How much is the *sum* (–a) + (–b)? We find the equalities (–a) + a = 0 and (–b) + b = 0. By adding member to member, and due to the properties lent to ℤ, it follows that: ((–a) + (–b)) + (a + b) = 0. This last equality gives the expected answer because it shows us that: (–a) + (–b) = –(a + b). All properties of ℤ as far as addition and subtraction operations are concerned and all compatibility relations between order and laws of composition can be obtained in this way. Furthermore, the *absolute value* function can then be defined and the triangular inequality established. ℤ differs from ℕ, because subtraction is always defined in it, and the property of good order is no longer satisfied only for the smallest sets but is sufficient to approximate the ℤ order to the ℕ order, which is discrete.

Table 3 has a simplified epistemological model to explain the extension from the system of natural numbers (ℕ) to the system of integers (ℤ). This extension is complex and is based on the theory of groups and rings of modern algebra (not addressed in this study). Table 3 portrays an important modeling study for those who teach mathematics in the final years of elementary school. But not only that, it is a process of mathematical modeling that produces mathematical objects crucial to school mathematics. In a not-so-simple but practical way, extending from ℕ to ℤ can be summarized as a graduated number line with positive integers on one side (right) and negative integers on the other (left), separated in the "middle" line by the digit zero, as shown in Figure 1.

Figure 1.

Graduated line with relative integers ($\mathbb{Z}$) (Prepared by the authors)



With the system of relative integers, the algebraic work stepped to another level. The limitation of the subtraction in $\mathbb{N}$ is overcome, which raises mathematical knowledge in relation to algebraic calculation but also creates problems for mathematics teaching in the school context. So much so, that Chevallard (1990) dealt briefly with the *Problèmes ouverts et curriculum* (Open Problems and Curriculum). The mathematician exemplified this obstacle by resorting to the problem related to the study and teaching of fractions and their generation through algebraic modeling. The modeling is intrinsically associated with open problems, bridging the gap between mathematical activity in schools and that of the mathematician (Chevallard, 1990).

The final two sections of the third article announce the *Premiers repères d'un programme de recherche* (First References of a Research Program) and the Problématique *du programme de recherche* (Problems of the Research Program) (Chevallard, 1990). We will not show the approaches in those sections here, however, the interested reader may consult Chevallard (1990). Instead, we will advance to the next section, returning to an initial idea – the didactic transposition.

Didactic transposition and school algebra

In 1994, Yves Chevallard published the article *Enseignement de l'algèbre et transposition didactique* (The Teaching of Algebra and Didactive Transposition) (Chevallard, 1994, p. 175), where he extended the discussions in the series *Passing from Arithmetic to Algebra in Mathematics Teaching in Middle School* (Chevallard, 1984, 1989, 1990). However, Chevallard (1994) focus was on the teaching of algebra and the process of didactic transposition. Over fifty pages long, the scholar divided his article into two themes: *Sur le processus de transposition didactique* (p. 175) (On the Didactic Transposition Process) and *Sur l'enseignement de l'algèbre élémentaire* (p. 180) (On the Teaching of Elementary Algebra). However, it is important to clarify that we will not provide an in-depth analysis of this article. Instead, we will focus on the most relevant ideas related to the didactic transposition process and expand on the discussions presented in Chevallard's (1984, 1989, 1990) articles.

The didactic transposition process has structural elements united on a characteristic universe: *society*. The *education system* is created within society. Within the constituted education system the *didactic systems* form, live, and disappear (Chevallard, 1994). The didactic systems components are *educators* (teachers), *educands* (students, apprentices, etc.) and *knowledge* (e.g., algebraic operations) (Chevallard, 1994).

Chevallard (1994) clarified that the education system *stricto sensu* is surrounded by an interface zone with society: *noosphere*[17]. To Chevallard (1994, p. 175, our translation), "the whole of the education system and its noosphere is designated as the education system *lato sensu*. The noosphere contains, in particular, active teachers, their associations, producers, and guardians of any didactic doctrine, etc." In addition, Chevallard drew attention to the social groups that are involved as structural elements of the didactic transposition.

> Inside society, and outside the education system *lato sensu*, two instances play an essential role in the mechanisms examined further ahead: the scholarly community related to the knowledge taught –here, the community of mathematicians– on the one hand, and the group of parents on the other.
> To these two social groups, several others can be added whose importance has hitherto been (at least as far as general education is concerned) relatively weak: in particular the groups of professionals (Chevallard, 1994, pp. 175-176, our translation).

The didactic transposition involves understanding the *L'écologie du savoir enseigné* (The Ecology of the Taught Knowledge) (Chevallard, 1994). This ecology is intertwined with the genesis of the taught knowledge and the conditions that enable the existence of this knowledge, i.e., it is "the set of those conditions (and the analyses that take them as an object)" (Chevallard, 1994, p. 176, our translation). The ecology of the taught knowledge leads to the process of didactic transposition, as announced by Chevallard (1994):

> The process of didactic transposition then emerges as the set of mechanisms by which the taught knowledge is generated in forms compatible with the set of conditions imposed on it and in relation to which it must prove its viability – unless it disappears

[17] Chevallard (1985/1991) introduced the notion of noosphere of the education system in the context of the theory of didactic transposition to designate the sphere where the functioning of the didactic system is thought. It is the true filter through which the interaction between the education system and society operates. The noosphere relates the institution producing knowledge with the school. The productions of the noosphere (official programs, textbooks, recommendations for teachers, teaching resources, etc.) strongly condition the characteristics and even the nature of the knowledge that must be taught in school (Farras et al. 2013, p. 3-4, our translation).

from the didactic systems (disappearance, which is, moreover, a banal and periodically observable phenomenon) (p. 176, our translation).

As the transposition process was announced, Chevallard himself (1994) posed and answered a question:

> Where does the taught knowledge come from? It comes, essentially, from the corresponding scholarly knowledge. To which elements of endogenous knowledge must be added, produced explicitly within the education system (*lato sensu*), which we will designate here as *didactic creations* (p. 179, our translation).

The idea of scholarly knowledge was associated with what was produced and legitimized by the scholarly (academic) community. This was the initial conception of the phenomenon of didactic transposition, later modified by Yves Chevallard himself in other articles and with the proposition of the anthropological theory of the didactic (Almouloud, 2007).

The influence of the process of didactic transposition of the *corpus* from algebra to school algebra can be seen in Table 4.

Table 4.
A summarized model to characterize a process of didactic transposition (Chevallard, 1994, p. 201, our translation)

Nothing is more revealing of this cultural pressure on mathematics teaching than the concepts that dominate the analyses made explicit and produced in the noosphere as much as the teaching practices concerning “the notion of number”, “the learning of numbers”, etc. To go a little deeper into this topic, let us first give a counterpoint, starting with the *negative* numbers. Far from arising from the relationship of an extramathematical reality, they appear within a certain mathematical practice, the resolution of equations. The natural integer is a solution to the equation x = 3. Now, what happens if we have the equation x + 3 = 0? No natural integer is the solution. There is a mystery, but an *intramathematical* mystery. And the key to this mystery is by no means looking for it into the extramathematical reality. It will take almost ten centuries, from the Arab algebraists of the 9th century to the algebraists of the 19th century and to Hermann Hankel (1867), for the relevant concepts to be made explicit: not “the number”, but the *number system,* and the *principle of permanence* associated with it. Since, in effect, the solution of an equation of this type cannot be a natural number, let us consider (adopting here a realistic point of view that, technically, can be immediately translated from a constructivist point of view) that there is a system of numbers that satisfy the “usual laws” of integers (for simplicity, we will not specify the details of those laws), which are solutions to the equations $x + n = 0$, where n is any non-zero natural integer. We will denote n^* the “number” solution of the equation $x + n = 0$, such that $n^* + n = 0$. The, how to calculate with such numbers? Let, for example, perform 5 + 7*. We have: (5 + 7*) + 7 = 5 + (7* + 7) = 5. From the equality (5 + 7*) + 7 = 5, it follows that (5 + 7*) + 2 = 0, that is, that 5 + 7* = 2*. Likewise, to calculate 5* × 7. We start from the equality 5 + 5* = 0. Multiplying by 7, you get: 5 × 7+5* × 7 = 0, or else 5* × 7 + 35 = 0, from which it follows that 5* × 7 = 35*. In the same way, one will obtain the value of 5* 7*by such manipulations

of equalities. Multiplying $7^* + 7 = 0$ by 5^*, we have that $5^* \times 7^* + 5^* \times 7 = 0$. As $5^* \times 7 + 5 \times 7 = 0$, we finally obtain, by adding 5×7 to the two members of the preceding equality, $5^* \times 7^* = 5 \times 7$. The famous *rule of signs*, which is a puzzle almost unseen by the dominant culture, finds its origin here: it is "forced" by a hypothesis of the mathematical work – the "principle of permanence" (in the transition from the natural integers to the relative integers) of the laws that govern the natural integers.

The mathematical modeling in Table 4 has a characteristic algebraic model to explain the rule of sign in the system of relative integers ($\mathbb{Z}$). This algebraic model is a simplified example of the didactic transposition process, which approximates the axiomatics of modern algebra with the habitus of school algebra practice.

Let us make Chevallard's (1994) ideas more explicit about the modeling in Table 4. Or rather, we will make a simple didactic transposition.

The equations in the form $x + n = 0$ (with $n \in \mathbb{N}$) can be equations of the type: $x + 6 = 0$; $x + 5 = 0$; $x + 8 = 0$; $x + 23 = 0$; $x + 140 = 0$. The solution $x = n^*$ for each is in the system of relative integers. And what is n^*? It is nothing more than the symmetric of $+ n$, i.e., $n^* = -(+n)$, therefore, in the equation $x + 140 = 0$, $x = n^* = 140^* = -(+(140))$. So, $n + n^* = 0 \Leftrightarrow 140^* + 140 = -(+(140)) + 140 = 0$.

Returning to the situation $(5 + 7^*) + 7 = 5 + (7^* + 7) = 5$ from Table 4. It is thus understood: $(5 + (-(+7))) + 7 = 5 + (-(+7) + 7) = 5 + 0 = 5$. From equality $(5 + (-(+7))) + 7 = 5$, we have:

$(5 + (-(+7))) + 7 + (-(+5)) = 5 + (-(+5)) \Leftrightarrow (5 + (-(+7))) + \mathbf{7} + (-(+5)) = 0 \Leftrightarrow$

$(5 + (-(+5) + (-(+2))) + (7 + (-(+5))) = 0 \Rightarrow (0 + (-(+2))) + (2 + 5 + (-(+5))) = 0 \Leftrightarrow$

$-(+2) + (2 + (5 + (-(+5))) = 0 \Rightarrow -(+2) + (2 + 0) = 0 \Leftrightarrow -(+2) + 2 = 0 \Leftrightarrow 2^* + 2 = 0$

What follows, therefore, is that $5 + 7^* = 2^*$. This operation is in school algebra, but in its final ("economic") form: $5x - 7x = (5 - 7)\,x = -2x$. It is an algebraic sum of relative integers with opposite signs: $(+5) + (-7) = -2$.

From the extension of the previous reasoning, we can calculate $5^* \times 7$, starting from $5 + 5^* = 0$. So,

$5 + (-(+5)) = 0 \Rightarrow (5 + (-(+5))) \times 7 = 0 \times 7 \Leftrightarrow (5 \times 7) + ((-(+5)) \times 7) = 0 \Leftrightarrow 35 + ((-(+5)) \times 7) = 0 \Leftrightarrow ((-(+5)) \times 7) + 35 = 0 \Leftrightarrow ((-(+5)) \times 7) + 35 + (-(+35)) = 0 + (-(+35)) \Leftrightarrow ((-(+5)) \times 7) + (35 + (-(+35))) = (-(+35)) \Leftrightarrow ((-(+5)) \times 7) + 0 = (-(+35)) \Leftrightarrow ((-(+5)) \times 7) = (-(+35)) \Leftrightarrow 5^* \times 7 = 35^*$.

In school algebra, this operation appears in the "economic" form, as we see in the examples: $-5\,(+7x) = -35x$; $-5(7x + 3y + 2b + 5xy) = -35x - 15x - 10x - 25xy$. It is the algebraic multiplication of relative integers with opposite signs: (−5) · (+7) = −35. Similarly, 5* × 7* = 5 × 7 is equivalent to: (−5) · (−7) = (+5) · (+7) = +35.

The continuation of Chevallard's (1994) analytical discourse revealed *cultural and ideological lamination* in algebra teaching, ensured mainly by the noosphere. This lamination is in teaching practice: "This cultural crushing of algebra, retransmitted even within the noosphere, as has already been said, and which weighs on teaching practices [...]" (Chevallard, 1994, p. 202, our translation). Chevallard (1994) pointed out that "[...] elementary algebra is, mathematically speaking, an authentic *tool for creating concepts* [...]" (p. 202, our translation, emphasis added). To show the transposition effect of the creation of concepts, the author extended the "forged" notion to the concept of "negative number system", as shown below:

> […] said how allows forging the concept of "negative number systems" (from the relative number system); however, it apparently provides the tool to create the concept of a "system of rational numbers": a and b being two natural integers (or two relative integers) and a being non-zero, the equation $ax = b$ has a unique solution, denoted by b/a. If k is a non-zero integer, the equations $ax = b$ and $kax = kb$ are equivalent so that $kb - ka = b/a$ (thus, the same rational number has an infinite number of names in the form b/a) [...] (Chevallard, 1994, p. 202, our translation).

Chevallard ended this 1994 article with a question: "[...] is it possible, within society, to reach a consensus on the importance and mastery of the algebraic tool and its correlative place in compulsory education?" (p. 208, our translation). However, the unfolding of this questioning in the French education system, according to Chevallard, was something unlikely to happen soon.

Final considerations

The ideas in this article pervade epistemological aspects contained in four articles by Yves Chevallard. The content analysis of those articles aimed to answer the following: What epistemological aspects of arithmetic and algebra are revealed in Yves Chevallard's 1984, 1989, 1990, and 1994 articles? To achieve this objective, we extracted the main ideas contained in these papers related to the subject: expose an analysis of Chevallard's articles on the transition from arithmetic to algebra that legitimizes the teaching of mathematical objects in the school curriculum, but which, for centuries, have undergone didactic transposition. This objective led us to a possible, albeit partial answer to the question posed.

The 1984, 1989, and 1990 articles are in the series entitled *The Transition from Arithmetic to Algebra in Middle School Mathematics Teaching*. The subtitles follow a thematic logic: *The evolution of didactic transposition*; *Curriculum perspectives*: *The Notion of Modeling*; and *Ways of Coping and Didactic Problems*. The 1994 article focuses on the teaching of elementary school algebra and its relationship with the process of didactic transposition: elementary algebra teaching and didactic transposition.

In Chevallard's articles, we identified epistemological aspects intrinsically linked to school algebra. Among them, we highlight 1) the historical-cultural context of mathematical knowledge; 2) arithmetic in the habitus of social practices; 3) algebraic symbolism; 4) the predominance of arithmetic in the curriculum of the French teaching system; 5) resistance to accepting the algebraic/numerical dialectic, with a predominance of the numerical/algebraic; 6) the numerical inadequacy of number systems: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, etc.; 7) the need to extend the system of natural integers ($\mathbb{N}$) to the system of relative integers ($\mathbb{Z}$); 8) the emergence of "artificial" numbers (negative numbers); 9) the extension from $\mathbb{N}$ to $\mathbb{Z}$ by mathematical algebraic modeling and 10) the extension of number systems by the operative properties derived from modern algebra through the theory of fields and rings.

The 1994 article resumes the ideas of the 1984, 1989, and 1990 articles, but from an epistemological perspective of the didactic transposition process. In the 1994 article, we identified several epistemological aspects related to elementary school algebra. Thus, we selected a fragment, Table 4, which brings a simplified epistemological model for the rule of the game of signs pertinent to the algebraic operations of the system of relative integers, mainly, for the additive and multiplicative operations. Another point is the extension of the ideas for creating mathematical concepts, for example, of rational and irrational numbers from studies of equations.

From Chevallard's articles, we infer that the epistemological aspects of school algebra follow an algebraic-numerical dialectic through equations. This dialectic reveals that mathematical modeling is intrinsically related to the process of didactic transposition of elementary algebra objects taught in the French education system. On the other hand, we see that school algebra, when understood as generalized arithmetic, initially allows a process of didactic transposition to be modeled with the knowledge of arithmetic, but the evolution of this transpositional process leads to adequate mathematical modeling independent from elementary algebra, i.e., the modeling of algebraic calculus through the functional character of polynomial algebraic expressions.

The functional modeling of algebraic expressions requires us to understand the epistemology of the properties of the extensions of number systems ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). It is in those understandings that the corpus of algebra gains some epistemological independence from the corpus of arithmetic and geometry. Thus, through our analysis of Yves Chevallard's 1984, 1989, 1990, and 1994 articles, we identified plausible understandings that shed light on various epistemological aspects of the transition from arithmetic to algebra. However, the issue is not exhausted. Other understandings in those articles cover the answer to the question we set out to answer. Furthermore, the ideas presented in this article do not encompass all the content of Chevallard's four articles. Therefore, further research is necessary to understand how these ideas are transposed to other education systems, such as the Brazilian educational system.

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