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Spatial geometric thinking and its articulation with the visualization and manipulation of objects in 3D

Pensamiento geométrico espacial y su articulación con la visualización y manipulación de objetos en 3D

La pensée géométrique spatiale et son articulation avec la visualisation et la manipulation d'objets en 3D

O pensamento geométrico espacial e sua articulação com a visualização e a manipulação de objetos em 3D

Zsolt Lavicza [1]
Department of STEM Education, Linz School of Education, Johannes Kepler Universität Linz Austria; PhD (Faculty of Education Cambridge)
https://orcid.org/0000-0002-3701-5068

Celina Aparecida Almeida Pereira Abar [2]
Pontifical University Catholic São Paulo, Brazil;
PhD in Mathematics
https://orcid.org/0000-0002-6685-9956

Mathias Tejera [3]
Department of STEM Education, Linz School of Education, Johannes Kepler Universität Linz, Austria
Master of Science, Educational Mathematics
https://orcid.org/0000-0002-9826-966X

Abstract

This paper aims to build a conceptual framework that addresses spatial geometric thinking and the respective visualization skills required at different levels of the schooling process. Studies indicate that spatial geometric sills are essential for scientific thinking. It encompasses a set of cognitive processes through which humans can construct and manipulate mental representations of objects in space and is a skill directed towards understanding objects and their relations in the 2D and 3D worlds. The use of these spatial reasoning skills involves drawing, manipulating, and explaining objects and their relationships and should be developed from the first years of schooling. Based on this theoretical context, partial research results on surface representations that can be manipulated in three dimensions (3D) and obtained through GeoGebra will be

[1] zsolt.lavicza@jku.at
[2] abarcaap@pucsp.br
[3] mathias.tejera@jku.at

presented in this paper. The underlying theory was Duval's Theory of Registers of Semiotic Representation, which allowed the analysis of activities developed by graduate students in Mathematics Education by observing and physically manipulating such representations to obtain the respective graphic and algebraic records. The conceptual framework constructed and presented in this article contributed to identifying other skills required in this study for the development of spatial geometric thinking.

***Keywords*:** Spatial Geometric Thinking, Visualization, GeoGebra, Mathematics Education.

Resumen

Este artículo tiene como objetivo construir un marco conceptual que aborde el pensamiento geométrico espacial y las respectivas habilidades de visualización requeridas en diferentes niveles del proceso escolar. Los estudios indican que el pensamiento geométrico espacial es esencial para el pensamiento científico, ya que abarca un conjunto de procesos cognitivos a través de los cuales el ser humano es capaz de construir y manipular representaciones mentales de objetos en el espacio y es una habilidad dirigida a la representación, uso de objetos y sus relaciones en los mundos 2D y 3D. Manipular y explicar objetos y sus relaciones y debe desarrollarse desde los primeros años de escolaridad. A partir de este contexto teórico, se presentan resultados parciales de una investigación sobre representaciones de superficies que pueden ser manipuladas en tres dimensiones (3D) y que fueron obtenidas a través de GeoGebra. La Teoría de los Registros de Representación Semiótica de Duval permitió el análisis de las actividades desarrolladas por los estudiantes graduados en Educación Matemática, mediante la observación y manipulación de dichas representaciones para obtener los respectivos registros gráficos y algebraicos. El marco conceptual construido y presentado en este artículo contribuyó para la identificación de otras habilidades requeridas en este estudio para el desarrollo del pensamiento geométrico espacial.

Palabras clave: Pensamiento Geométrico Espacial, Visualización, GeoGebra, Educación Matemática.

Résumé

Cet article vise à construire un cadre conceptuel qui aborde la pensée géométrique spatiale et les compétences de visualisation respectives requises à différents niveaux du processus de scolarité. Des études indiquent que la pensée géométrique spatiale est essentielle pour la pensée scientifique, car elle englobe un ensemble de processus cognitifs à travers lesquels l’être humain est capable de construire et de manipuler des représentations mentales d'objets dans l'espace et est une capacité dirigée vers la représentation, l'utilisation des objets et leurs relations dans les mondes 2D et 3D. manipuler et expliquer les objets et leurs relations et devrait être développé dès les premières années de scolarité. Sur la base de ce contexte théorique, les résultats partiels d'une recherche sur les représentations de surfaces manipulables en trois dimensions (3D) et obtenues grâce à GeoGebra sont présentés. La théorie des enregistrements de représentation sémiotique de Duval a permis l'analyse des activités développées par les étudiants diplômés en enseignement des mathématiques, en observant et en manipulant ces représentations pour obtenir les enregistrements graphiques et algébriques respectifs. Le cadre conceptuel construit et présenté dans cet article a contribué à l'identification d'autres compétences requises dans cette étude pour le développement de la pensée géométrique spatiale.

Mots-clés : Pensée géométrique spatiale, visualisation, GeoGebra, enseignement des mathématiques.

Resumo

Neste artigo pretende-se construir um quadro conceitual que aborde o pensamento geométrico espacial e as respectivas habilidades de visualização requeridas nos diferentes níveis do processo de escolarização. Estudos indicam que o pensamento geométrico espacial é essencial para o pensamento científico, uma vez que engloba um conjunto de processos cognitivos por meio dos quais o ser humano é capaz de construir e manipular representações mentais de objetos no espaço e é uma habilidade direcionada à representação, uso de objetos e suas relações nos mundos 2D e 3D. O uso dessa habilidade de raciocínio espacial envolve desenhar, manipular e explicar os objetos e suas relações e deve ser desenvolvida desde os primeiros anos de escolarização. Com base nesse contexto teórico são apresentados resultados parciais de uma pesquisa sobre representações de superfícies que podem ser manipuladas em três dimensões (3D) e que foram obtidas por meio do GeoGebra. A Teoria dos Registros de Representação Semiótica de Duval, permitiu a análise das atividades desenvolvidas por pós-graduandos em Educação Matemática, ao observar e manipular tais representações para obter os respectivos registros gráficos e algébricos. O referencial conceitual construído e apresentado neste artigo, contribuiu para a identificação de outras habilidades requeridas neste estudo para o desenvolvimento do pensamento geométrico espacial.

***Palavras-chave*:** Pensamento Geométrico Espacial, Visualização, GeoGebra, Educação Matemática.

Spatial geometric thinking and its articulation with the visualization and manipulation of objects in 3D

Spatial geometric thinking is the ability to visualize, represent, and manipulate three-dimensional objects in our minds. Spatial visualization skills are those cognitive skills that an individual can acquire to process images in his mind and give solutions to various problems (Gutiérrez, 1996; Presmeg, 2006). They are fundamental skills for understanding and solving problems in spatial geometry and related areas such as architecture, engineering, physics, and design. It involves the perception of shapes, sizes, positions, and orientations of objects in three-dimensional space, as well as the ability to perform spatial transformations such as rotations, reflections, and translations. These skills are important for both conceptual understanding and practical problem solving, from creating mathematical models to planning physical buildings.

The development of spatial geometric thinking is influenced by factors such as experience, education, and cultural environment. For example, people who live in societies with more elaborate architecture and design tend to have better skills in spatial geometric thinking than those who do not have these experiences.

Howard Gardner (1983), in his theory of multiple intelligences, mentions spatial intelligence and considers spatial thinking essential to scientific thought since this is the set of cognitive processes through which the human being can construct and manipulate mental representations of objects in space. Duval (1998) points out that the study of geometry involves three types of cognitive processes fundamental to geometric learning: visualization processes, construction by tools, and reasoning. In this process, visualization is understood as a heuristic exploration of a complex situation. Configuration construction can function as a model, and reasoning concerns knowledge, proof, and explanation. While Pittalis and Christou (2013) argue that in interpreting 3D figure representations, they utilize two capabilities: a) recognize the properties of 3D shapes and compare 3D objects, and b) manipulate different representational models of 3D objects.

Based on van Hiele's model, (van Hiele, 1999) on the development of geometric thinking and used to describe and analyze students' thinking in 2D geometry, Gutiérrez (1992) proposed levels of 3D thinking and stated that in Mathematics, visualization is a type of activity based on the use of visual or spatial elements, whether mental or physical, performed to solve problems or prove properties (Gutiérrez, 1996). Fujita et al. (2014) refined the work of Gutiérrez (1992) and argue that to capture the geometric thinking of students involved with 3D shapes should be considered levels of situations such as students' thinking is influenced by 2D representations; students begin to use 3D properties of shapes but without effectiveness;

students use 3D properties and manipulate the figure, but it is still not appropriate; students use 3D properties and manipulate the figure appropriately, but with an incorrect answer; students utilize 3D properties and manipulate the figure appropriately to get the correct answer.

These levels proposed by Fujita match van Hiele's levels. The van Hiele levels were proposed by Dutch educator couple Pierre and Dina van Hiele in the 1950s to understand how students develop their understanding of geometry. They proposed that students go through five levels of development, each with its distinct characteristics. These levels are:

Level of visualization: Students primarily use their observation and visualization ability to understand geometry at this level. They are able to recognize and name simple geometric figures, such as triangles, squares, and cubes, and can draw and manipulate them in two dimensions.

Level of analysis: At this level, students begin to analyze the geometric properties of the figures. They are able to identify relationships between the figures, such as congruence and similarity, and can begin to prove geometric properties using logical arguments.

Level of ordering: At this level, students begin to organize their geometric ideas into broader, more abstract categories. They can classify figures based on their properties and use diagrams and models to help understand the relationships between the figures.

Deductive Analysis: Characterized by the ability to analyze a system, identify its properties and relationships, and then use that information to make predictions and generalizations. Students can understand the logic behind a geometric construction, using deduction to predict and justify other properties of the figure.

Abstraction: The most advanced level, where the student can make connections between different systems and see underlying patterns and structures, can understand mathematical concepts in general, and transfer their skills and knowledge to new situations. They can create and manipulate complex mathematical models to solve problems and abstractly express their ideas.

It is worth mentioning that all students do not necessarily achieve these levels, and that the process of developing the understanding of geometry can vary according to the age, previous experience, and individual skills of each one.

To understand the nature of students' geometric thinking when interacting with three-dimensional objects, Fujita et al. (2014) rely on an analytical model and articulate the constructs of spatial thinking levels and the capabilities involved in interpreting three-dimensional objects with three levels characterized by two dimensions: (i) recognition of properties of three-dimensional figures and (ii) manipulation of different representations of three-dimensional objects.

Spatial visualization skills develop throughout life and vary by age group. In general, from the age of 6, it is possible to expect the child to develop more complex visualization skills, with which he can:

Identify patterns, recognize and identify objects, images, and sequences. (pattern recognition)

Being able to remember images and visual details for longer periods. (visual memory)

Develop a better understanding of spatial concepts such as distance, direction, proportion, and perspective. (Spatial understanding)

Imagine and visualize objects, situations, and scenarios that are not present. (imagination)

Use visual thinking to solve problems, make connections, and formulate hypotheses. (visual reasoning)

Draw or sketch images that represent your ideas and thoughts. (graphic representation)

Understand the reading of maps, diagrams, and other visual representations to understand and interpret information.

Spatial geometric thinking involves understanding and manipulating shapes, positions, and movements in space. Several theories seek to explain how people develop and use spatial geometric thinking, some of which are:

Mental Representation Theory: This theory suggests that spatial geometric thinking is a form of mental representation that allows people to create mental images of objects and concepts not present in the physical environment.

Spatial Cognition Theory: This theory emphasizes the importance of spatial cognition in different areas of cognition, such as memory, language, and thought. She suggests that understanding space is fundamental to constructing knowledge in other areas.

Situated Learning Theory: This theory highlights the importance of contextualized and situated learning in developing spatial geometric thinking. It suggests that understanding geometry is best gained when related to concrete real-world situations.

Deductive Analysis Theory emphasizes the importance of deduction in spatial geometric thinking. She suggests that individuals develop their understanding of geometry by analyzing the properties of objects and deducing new properties from those already known.

Social Interaction Theory: This theory emphasizes the importance of social interaction in developing spatial geometric thinking. She suggests that exchanging ideas and solving problems in groups can help improve understanding of geometry.

Different elements can be chosen and brought together more or less harmoniously in the interaction between theories to investigate a particular research problem.

These theories have influenced the teaching of geometry and research in the area, helping to understand how individuals develop spatial geometric thinking skills and how this can be enhanced.

There are several strategies for developing spatial geometric thinking, including the physical manipulation of objects, the creation of such objects with technological resources, mental visualization, graphic representations, and working with problems and challenges that require these skills. Education in spatial geometry is fundamental for the formation of individuals capable of dealing with space-related issues in various areas of knowledge. The use of technology in mathematics education brings a new set of possibilities to this endeavor.

Possible Technological Resources for the development of spatial geometric thinking

Technological resources can be helpful in the development of spatial geometric thinking since they allow the creation and manipulation of three-dimensional objects virtually, offering new possibilities for exploration and learning. Here are some ways technology can be used in this context:

3D modelling software: this software allows the creation of three-dimensional objects virtually, making it possible to visualize and manipulate these objects from different perspectives. In addition, this software can also be used to create animations, which can make it easier to understand more complex concepts.

Games and educational applications: some several educational games and applications aim to develop spatial geometric thinking, offering challenges and problems to be solved by users. These games and apps can be used both in and outside the classroom.

Virtual reality: Virtual reality technology allows the creation of virtual environments in 3D, enabling the exploration and interaction with objects and spaces in an immersive way. This technology can be very useful for understanding complex concepts such as Euclidean geometry.

Augmented Reality: AR is a technology that superimposes virtual objects (augmented components) on digital representations of the real world. Virtual objects then appear to coexist in the same space as real-world objects.

3D printers: 3D printers allow the creation of three-dimensional objects physically, from digital models, which can be very useful for understanding geometric concepts and allows students to visualize and manipulate three-dimensional objects tangibly and concretely.

These resources can aid in the development of spatial geometry thinking. Moreover, they offer innovative and engaging ways for students to explore and understand complex mathematical concepts. Building on this idea, we explored the literature on 3D representations and GeoGebra's potential for enhancing students understanding and problem-solving abilities in the context of 3D geometry.

Research reports on surface representations through GeoGebra.

Research on spatial reasoning in mathematics education states that training in spatial reasoning ability improves students' mathematical performance in primary schools (Cheng & Mix, 2014; Lowrie et al., 2016). A teacher who has good spatial reasoning ability can improve the teacher's ability in the STEM field (Mulligan, Woolcott, Mitchelmore & Davis, 2018). In addition, spatial reasoning is also a predictor of someone's success in the STEM field (Newcombe et al, 2013) and thus, spatial reasoning is an issue that should be considered in learning primary school mathematics.

The research presented here used Standard Template Library - STL extension files generated directly in GeoGebra. 3D prints of some quadratic surface representations were developed to support the proposed activities. Based on Duval's Theory of Semiotic Representation Records, the objective was to verify whether mathematics students, when observing and manipulating 3D printed objects representing quadric surfaces, could convert such representations into other representations with different registers such as graphs and algebraic configuring them as semiotic resources. Initially, participants were asked to identify and represent such surfaces on paper.

Then, they were instructed to create a representation of them in the 3D visualization window of GeoGebra employing an algebraic command of their choice in the Input field of the software. It was verified that to obtain GeoGebra 3D respective representations, the participants used different algebraic records of the representation of the same surface configuring a treatment in Duval's conception.

We understand that manipulable materials can help in the study of surfaces, because, according to Passos (2006), the use of this type of material is characterized:

[...] by the physical involvement of students in an active learning situation.

> [...] I consider that these materials should serve as mediators to facilitate the teacher/student/knowledge relationship at the moment when knowledge is being built (Passos, 2006, p. 78).

The work with the manipulable object can enable the study of several registers of representation of the same mathematical object that converge, according to Duval & Moretti (2012), for an understanding of Mathematics if students incorporate, in their cognitive architecture, at least two different records of a mathematical object. In addition to treatment, Duval indicates conversion as a fundamental cognitive activity when dealing with representations. Thus, we ask: can students perform treatments and conversions of semiotic representation records from a manipulable object in 3D?

With the option of obtaining directly from GeoGebra files with the extension STL, it was possible to print in 3D some representations of quadric surfaces used in the proposal of the activities. According to Moyer (2001), manipulable materials are objects designed to represent abstract mathematical ideas explicitly and concretely. They have visual and tactile appeal and can be manipulated by students using their hands when conducting experiments (p. 176). Manipulable materials involving mathematical ideas have been used for some time. Piaget (1952) suggested that children who are not mature enough, mentally, to understand abstract mathematical concepts in words or symbols need experiences with concrete materials and drawings to learn. George Cuisenaire (1891-1975), a Belgian educator, is famous for having created the Cuisenaire bars around 1950; they were used to help teach the concept of fractions and other Mathematics ideas.

Another example is Numicon (Figure 1), from England – a multisensory material that facilitates the visualization of numbers and assists in learning Mathematics for children with special needs.

Figure 1.

Numicon (https://www.oup.com.au/primary/mathematics/numicon

The above examples are preferably addressed to Mathematics teaching at Elementary School. Some of the most common three-dimensional manipulable materials for higher education Mathematics students are representations of solids and their planes.

According to Lorenzato (2006), didactic manipulable materials can perform several functions, depending on their purpose: presenting a subject, motivating students, helping to memorize results, and facilitating rediscovery. The author also notes that "as good as they may be, teaching materials never go beyond the category of teaching aids, of methodological alternatives, and are not a guarantee of good teaching or meaningful learning" (p. 18).

Regarding teaching materials, Passos (2006) reveals that:

> Any material can be useful to present situations in which students face relationships between objects that can make them reflect, conjecture, formulate solutions, ask new questions, discover structures. However, the mathematical concepts that they must construct, with the help of the teacher, are not in any of the materials to be abstracted from them empirically. The concepts will be formed by the internalized action of students, by the meaning they give to these actions, to the formulations that they enunciate, to the verifications that they perform (Passos, 2006, p. 81).

We consider that learning Mathematics is not just about the manipulation of materials by students because mathematical concepts go beyond actions concerning them. Between experimentation and reflection about objects, mental action by students is required, mediated by teachers, to understand the mathematical concepts that involve the manipulated object. We consider, too, that three-dimensional representations on computer screens remain flat, and that the use of manipulable objects can complement the understanding of the mathematical object.

In this section, we discussed the significance of technology, like GeoGebra and 3D printing, in enhancing spatial reasoning in mathematics education. These tools allow students to visualize and interact with complex concepts through multiple representations. In the next section, we focus on Duval's theory on semiotic representations, providing the foundation for understanding the role of representations in developing spatial geometric thinking.

Registers of Semiotic Representations

Semiotics, the basis of the Theory of Registers of Semiotic Representations by French researcher Raymond Duval, seeks to understand how an individual attributes meaning to everything around him.

According to Duval,

> Access to mathematical objects necessarily includes semiotic representations. (...) We can then formulate the paradox of understanding in Mathematics in the following way:

how can we not confuse an object and its representation if we do not have access to this object except through its representation? (Duval, 2011, p. 21).

To research the conversion of a mathematical object's representation through the respective representation registers obtained is important "because to move from one register of representation to another is not only changing the mode of treatment, it is also explaining the properties or the different aspects of a same object" (Duval, 2011, p. 22).

When we mention registers, what is even more relevant, mathematically, are the transformations that can be carried out based on a semiotic representation. These transformations can be of two types, namely: processing, a transformation from an internal representation to a register of the representation, and conversion, which is a transformation that makes a representation "move" from one register of representation to another, thus proposing the coordination of two or more registers (Duval, 2000, 2006, 2011). The coordination of semiotic representation registers is essential for the conceptual understanding of objects so that the object is not confused with its representations and is recognized in each possible representation (Duval, 2000, 2006, 2011). Thus, the understanding of the mathematical object is related to the ability to mobilize at least two registers of representation, and, once we are discussing the teaching and learning of spatial geometry, the geometric register must be frequently mobilized.

In this sense, Duval (2011, p. 86-87) states that "geometric figures are distinguished from all other visual representations by the fact that there are always several ways of recognizing figurative forms or units, even though recognizing some excludes the possibility of recognizing others." Thus, to "geometrically see" a figure means "to operate a dimensional deconstruction of the forms that we recognize immediately into other forms that we do not visualize at first sight, and without any changes being made to the figure shown in the monitor or constructed on paper."

To Duval, visualization is essential to fully understanding geometric properties and is also associated with the many understandings and ways of visualizing geometric figures. To Duval (apud Kluppel, 2012, p. 38), "the cognitive activity that Geometry requires is more demanding than other fields of knowledge, since it requires that discursive and figurative treatment be performed simultaneously and interactively." Considering this, is that we take advantage of the possibilities of technology to create new representations of mathematical objects that allow students with new ways of interaction and visualization.

Building Relevant Manipulatives

In general, a surface is a set of points in space whose coordinates (x, y, z) satisfy an equation of form $G(x, y, z) = 0$ (implicit form) or $z = F(x, y)$ (explicit form), or coordinates of the points are given in terms of 2 parameters (parametric form).

For example, a spherical surface of radius r, such as in Figure 2, can be represented algebraically as follows:

- in implicit form by $x^2 + y^2 + z^2 = r^2$; and Entry command in GeoGebra: $x^2+y^2+z^2 = 4$

- in explicit form by $z=\sqrt{r^2 - x^2 - y^2}$ and $z=-\sqrt{r^2 - x^2 - y^2}$; and Entry commands in GeoGebra: $z = sqrt(4 - x^2 - y^2)$ and $z = -sqrt(4 - x^2 - y^2)$

- in parametric form $\begin{cases} x = rcos(t)sin(u) \\ y = rcos(t)\cos(u) \\ z = rsin(t) \end{cases}$ where t is the polar distance and $0 \leq t \leq 2\pi$, u is the geographic longitude and $0 \leq u \leq \pi$. The respective Entry command in GeoGebra is:

Surface(2cos(t) sin(u), 2cos(t) cos(u), 2sin(t), t, 0, 2π, u, 0, π)

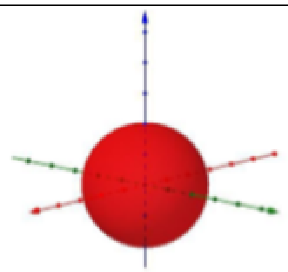
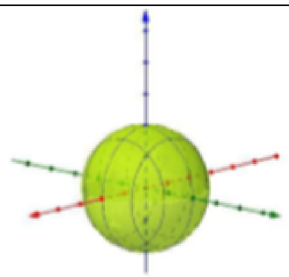
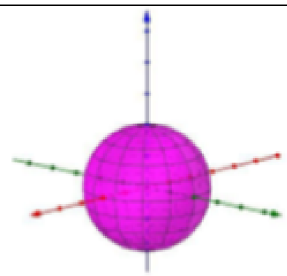
Implicit	**Explicit**	**Parametric**
		

Figure 2.

Three form representations of a sphere in GeoGebra (GeoGebra Screenshot)

Representations of quadratic surfaces in manipulable materials used in the activities were obtained from their 3D representations in GeoGebra.

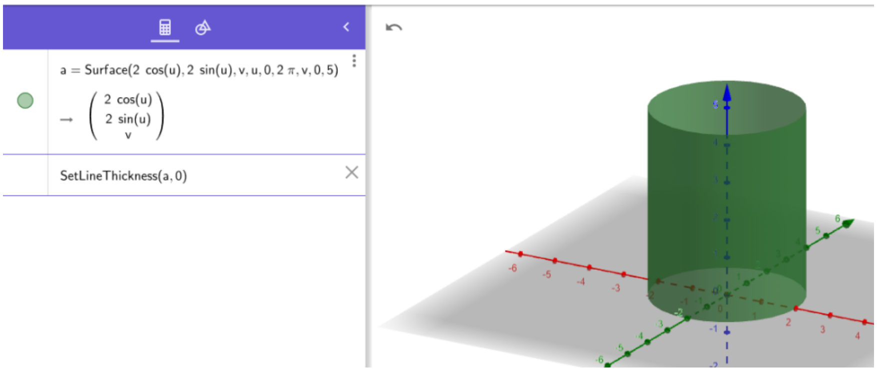


Figure 3.

Cylindrical surface obtained in GeoGebra 3D Graphic Calculator (Source: GeoGebra Screenshot)

Next, with access to the 3D Graphic Calculator at https://www.geogebra.org/3d, the Surface command was used, in which each Expression is a parametric equation of the corresponding surface. The command that was used was the following:

Surface (<Expression>, <Expression>, <Expression>, <Parameter 1 Variable>, <Initial Value>, <Final Value>, <Parameter 2 Variable>, <Initial Value>, <Final Value>) and obtained such as in Figure 3.

The command SetLineThickness(a,0) allows obtaining the quality of the objects or fully compactly or if SetLineThickness(a,20) with spaces. The results of the print obtained with the above commands were described for the students and did not interfere in the activities.

After obtaining the representation of the surface by clicking STL (Standart Template Library) and Export, one can directly obtain the file, which may then be saved and then opened by any software that communicates with the 3D printer that will be used.

This GeoGebra alternative facilitates the construction of 3D objects and does not require any other software to obtain the file in extension STL.

The manipulable objects used in the activities were obtained in the Da Vinci Mini 3D Printer. To this end, each file of the surface representation, in extension STL, was opened in the free XYZ Printing software that communicates with the printer.

In this software, it is possible to change the size and position of the representation of the object – which, in some cases, facilitates the printing of the object, as shown in Figure 4

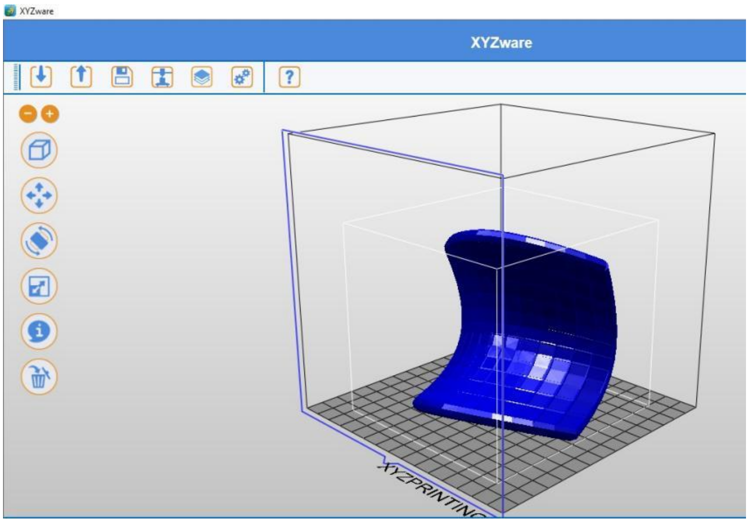


Figure 4.

Quadratic Surface (Source: XYZ Printing Software Screenshot)

Figure 5 presents some manipulable objects – quadratic surface representations obtained in the Da Vinci Mini 3D Printer and presented to students participating in the activities.

Figure 5.

Quadratic Surface Representations (Source: The authors)

To carry out the activities described below, the representations of the quadratic surfaces. Some represented in Figure 5 were used: circular cylinder; sheet hyperboloid; circular paraboloid; cone; and hyperbolic paraboloid.

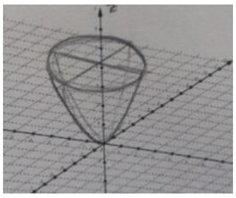
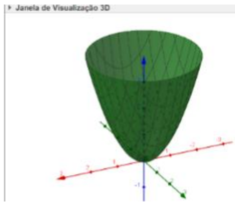
Activities carried out

This section describes one of the suggested activities performed by seven students. All participants are Mathematics teachers studying for a master's degree in mathematics education. We thus conjectured that they were prepared to carry out the proposed activities and that any difficulties that might occur would be in the algebraic choices for surface representation in GeoGebra 3D. Everyone was aware of GeoGebra commands and how to use them.

By manipulating some manipulable representations of quadric surfaces in 3D, students were asked to look at the object printed in 3D and, as they identified it, to represent it by sketching it on a sheet of paper. To conclude, they were asked to represent that surface in GeoGebra’s 3D Visualization Window using an algebraic register.

For example, one of the objects manipulated and printed in 3D was a paraboloid. Table 1 shows the different representations developed by one of the students.

Table 1: *Example of one of the developed activities that required representing a paraboloid.*

Representation of a manipulable paraboloid printed in 3D:	Graphical representation made with a pencil on paper:	Algebraic expression:	Graphical representation obtained by using the GeoGebra software program:
		$x^2 + y^2 - z = 0$	

We observed that some participants, after identifying the respective surfaces presented in 3D, were able to perform the suggested activities, i.e., converting the respective representation to obtain the register of the manipulable material into a graphic register on paper, then into an algebraic register, and representing the surface in the GeoGebra 3D viewing window.

However, during those activities, some students found it difficult to process the algebraic register to obtain a result similar represented by pencil on paper, and an intervention was necessary so that they could reflect on the possible forms of equations to obtain one and the same graphical representation, the implicit one, the explicit one and the parametric one. We had to teach them some GeoGebra commands to help them perform that task, especially regarding the possible forms of equations to obtain one and the same graphical representation.

In this section, we described an activity involving seven master's degree students in mathematics education who are also mathematics teachers. The participants were asked to identify, sketch, and represent 3D-printed quadric surfaces using GeoGebra's 3D visualization window. Although familiar with GeoGebra commands, some students encountered difficulties transitioning between different representations, particularly when converting between algebraic registers. This required additional guidance and instruction on GeoGebra commands and equation forms.

Discussion

The study aimed to understand how students develop spatial thinking and problem-solving strategies when engaging with multiple representations and hands-on experiences using computer software and manipulative materials. For that purpose, we developed a series of tasks and implemented them with a group of students in a master's course in mathematics education. In this section, we analyze the outcome of this project through the glasses of our proposed conceptual framework.

The hands-on experience, including multiple representations, allowed the participants to engage in relevant mathematical activities. Mathematics teachers participating in this study were given 3D manipulatives representing quadratic surfaces and asked to identify and represent them with pen and paper. These activities relate to National Council of Teachers of Mathematics - NCTM (2000) standards by emphasizing problem-solving, communication, connection, and representation while covering essential areas such as geometry and algebra. Furthermore, the activities engage participants in exploring quadratic using different representation registers, including manipulative materials, graphical representations, algebraic expressions, and GeoGebra dynamic representations. Through the treatment of the representation and conversion between different representations, participants are encouraged to make connections that can deepen their understanding of the involved mathematical concepts (Duval & Moretti, 2012).

According to Hitt (1998), understanding multiple representations and converting between them is crucial to developing higher-order mathematical skills. In our study, participants struggle to select the appropriate equation form (implicit, explicit, or parametric) and convert between these forms. Furthermore, they encountered issues when using GeoGebra commands and creating accurate 3D visualizations. The challenges students experienced in the study highlight the need for more focused instruction and practice in working with different equation forms, converting between them, and coordinating various representation registers.

We found that providing targeted interventions, such as guidance on equations and GeoGebra commands, helps students overcome challenges. These interventions effectively guided students' reflection on the proper equation forms and utilized technology to understand multiple representations. The study emphasizes the importance of targeted instruction in facilitating transitions between representation registers, consistent with Hitt's findings. By offering students guidance and practice in converting between equation forms and coordinating multiple representations, educators can promote a more profound comprehension of mathematical concepts and support the development of higher-order mathematical thinking skills.

GeoGebra software and 3D manipulable materials positively impact students' spatial thinking and visualization skills. The students in this study could better comprehend mathematical concepts by incorporating multiple representation registers, such as 3D printed materials, graphical representations, algebraic expressions, and GeoGebra representations. This approach echoes Gutierrez's (1996) work, which emphasizes fostering spatial thinking and visualization skills to enhance mathematical learning.

Moreover, our study suggests that employing these tools and materials may support the development of problem-solving strategies. Duval's (2000, 2011) ideas on multiple representations stress the need for students to coordinate various representation registers to understand mathematical objects fully. Based on the findings from this case study, it appears that integrating GeoGebra software and 3D manipulable materials in teaching 3D geometry aligns with Duval's theoretical framework and may potentially aid in developing problem-solving strategies, ultimately leading to a more profound understanding of mathematical concepts within this specific context.

The findings suggest that incorporating these tools and materials in the classroom may enhance students' spatial thinking, visualization skills, and problem-solving strategies, leading to a deeper understanding of mathematical concepts. These results are relevant for educators seeking effective methods to improve student engagement and comprehension in 3D geometry. Furthermore, we recommend that teachers incorporate multiple representations, including graphical, algebraic, digital representations, and physical 3D manipulable materials, when teaching 3D geometry concepts. By providing students with varied opportunities to explore and engage with the material, educators can foster a deeper understanding of the subject matter and support the development of problem-solving strategies. Additionally, teachers should prepare to offer targeted instruction and interventions when necessary to facilitate the transition between representation registers and to help all students successfully navigate and coordinate the different registers in their learning process.

While offering valuable insights into using multiple representation registers and hands-on experiences in teaching 3D geometry, this case study has limitations. The small sample size, consisting of seven students who are mathematics teachers pursuing a master’s degree in mathematics education, might not be representative of a broader population of learners. Furthermore, the study focuses explicitly on quadratic surfaces, which may limit the applicability of the findings to other geometric forms.

To address these limitations, future research could explore the use of similar strategies with more extensive and diverse student populations and investigate the effectiveness of these approaches in teaching other geometric forms beyond quadratic surfaces. Additionally, future studies might examine the impact of alternative instructional strategies, including using different software programs, teaching methods, or classroom settings, to determine their influence on students' spatial thinking and problem-solving skills in 3D geometry. These would contribute to a more comprehensive understanding of the most effective ways to foster spatial thinking and develop problem-solving strategies in the context of 3D geometry education.

Final Thoughts

In conclusion, this case study investigated the impact of using multiple representation registers and hands-on experiences with GeoGebra software and 3D manipulable materials on students' spatial thinking, visualization skills, and problem-solving strategies in 3D geometry. The study shows that incorporating these tools engages students in relevant mathematical activities and enhances their understanding of mathematical concepts, aligned with the NCTM standards and the work of Duval. Moreover, targeted instruction and interventions proved beneficial in helping students overcome challenges and facilitating transitions between representation registers.

The conceptual framework constructed and presented in this article contributed to identifying other skills required for the development of spatial geometric thinking. This framework can be a foundation for future research and educational practice in 3D geometry education. Other activities with 3D objects, built with the help of GeoGebra, are being developed for high school students. The research results will be presented in the future, contributing to a more comprehensive understanding of effective teaching practices in 3D geometry education.

Despite the limitations of the small sample size and specific focus on quadratic surfaces, this study offers valuable insights for educators looking to improve student engagement and comprehension in 3D geometry. Future research could expand upon these findings by exploring the effectiveness of similar strategies with diverse student populations, different geometric forms, or alternative instructional strategies, further contributing to the knowledge base on effective teaching practices in 3D geometry education.

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