

The division algorithm in the initial mathematics teacher training

El algoritmo de la división en la formación inicial del profesor de matemáticas

L’algorithme de division dans la formation initiale du professeur de mathématiques

O algoritmo da divisão na formação inicial do professor de matemática

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Abstract

Among studies on teacher knowledge and mathematical training, in recent years we have seen the emergence of interest in training in Elementary Theory of Numbers, Mathematics field whose concern is integers numbers and their operations. In this context, the present investigation aimed to discuss issues related to Euclidean division, more specifically to the division algorithm, within the scope of initial training, as well as to identify and debate, difficulties evidenced by undergraduate students in Mathematics degree at a public university in Ceará in relating This topic is addressed in the subject of Theory of Numbers with the teaching of the mathematical operation of division in Basic Education. The adopted theoretical framework points to the need to review the way in which the subject of Theory of Numbers has been worked on in the teaching degree, while the results of the questionnaire applied to 18 (eighteen) students in Mathematics degree indicated that basic ideas related to Euclidean division, and more specifically to the division algorithm, are still not well settled and articulated in these students, which may signal a possible weakness in the future practice of these teachers. It is concluded, therefore, that an approach more focused on the meanings of calculations and their implications than on the memorization and execution of algorithms is necessary to qualify the mathematical training in Elementary Theory of Numbers of future Mathematics teachers.

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Keywords: Mathematics teacher training, Elementary theory of numbers, Euclidean division, Division algorithm.

Resumen

Entre los estudios sobre el conocimiento docente y la formación matemática, en los últimos años hemos visto surgir el interés por la formación en Teoría Elemental de Números, área de las Matemáticas cuya preocupación son los números enteros y sus operaciones. En este contexto, la presente investigación tuvo como objetivo discutir cuestiones relacionadas con la división euclidiana, más específicamente con el algoritmo de la división, en el ámbito de la formación inicial, así como identificar y debatir, las dificultades evidenciadas por los estudiantes de licenciatura en Matemáticas en una audiencia pública. universidad de Ceará en relacionar Este tema es abordado en la asignatura de Teoría de Números con la enseñanza de la operación matemática de división en la Educación Básica. El marco teórico consultado apunta a la necesidad de revisar la forma en que se ha trabajado la asignatura de Teoría de Números en la carrera docente, mientras que los resultados del cuestionario aplicado a 18 (dieciocho) estudiantes de licenciatura en Matemáticas indicaron que las ideas básicas relacionadas a la división euclidiana, y más específicamente al algoritmo de división, aún no están bien asentados y articulados en estos estudiantes, lo que puede señalar una posible debilidad en la práctica futura de estos docentes. Se concluye, por tanto, que es necesario un enfoque más centrado en los significados de los cálculos y sus implicaciones que en la memorización y ejecución de algoritmos para cualificar la formación matemática en Teoría Elemental de Números de los futuros profesores de Matemáticas.

Palabras clave: Formación de profesores de matemáticas, Teoría elemental de números, División euclidiana, Algoritmo de división.

Résumé

Parmi les études portant sur les savoirs des enseignants et la formation mathématique, on a pu constater, ces dernières années, un intérêt pour la formation en Théorie élémentaire des nombres, un domaine des Mathématiques qui s'intéresse aux nombres entiers et à leurs opérations. Dans ce contexte, la présente enquête visait à discuter des questions liées à la division euclidienne, plus spécifiquement à l'algorithme de division, dans le cadre de la formation initiale, ainsi qu'à identifier et débattre, les difficultés mises en évidence par les étudiants de premier cycle en mathématiques à un public université de Ceará en relation Ce sujet est abordé dans le sujet de la théorie des nombres avec l'enseignement de l'opération

mathématique de division dans l'éducation de base. Le cadre théorique consulté indique la nécessité de revoir la manière dont le sujet de la théorie des nombres a été travaillé dans le diplôme d'enseignement, tandis que les résultats du questionnaire s'appliquaient à 18 (dix-huit) étudiants de premier cycle en mathématiques ont indiqué que les idées de base liées à la division euclidienne, et plus particulièrement à l'algorithme de division, ne sont pas encore bien installés et articulés chez ces élèves, ce qui peut signaler une éventuelle faiblesse dans la pratique future de ces enseignants. On en conclut donc qu'une approche plus centrée sur le sens des calculs et leurs implications que sur la mémorisation et l'exécution d'algorithmes est nécessaire pour qualifier la formation mathématique des futurs professeurs de mathématiques en théorie élémentaire des nombres.

Mots-clés: Formation des professeurs de mathématiques, Théorie élémentaire des nombres, Division euclidienne, Algorithme de division.

Resumo

Entre os estudos sobre os saberes e a formação matemática do professor, vimos, nos últimos anos, surgir o interesse sobre a formação em Teoria Elementar dos Números, área da Matemática cuja preocupação são os números inteiros e suas operações. Nesse contexto, a presente investigação teve como objetivo discutir questões relativas à divisão euclidiana, mais especificamente ao algoritmo da divisão, no âmbito da formação inicial, bem como identificar e debater dificuldades evidenciadas por licenciandos em Matemática de uma universidade pública cearense em relacionar esse tópico abordado na disciplina de Teoria dos Números com o ensino da operação matemática de divisão na Educação Básica. O referencial teórico adotado aponta à necessidade de rever a forma como a disciplina de Teoria dos Números vem sendo trabalhada na licenciatura, enquanto os resultados do questionário aplicado a 18 (dezoito) licenciandos em Matemática indicaram que ideias básicas relacionadas à divisão euclidiana, e mais especificamente ao algoritmo da divisão, ainda não estão bem assentadas e articuladas nesses estudantes, o que pode sinalizar possível fragilidade nas futuras práticas docentes. Conclui-se, assim, que uma abordagem mais voltada aos significados dos cálculos e suas implicações do que à memorização e execução dos algoritmos é necessária à qualificação da formação matemática em Teoria Elementar dos Números dos futuros professores de Matemática.

Palavras-chave: Formação de professores de matemática, Teoria elementar dos números, Divisão euclidiana, Algoritmo da divisão.

The Division Algorithm in the Initial Mathematics Teacher Training

Investigating what Mathematics teachers should know has been, in recent years, a fruitful theme of research in the field of teacher training. Such studies, such as Shulman (1986) and Carrillo-Yañez et al. (2018), are practically unanimous in assuring that knowing the content to be taught is a *sine qua non* condition, that is, indispensable for the future teacher, after all, it is not possible to teach what one does not know. This is, however, an often-generic conception, insufficient to delineate what knowledge to be taught is necessary for teachers to have in order to carry out their work properly. With regard to this teacher, studies such as those by Moreira and David (2005) and Valente (2022) shed light on what Mathematics this professional should know, and, consequently, what Mathematics should be worked on in initial training courses, the licentiate degrees.

It is possible to consider, from the outset, that such studies confirm teacher training in Mathematics that is still very similar to that of the 20th century, that is, based on scientific knowledge *per se*, disconnected not only from the other professional knowledge necessary for teaching (Fiorentini, 2005), but also of school Mathematics present in the curricula and pedagogical practices of Basic Education (Moreira & David, 2005). This scenario is certainly related to the fact that, even with the investments and curricular changes implemented in recent years, the mathematical performance of Basic Education students in assessments such Program for International Student Assessment has not shown considerable progress for at least 15 years (OECD, 2019).

The present study is therefore part of this field of investigative interest, as it focuses on the subject of mathematical training in Elementary Number Theory for future Mathematics teachers. We take here as Elementary Theory of Numbers (or just Theory of Numbers) “[...] the study of integers, their properties and the relationships between them” (Resende & Machado, 2012, p. 273). Resende and Machado (2012, p. 259) also state that “[...] although the study of numbers, especially integers, occupies a large part of the Mathematics curriculum in elementary school, it does not seem to deserve it in the training of students. teachers a treatment that corresponds to the demands that the teaching of this subject presents [...]”. This finding raises a series of questions that can be summarized in the following: “Which Number Theory could be conceived as a knowledge to teach in Mathematics Degree, aiming at teaching practice in basic school?” (Resende & Machado, 2012, p. 262).

We consider, as Ponte (2006) points out, that we often assume that numerical concepts are an excessively easy subject, when, on the contrary, they are “extremely complex and

ingenious intellectual constructions” (p. 7), which is why stresses the importance of analyzing the Theory of Numbers and the topics that compose it. It should also be considered that, in the list of this area of knowledge, the Euclidean division – commonly known as “division with remainder” – is a fundamental issue that has consequences in the study of integers, and that “[...] difficulties presented by students in understanding of Number Theory has roots in the thought of division with remainder, as this subject is not treated in basic school as something that is fundamental in the set of integers” (Resende, 2007, p. 76).

Given this scenario, this study aimed to discuss issues related to Euclidean division, more specifically to the division algorithm, within the scope of initial training, as well as to identify and discuss difficulties evidenced by undergraduate students in Mathematics degree at a public university in Ceará in relating this topic addressed in the subject of Number Theory with the teaching of the mathematical operation of division in Basic Education. A better understanding of these issues may help, eventually, in the outlining of actions aimed at qualifying the training of Mathematics teachers.

To do so, we initially discussed the Elementary Theory of Numbers, addressing its presence in regulatory documents and its importance and configuration in initial training, as well as exposing some mathematical results that emphasize such analysis. Next, we present the data obtained from a questionnaire applied to teachers in training, discussing their difficulties and pointing out didactic issues in order to overcome them.

The Elementary Theory of Numbers and the Algorithm of Division

We can say that numbers are to Mathematics what the letters of the alphabet are to language. That is, the second ones are neither defined nor limited by the first ones, but, without a doubt, they have in them the founding base of the relations and developments of its multiple areas. In this sense, the field of study of the fundamentals of integers, called Elementary Theory of Numbers, is highly relevant in the mathematical field. A symbolic statement of this importance comes from a maxim attributed to the German mathematician Carl Friedrich Gauss, who says that Mathematics is the queen of Sciences and Number Theory is the queen of Mathematics.

Referring to the history of Mathematics, while Algebra and Analysis, for example, were articulated as fields of knowledge that bring together interrelated objects very recently, the raw material of Number Theory has been laid since remote times, and “[...] the study of properties and relationships involving integers was carried out, even if still in a non-formal and non-systematized way, throughout the history of civilizations” (Resende, 2007, p. 68).

Far from being an outdated field, however, the study of numbers – especially natural and integer numbers – has reverberated in contemporary themes that are essential to technological evolution in our society. Almouloud et al. (2021), for example, point out that “[...] many principles of modern Cryptography are sufficiently described by mathematical contents of Elementary or High School, such as Prime Numbers, Divisibility, Factoring, Potentiation, Affine Functions, etc.” (p. 23), and also that “Modern cryptographic methods, in particular, are based on Number Theory” (p. 23).

It is not by chance that the study of numbers has been part of the school curriculum for a long time, being, as a rule, the beginning of the systematization of students’ mathematical learning. Undoubtedly, it represents a cognitive turning point when the young child starts to be able to represent quantifications, orders, groupings etc. – that is, numbers –, which previously could only be done via concrete objects, in the form of numerals. This allows them to start their process of developing abstract thinking, articulating the logical and also arithmetic relationships of numbers based on their representations.

In this sense, the Brazilian National Curricular Parameters (PCN, in Portuguese) of Mathematics of the third and fourth cycle (which, in the current nomenclature, are equivalent to the final years) of Elementary School state that:

Although the study of numbers and operations is an important theme in elementary school curricula, it is often seen that many students reach the end of this course with insufficient knowledge of numbers, how they are used and without having developed a broad understanding of the different meanings of operations. This probably occurs due to an inadequate approach to the treatment of numbers and operations and the little emphasis that is traditionally given to this subject in the third and fourth cycles. (Brazil, 1998, p. 95)

The finding presented in the document, still at the end of the 20th century, concerns a possible mechanization of the teaching of operations, more concerned with the assimilation of resolution processes and algorithms than with the different meanings of each one of them and the different types of calculation (exact, approximate, mental and written). In other words, the beginning of the child’s numeralization process and the development of his arithmetic thinking must be concerned with the child’s numerical reasoning, even when the resolution “mechanisms” are presented and inserted, that is, the algorithms.

Also, according to the PCN, priority should be given to “[...] activities that make it possible to expand the numerical sense and understanding of the meaning of operations, that is, activities that allow establishing and recognizing relationships between different types of numbers and between different operations” (Brazil, 1998, pp. 95-96). Contrary to this, what can

be seen are aspects that compromise the student's numerical learning, reflected in the "[...] work centered on algorithms, such as calculating the least common multiple (lcm) and greatest common divisor (gcd) without understanding the concepts and relationships involved and the identification of regularities that make it possible to broaden the understanding of numbers" (Brazil, 1998, p. 97).

A similar concern is also presented in the Brazilian National Common Curricular Base (BNCC, in Portuguese), albeit in lesser detail. According to the document, in relation to the final years of Elementary School, "[...] the expectation is that students solve problems with natural, integer and rational numbers, involving fundamental operations, with their different meanings, and using strategies diverse, with an understanding of the processes involved" (Brazil, 2018, p. 269).

Euclidean division, in the context of operations with natural numbers, appears as one of the first objects of knowledge in the thematic unit Numbers, in the Mathematics knowledge area of the 6th year, highlighted in relation to addition, subtraction and multiplication, having as a specific skill related the EF06MA03, which expects from the student: "To solve and elaborate problems that involve calculations (mental or written, exact or approximate) with natural numbers, through varied strategies, with understanding of the processes involved in them with and without the use of a calculator" (Brazil, 2018, p. 301).

The appointment of the skill involving the operation of division in the 6th grade, in curricular terms, already demands that the teacher licensed in Mathematics, the one who will teach this discipline in the last seven years of schooling in Basic Education, be trained to teach not only the resolution mechanisms, but the meanings of the operations and their algorithms. This demand becomes, however, even more latent when we know that the curriculum considers an "ideal scenario" and that, in practice, the Brazilian school reality has students who reach the final years of Elementary School with difficulties in previous operations (addition, subtraction and multiplication) and often conclude this teaching stage still having difficulties in dividing rational numbers ("decimals with a comma", such as, for example, 20.7 divided by 3.16) or even integers with many digits (for example, 125468952 divided by 2654) (Rodrigues, 2019).

With this in mind, we reiterate Resende and Machado's (2012) consideration that one of the fundamental guiding principles for a teacher training course, among those listed in the Brazilian National Curriculum Guidelines (DCN, in Portuguese) for the Training of Basic Education Teachers, is the need for coherence between the training offered and the expected practice of the teacher. After all, "[...] a discipline of the degree should not be thought of, looking only at the wise knowledge that gives rise to it, but also at the demands that are

presented to the teacher in basic school to teach the subjects related to the field” (Resende, 2007, p. 227).

In the case of Number Theory, a field of knowledge that provides a mathematical foundation for working with integers and their operations, it is necessary to reflect on its role in teacher education and the means by which it is embodied. Even though it seems trivial, given all the arguments already presented here, it is always healthy to consider – and, in this case, defend – the importance of a robust mathematical education for teachers. Because, when we say that the teacher must know the content he is going to teach, we do not mean that he must know only that content and such and such as he will teach it, but rather that he must know in depth, including its principles, to understand didactic phenomena, such as, for example, didactic and epistemological obstacles (Pais, 2002), which may arise around mathematical content during class.

In their study on the discipline of Number Theory in teacher education in Mathematics, Resende and Machado (2012, pp. 274-275) found that:

[...] the definition of this discipline, both with regard to the objectives, as well as the selection of content and approaches to be made, must consider that: 1) topics of Theory of Numbers are present in basic education [...] 2) Number Theory is a propitious space for the development of relevant mathematical ideas related to natural numbers and some also extended to integers, present in school mathematics [...] 3) Number Theory is a propitious field for a broader approach to proof [...] 4) Number Theory is a propitious field for mathematical investigation.

Briefly traversing each of the ideas listed by the authors, we consider it possible, in view of what has already been stated, to synthesize the role of the Mathematics teacher in view of the concepts of numbers and operations. The discourse defended by some that this responsibility is exclusive to the pedagogues who teach Mathematics in Early Childhood Education and in the early years of Elementary School, in addition to being wrong and laden with blame, does not help in resolving the issue. The concern with the mathematical training of younger students is a subject inherent to the work of the Mathematics teacher, due to the fact that we must also be committed to recovering these students’ basic learning for a more effective Mathematics education. Hence the reason why knowledge related to Number Theory is, in fact, present in school education, necessary in the development of concepts about integers.

In addition, Number Theory is also a favorable space for mathematical investigation and the approach to demonstrations, “[...] present, not as a topic, but as a form of validation or justification of mathematical results” (Almeida & Ribeiro, 2019, p. 129), essential for verification in Mathematics and the development of mathematical reasoning in future teachers.

In this regard, it is worth considering the role of demonstration, not only in the subject of Number Theory, but in the various mathematical subjects of the degree.

In the terms of Moreira and David (2005), when it comes to Scientific Mathematics – that is, the science called Mathematics, present in Higher Education, including baccalaureate courses – its axiomatic structure requires that the demonstrations be developed supported by definitions and previously established theorems, which requires a precise formulation, so that contradictions do not occur in the theory from ambiguities in the characterization of a mathematical object. In other words, the mathematical vocabulary and rigor in the demonstration processes guarantee the validity and accuracy of the results, which implies the improvement of the student's logical-deductive thinking.

However, it is known that mathematical demonstrations, and even theorems and postulates, are not fully present in School Mathematics, especially if we consider that the subject of numerical sets and their operations appears when students are still at a young age and do not have maturity. cognitive to accompany demonstrations in this regard. Which does not mean, in turn, that there is no room for demonstrations in Basic Education, as the authors assert when they say that:

The fundamental issue for School Mathematics [...] refers to learning, therefore to the development of a pedagogical practice aimed at understanding the fact, the construction of justifications that allow the student to use it in a coherent and convenient way in their lives. school and extra school. (Moreira & David, 2005, pp. 23-24)

This discussion reverberates, in effect, in the role and purpose of mathematical demonstrations in the subject of Number Theory, in the initial training of the Mathematics teacher, who must meet the formative purposes and, at the time in which he presents the student with the refined and rigorous thinking that it validates the mathematical knowledge under study, it also broadens your understanding of the phenomenon itself, and enables the elaboration of strategies to approach the theme with your future students, in Basic Education. In his study of Number Theory courses in Brazil, however, Resende (2007) points out that this is not exactly what has happened.

In many cases, the discipline in question is just a prelude to the discipline(s) of Algebra, in which the set of integers with their operations is just one example, among many, of an algebraic structure. In the understanding of the authors consulted and in ours, the discipline of Number Theory must have its importance recognized in itself, not as a prerequisite, given its objective of exploring mathematical ideas related to integers, such as: “[...] the idea of recurrence through which many notions are defined; mathematical induction; the question of

divisibility; issues related to prime numbers and the multiplicative structure of integers” (Resende & Machado, 2012, pp. 273-274).

The research scenario on this topic, however, points out that the Theory of Numbers addressed in most universities “[...] is not concerned with the training of basic school teachers, since the approach to content is axiomatic, in a predominantly symbolic-formal language, with an emphasis on demonstrations, which allows its teaching to be framed within the classical formalist tendency” (Resende, 2007, p. 7). This goes against our defense that this discipline (or disciplinary field) has its own space in the curricula of undergraduate degrees in Mathematics, “[...] so that the characterizing aspects of integers, present in the basic school curricula, can be duly treated both as knowledge of the content, as pedagogical knowledge of the content and as curricular knowledge” (Resende, 2007, p. 227).

Entering a more specific discussion on the subject of this vast area that is Number Theory, we have that operations between integers, their properties, problems, and, in particular, the division algorithm, were studied, within the scope of school Mathematics, in a discipline called Arithmetic. Today, the division algorithm, in Mathematics degree courses, is studied in the Elementary Theory of Numbers discipline, which has Arithmetic as one of its components.

According to Boyer (1996), Arithmetic has as its main starting point Euclid’s “The Elements” (approx. 300 BC). For Hefez (2004), this field of study reaches its peak with the works of Pierre de Fermat (1601-1665) and Leonhard Euler (1707-1783), which made it become one of the main pillars of Mathematics. Other mathematicians, over the years, made their contributions transform Arithmetic into Number Theory, as is the case of Carl Friedrich Gauss (1777-1885), from the beginning of the 19th century, who made relevant contributions to this occurred, creating the Algebraic Theory of Numbers.

Scholars such as Johann Dirichlet (1805-1859) and Bernhard Riemann (1826-1866), in turn, using tools of Mathematical Analysis, helped to create the Analytical Theory of Numbers. Already in an approach of Algebraic Geometry, Emil Artin, Helmut Hasse, Louis Joel and André Weil, used their methods to create Arithmetic Geometry, already in the beginning of the 20th century. It was based on this last approach that, only in 1995, the English mathematician Andrew Willes published the proof of Fermat’s Last Theorem (Hefez, 2004).

The ordinary process of dividing any positive integer by another non-zero positive integer has been known since the 6th year of Elementary School. As we know, this division is not always possible, and the relation that expresses this possibility is called the divisibility relation. When this division is possible, it means that, when dividing the largest of these integers by the smallest, one obtains a quotient (positive integer) and a remainder of zero – exact division

– and, in this case, the first of these numbers is said to be divisible by the second. Even if there is no such relationship between two integers, it will still be possible to perform a division, called *division with remainder* or *non-exact division*, or even *Euclidean division*.

Generally, in Brazil when talking about division between two positive integers, say D and d , $D \geq d > 0$, since the early years of elementary school, we (both students and teachers) associate it with the following practical device:

$$\begin{array}{r} D \quad | \quad d \\ \hline \quad \quad ? \end{array}$$

and that, due to such association, this device started, erroneously, to be called division algorithm. This confusion is due to the fact that, even in Elementary School, students have not absorbed the idea of division between two integers. If these same students choose to enroll in a degree in Mathematics, they will consequently carry such deficiencies for their training as future teachers. If these misconceptions are not worked on during graduation and remain unreflected, they will naturally be taken to their future students.

It is urgently necessary to change the teacher's approach, seeking to remedy deficiencies throughout their training so that they are not taken into the classroom. In conducting this process, it is essential that appropriate didactic/methodological approaches occur, causing problems to stop turning them in a circle, not allowing poorly formed generations to compromise the following generations.

At some point, throughout school life, the student will be faced with a situation in which there is a need to perform the division between any two integers, with the second different from zero and, therefore, extend the application of this algorithm (previously applied to positive integers) for integers. The algorithm we are talking about and which is often confused with the device mentioned above is the division algorithm, which, formally, can be stated like this:

Division algorithm 1: If D and d are integers, with d positive, then there exist integers q and r such that

$$D = qd + r \text{ and } 0 \leq r < d. \quad (1)$$

In order for the student to understand what the division between two integers means, with the second integer positive, it is extremely important that the teacher has this understanding that this algorithm gives meaning to the ordinary division process, without necessarily having to make an association with the aforementioned device, causing the student to confuse the division operation with a mechanism created to carry it out. After this understanding, as a resource to facilitate the division calculation, the device can be used. That said, and to expand

the concepts related to the division of integers, the student should be presented with cases in which the division between any two integers, with the second different from zero, is also possible.

First, note that if D and d are integers, with d positive, then D is a multiple of d or D lies between two consecutive multiples of d , that is,

$$qd \leq D < (q + 1)d, \quad (2)$$

where q is an integer. This result is, as asserted by Santos (1999), known as the Theorem of Eudoxius (408-355 BC). Note, now, that from (2) follows $0 \leq D - qd < d$, or even, $r = D - qd$ and $0 \leq r < d$ which gives us

$$D = qd + r \text{ and } 0 \leq r < d, \quad (3)$$

meaning that when we divide the integer D by the integer d , we get the integer q and leave r , also integer. This means that d matches $|q|$ times in D , and left over r .

Note, too, that the integers q and r , which satisfy the conditions in (2), must be unique. This is because, if there are other integers, say, q_1 and r_1 , such that $D = q_1d + r_1$ and $0 \leq r_1 < d$, then $qd + r = q_1d + r_1$, and hence $d|q - q_1| = |r_1 - r|$, since $0 \leq r_1 < d$ and $-d < r \leq 0$, and, since $d > 0$, it follows that $0 < d|q - q_1| < 1$, and thus $|q - q_1| = 0$. Therefore, $q = q_1$ and $r = r_1$, showing the uniqueness of the integers q and r .

One result, which is a consequence of the algorithm presented in (1) and which tells us that it is possible to divide between two integers when the second is different from zero, is the following:

Division algorithm 2: If D and d are integers, with $d \neq 0$, then there exist integers q and r such that

$$D = qd + r \text{ and } 0 \leq r < |d|. \quad (4)$$

The numbers D , d , q and r are called, respectively, *dividend*, *divisor*, *quotient* and *remainder* when dividing D by d . This means that the division algorithm applied to the integers D and d , with $d \neq 0$, obtains q for *quotient* and r for *remainder*. In the case where D and d are positive integers, a situation seen by students and teachers from the 6th year of elementary school onwards, the quotient q means that d “fits” exactly q times in D , if $r = 0$ (exact division); if $r > 0$, d “fits” q times in D , but with a small remainder. Note that in this case the quotient q is a positive integer.

In the calculation of the quotient and the remainder, when D and d are any integers and $d \neq 0$, for some students, the task of determining them is no longer an easy task as in the previous situation. When $D < 0$ and $d < 0$, for example, there is a situation that was not

explored in Elementary School, just the case where D and d are positive integers. In the discipline of Number Theory, already in the degree courses, in which one has the first contact with the division of any integers, with the second different from zero, through the division algorithm, some manage, without great difficulties, to correctly carry out the division (correctly determine the quotient and remainder in this division). Others, however, demonstrate little understanding, presenting the rest, for example, with a negative number, contrary to the condition $0 \leq r < |d|$ of the division algorithm.

Applying the division algorithm to the integers D and d , $d \neq 0$, let's analyze the possible cases for D and d : $D > 0$ and $d > 0$; $D < 0$ and $d > 0$; $D > 0$ and $d < 0$; and $D < 0$ and $d < 0$. In the case where $D > 0$ and $d > 0$, we have, by (1), that $D = dq + r$, where the quotient is $q > 0$ and the remainder is r , $0 \leq r < d$. When $D < 0$ and $d > 0$, consider $-D > 0$; applying (1) to $-D > 0$ and $d > 0$, from $D = dq + r$ it follows $-D = d(-q) + (-r)$ (here, the quotient is not $-q$, nor the remainder is $-r$) and therefore $-D = d(-q) + (-d) + d + (-r)$, which gives us $-D = d[-(q + 1)] + (d - r)$, thus concluding, that the quotient is $-(q + 1)$ and the remainder is $d - r$, since $0 < d - r < d$. The other cases can be analyzed in a similar way. The possible quotients and remainders in the division of non-zero integers, D and d , are, in summary, as shown in the table below:

Table 1.

Possible quotients (q) and reminders (r) in the division of D by d , $d \neq 0$.

Dividend	Divisor	Quotient	Reminder
$D > 0$	$d > 0$	$q > 0$	$0 \leq r < d$
$D < 0$	$d > 0$	$-(q + 1) < 0$	$0 < d - r < d$
$D > 0$	$d < 0$	$-q < 0$	$0 \leq r < d$
$D < 0$	$d < 0$	$q + 1 > 0$	$0 < d - r < d$

From what was seen in the considerations on the division algorithm, the relationship between the division operation and the multiplication operation is evident: when dividing D by $d \neq 0$, one obtains a quotient q and a remainder r , so that $dq + r = D$, and that multiplication, too, can be thought of as adding equal parts.

In order to facilitate the understanding of division, mainly by undergraduate students, who will be future Mathematics teachers and, consequently, base their mathematical knowledge, it is important to broaden the notion of division between integers and seek other possibilities to carry it out. Seeking didactic resources, we present here another algorithm to divide positive integers, now based on the use of repeated differences. This algorithm consists of always subtracting the smallest from the largest of the given numbers and, again, from this

difference subtracting the smallest of the given numbers. This process, which is finite, always continues subtracting the smallest of the numbers from the last differences obtained, until the last of the differences is equal to or less than zero.

This algorithm, which can be found in the lecture notes “Introduction to the Theory of Numbers”, written at the Institute of Pure and Applied Mathematics (IMPA, in Portuguese), by Said Sidki, for a course taught at the 10th Colloquium of Brazilian Mathematics, in January 1975 (Sidki, 1975), can be established as follows:

Division by difference algorithm: To divide the integers D and d , say $D \geq d > 0$, we can apply repeated differences, proceeding as follows:

I) Make $D = D_1$;

II) Calculate $D_{i+1} = D_i - d$;

III) (a) If $D_{i+1} > 0$, then go back to II), replacing i with $i + 1$;

(b) If $D_{i+1} \leq 0$, stop. The result will be:

$$D = id, \text{ if } D_{i+1} = 0, \text{ and}$$

$$D = (i - 1)d + D_i, \text{ if } D_{i+1} < 0.$$

It is inevitable that eventually, in relation to the two methods presented, questions like “Which one brings more advantage in terms of a better understanding of the division?” and “What relationship is there between them?” may arise. This algorithm, unlike the previous one, is best applied to the case where you want to divide two positive integers; for the other cases, an adaptation can be made, but with more work, making it computationally inefficient. Furthermore, it offers no advantages didactically, making the division algorithm more suitable for carrying out the division in the case where there are any two integers, with a non-zero divisor.

Note, in the case of the division algorithm, that if $r = 0$ (exact division), then $D = dq$, that is, D is divisible by d , and if $0 < r < d$ (division with remainder), then $D = dq + r$. When $D \geq d > 0$, applying the division algorithm based on repeated differences to D and d , the number of steps taken (iterations) until reaching a difference smaller than or equal to zero is equal to i (these differences are represented by D_{i+1}). Thus, the quotient q and the remainder r , obtained by using the algorithm of dividing D by $d > 0$ are, $q = i$ and $r = D_{i+1} = 0$; when $D_{i+1} < 0$, we have $q = i - 1$ and $0 < r = D_i < d$.

After this brief reflection on the presence and ways of approaching the division algorithm in the context of Number Theory in initial training for teaching Mathematics, we present, below, the specific case of a group of undergraduate students to add to our

understanding of the theme. It is necessary, however, to present the methodological course of the investigation carried out, to which the next section is dedicated.

Design of the data production instrument, participants and study locus

The data under analysis in this discussion come from a questionnaire composed of open questions (Fiorentini & Lorenzato, 2009) applied to Mathematics degree undergraduates from a public state university, in the interior of Ceará. The choice for this data production instrument aimed to capture the reasoning and written argumentation capacity of teachers in training. The questionnaire consisted of four questions related to divisibility, the division of two integers, the Euclidean division algorithm and the division algorithm based on repeated differences. In it, we tried to question the meaning of the division, its relationship with the device commonly used to perform it and the procedure adopted to perform the division through this device.

Participants were a group of 18 (eighteen) students from the Mathematics degree course at the aforementioned university, who volunteered to participate in this study. Of these, eight were at the beginning of the course and had not yet taken the subject of Number Theory; seven were approaching the end of the course and, therefore, had already taken this course some time ago; and the remaining three were taking the discipline during the research period. The choice for this diversity of profiles was intended to raise different understandings on the subject, related to the students' previous experience and trajectories in initial training.

Before moving on to the students' responses, however, we must characterize the research *locus*, that is, the course in which this investigation took place, briefly discussing the Pedagogical Project of the Course (PPC), specifically with regard to the composition of the discipline of Number Theory, which appears in the 4th semester. Dating from 2008 (which reveals a considerable mismatch, since, since then, two new DCN for initial teacher education have been enacted), the PPC has only one compulsory subject related to the mathematical topic in question, called "Number Theory I". It does not have prerequisites for enrolling, and any student from the second semester onwards can take it, whereas it is a prerequisite to take Algebraic Structures I (5th semester) and Number Theory II, the latter being optional and, based on our experience since the creation of the course, it has never been offered.

In the course's syllabus we have: "Natural numbers and whole numbers. Mathematical induction. Divisibility. Prime numbers. Congruence", its objective being "To give students a logically organized body of integers, even if incomplete". The expression "although incomplete" signals the deepening proposed in the subject of Number Theory II, which has as its syllabus "Divisors of an integer. Arithmetic Functions. Function and Euler's Theorem.

Perfect numbers, Fibonacci numbers and Pythagorean triples. Residual classes. Primitive Roots”.

In Number Theory I, there is, among the syllabus, the topic of divisibility, which comprises: “Definition, properties and algorithm of Euclid’s division; common divisors of two integers, greatest common divisor of two numbers, Euclid’s algorithm for gcd, existence and uniqueness of gcd, gcd properties, gcd of several integers; common multiple of two integers, least common multiple of two integers, relationship between gcd and lcm”. In this sense, it is pertinent to consider the concept and algorithm of Euclidean division, in its operational and didactic aspects, as a topic of interest and training for future Mathematics teachers in this program.

Observation reveals that the discipline’s syllabus shows adequate content, which allows a consistent mathematical training for these future teachers. As long as appropriate approaches are taken, it is possible that, at the end of the first Number Theory course, undergraduates will be able to solve the problems present in the applied questionnaire, which are problems present in the daily lives of students and teachers. This requires, however, that the trainer who works in the degree has a vision and attitude that allow him to put himself in the place of those who are being trained, understanding the training purposes.

Other information that stands out in the course program, however, is related to the methodological nature. In terms of procedures (methodologies, strategies etc.) and assessment in the discipline, the PPC recommends “expository theoretical classes, with exercise resolution” and “individual written tests and individual or collective work”. Such information reinforces Resende’s (2007) perception that the didactic and evaluative methods of Number Theory courses accentuate the classical-formalist character of this training.

Considering this preamble sufficient for us to understand in general terms the context from which the responses emerge, we move on to the answers to the questionnaire.

Results and discussion

For the proposed discussion, we will present the wording of each question and then announce the answers of some undergraduate students, commenting on them and presenting didactic suggestions that may contribute to resolve errors and identified problems, aiming at qualifying the training of the prospective Mathematics teacher. Naturally, in order to preserve the anonymity of the participants, with their responses being our focus, these were named Student 01, Student 02, ..., Student 18.

At first, we had:

Question 01: Based on what you learned about division of integers, since Elementary School, answer:

a) What does divisibility mean?

b) Is there a relationship between what you studied in Elementary School and what you are studying now, in Number Theory? If so, how does this relationship work?

Students 01, 02, 04 and 16 were in the 1st semester of the course, and, of these, Students 02, 04 and 16 did not answer this question, which can be explained by the fact that they had not yet taken the subject of Theory of Numbers, having in Elementary or High School only contact with the notion of divisibility in the numerical set of natural numbers. Unless the student has participated in Olympic training under the Junior Scientific Initiation Program of the Brazilian Mathematical Olympiad for Public Schools (OBMEP, in Portuguese), where various topics in Number Theory are studied, such as the notion of divisibility in set of integers, this first contact occurs only in Mathematics degree, in a subject of Elementary Theory of Numbers.

This is the case of Student 01, who gave the following answer: “It means method to facilitate the process of division between integers. Addressing specific properties and principles for even numbers, primes etc.”. In his answer, this licensee considered divisibility as a method to carry out a division of integers, and not a condition. In fact, an integer, not null, is a divisor of another integer, if and only if there is a third integer that multiplied by the first is equal to the second. This is the divisibility ratio between two integers.

The eight students who are at the beginning of the course and who did not take the Number Theory course showed that they did not really know what divisibility means, always associating it with the method for carrying out the division or with the division operation itself between two integers. Among the answers that follow this line of reasoning are: “A better way to divide integers and show whether an integer is a divisor of another” (Student 02), “Sharing something” (Student 05), “It is the partition of an number into equal parts, being the divisor and the dividend” (Student 06), “Dividing a number by another, where the result is a whole number and the remainder is zero” (Student 13) and “Dividing something into equal parts, whether the result is accurate or not” (Student 16). It draws attention to the lack of mention of the divisibility criteria by 2, 3, 5, 10, for example, which are studied in Elementary School. On the contrary, we had the following report from Student 18: “I don’t know, I didn’t hear that word during my Elementary School”, which causes us to reflect on the problem.

Of the three students who were taking the course at the time of the research, who had recently come into contact with the first notions of divisibility, Student 14 did not respond and Student 10 gave the following response: “It is a property that some numbers have of be divided

by another without there being a remainder.” Student 03, in turn, gave a similar answer, but with greater detail, as shown below:

Divisibility is the mathematical property that allows a number to be divided by another without leaving a remainder, that is, the division is exact. A number is divisible by another if the result of the division is an integer and has no remainder. For example, 6 is divisible by 2, because the division $6/2$ is exact and has no remainder. The number 9 is not divisible by 2, because division over remainder. (Student 03)

Of the seven undergraduates who had already taken the Number Theory course, three gave answers that had little or nothing to do with the concept of divisibility, such as, for example: “It means knowing how many times you can add the same number so that the remainder is zero” (Student 08), “Divisibility means discovering the maximum number of times we can add a quantity by itself, aiming for another quantity greater than or equal to the first” (Student 11), and “It is a form of equation that works with way of sharing...” (Student 15). Two confused her with the division operation itself or with a method to carry out the division, and their answers were: “The division of an integer by another, that is, ‘breaking’ an integer into equal parts without any remainder” (Student 07) and “It is a division of one integer by another, that is, ‘breaking’ the whole number into parts” (Student 09). Only two provided answers that, despite being succinct, came close to the definition of divisibility: “Divisibility is the characteristic given to two numbers that, when divided, will have a remainder of zero” (Student 12) and “Dividing one number by another and leaving zero remainder” (Student 17).

It is noted, in view of what was presented, that such undergraduate students correctly associate the concept of divisibility, demonstrating that they mobilize such knowledge, even though the way in which they expose it is intuitive, very related to experiences with division in Basic Education. With the exception of Student 10, who had not yet taken the subject in focus, the other students should theoretically be in a position (even because they had been in contact for less time) to conceptualize, informally, divisibility as the relationship between two integers, one of them nonzero, where there is a third integer that multiplied by the first (the nonzero) equals the second. Mathematically, we have that “given two integers a and b , with $a \neq 0$, we say that a divides b when there is an integer c such that $ac = b$ ”. It is also said that b is divisible by a , or that a is a divisor of b , thus establishing the divisibility relation.

Thus, the most basic ideas of the division, as they are worked on in schools, are shown to be more consolidated in part of the undergraduate students. Therefore, it is worth focusing on the answers to item b), which questioned precisely the relationship between the divisibility studied in Elementary Education and in the teaching degree. Naturally, those who did not attend

the course did not answer this question, considering their experiences in Number Theory. Of the ten students who took or were taking the discipline, only one did not answer and two simply answered “No”, not showing that they related the division of Basic Education with that studied in the degree.

The other answers were as follows: “Yes, in the division algorithm, in the definition of remainder, in ‘taking the test’ through multiplication” (Student 07), “Yes; in the definition of rest, in the test” (Student 09), “There is indeed a relationship, because it can be said that the concept studied in Elementary School is the basis for Number Theory, which broadens and deepens this understanding” (Student 03), “Yes, there is a relationship between the two and that Number Theory shows other ways and demonstrates with other methods the divisibility of two integers” (Student 10), “Yes, in the most primitive notions and concepts during the course” (Student 12) and “Yes. In Higher Education, the division algorithm is studied more rigorously” (Student 17).

Faced with the answers, which certainly somehow relate the divisibility of these two levels of education, with greater or lesser precision, some considerations are possible. While Students 07 and 09 address similar topics, such as the definition of rest and “proof”, Student 10 points out the difference in the methods with which they deal with the content. Students 12 and 17, on the other hand, distinguish the notions addressed in Basic Education and in the teaching degree by their primitiveness and rigor, stating that the division is treated with more rigor in Higher Education, which we understand as the division being treated in its foundations, and that the Number theory topics that come close to those of Education are the most primitive ones, that is, those that are taught in the school curriculum. Student 03’s response also draws attention, which states that Elementary School topics are the basis for the Theory of Numbers studied in the degree; in our understanding, it is essentially the opposite: it is the Theory of Numbers that supports subjects related to the operation of integers in Basic Education; underpins and, in fact, broadens and deepens this understanding.

Let’s now analyze Question 02, which was:

Question 02: To carry out the division of positive integers, say $D \geq d > 0$, one of the ways we know is the division algorithm, which guarantees that there are unique integers q and r , such that $D = dq + r$ and $0 \leq r < d$. This means that, in this division, the quotient is q (d “fits” q times in D), and r is the remainder, $r < d$. Usually, when we talk about division of integers, since the first years of elementary school, we associate it with the following practical device:

$$\begin{array}{r} D \quad | \quad d \\ \hline \quad \quad q \\ \quad \quad r \end{array}$$

with D , d , q and r , satisfying the conditions $D = dq + r$ and $0 \leq r < d$. So please answer:

- a) What is division? What does it mean, in practice, to divide two integers?*
- b) Explain the reason for the association of the division with the aforementioned device.*
- c) The algorithm is also valid when $d \in \mathbb{Z}$, with $|D| > |d|$ and, in this case, we have $0 \leq r < |d|$. In this case, what is the quotient (q)? We can also carry out the division of D by d , when $|D| < |d|$. In this situation, what is the quotient (q) and remainder (r)?*
- d) Considering the answers to item c), determine q and r , when:*
 - i. $D = 141$ and $d = 15$;*
 - ii. $D = -141$ and $d = 15$;*
 - iii. $D = 141$ and $d = -15$;*
 - iv. $D = -141$ and $d = -15$.*

This question dealt with division and the application of the division algorithm to two integers, with the purpose of verifying which students mobilized knowledge about the division of any integers (not just positive ones), considering that those who had already taken the Theory of Numbers had contact with theorizing about these cases of division, while those who did not attend the discipline might not have had such contact. In the answer to item a), when asked “what is division?”, we had some students who took the subject of Number Theory answers such as “distribute, break into equal parts...” (Student 07), “division is the inverse operation of multiplication” (Student 08) or “the act of dividing two numbers” (Student 09), “Dividing is an arithmetic operation that appears as the inverse of multiplication” (Student 11) and “Inverse of multiplication. Separate into equal parts” (Student 17). One can notice, in such formulations, ideas underlying the conception of division, but which, in addition to practically repeating what they said about divisibility, have little depth, especially if we consider that the respondents have already attended the discipline in focus.

When asked about the relationship between the division and the usual practical device mentioned above, in general, they did not respond. This suggests that the students are even able to operate the division through the device, but they have not yet mobilized knowledge regarding the distinction between the division algorithm and this mechanism to carry it out. In other words, it is noted that the perception still remains that when dealing with the division we are immediately approaching the mentioned device, as if they were the same thing. We only have one exception, again Student 03, who, regarding items a) and b), said:

a) Division consists of finding how many times a number (the divisor) goes into another number (the dividend), and the result of this operation is called a quotient. In practice, when dividing two integers, one seeks to know how many times a number is contained in the other, that is, a number is divided into equal parts. For example, if you divide the number 14 by 7, you want to know how many times the number 7 goes into 14.

b) The aforementioned device is important to better understand how the division takes place. Placing the dash between the dividend and the divisor is a mathematical convention that helps visualize the division operation. It is used as a visual tool to organize the division operation and ensure that the result is accurate and correct. (Student 03)

We note, in this answer, the understanding of the division algorithm, which encompasses both exact divisions (with zero remainder) and inexact divisions, in addition to the distinction between the division algorithm and the device that serves in the organization and better visualization of the calculation. This understanding enables the foundation of the future teaching practice of Student 03 in teaching division, since the idea of the algorithm itself is mobilized and appears (must appear) before the device. In other words, first the child is taught the concept, the notion of what it means to divide one number by another, and then the ways (note, in the plural) of carrying out such an operation are worked on. So much so that the mentioned device is more effective (that is, it facilitates the resolution) for larger numbers; if I divide 8 by 4, for example, it is much more practical to think how many times 4 “fits” in 8, than to arm and carry out the operation through the device.

In Question 02, the division algorithm for the integers $D \geq d > 0$ was also stated. In item c), we asked about the quotient and remainder in cases where $|D| > |d|$, but $D < 0$ and $d \neq 0$. The responses, in general, showed some misunderstandings regarding the operation of division between integers, evidencing the non-assimilation of the division algorithm, especially in the case of those who had already attended the discipline of Number Theory. That is, they failed to understand that dividing an integer by another non-zero integer means finding an integer (quotient) and a remainder (equal to or greater than zero), so that multiplying the smallest of these numbers (divisor) by the quotient and of this result adding the remainder

obtained is the largest of the numbers (dividend), which is nothing more than the application of the division algorithm.

As for item d) of this question, most of those evaluated showed difficulties in applying the division algorithm in the requested cases, especially when the dividend is negative, as in the case of sub-item *ii.*, which asks to divide -141 by 15 . Applying the division algorithm to the integers -141 by 15 , we have

$$141 = 15 \times 9 + 6, \quad (5)$$

And so from (5) comes $-141 = 15 \times (-9) + (-6)$. Here some, erroneously, showing that they did not understand the division algorithm, say that the quotient is -9 and the remainder -6 ; however, the rest cannot be negative, according to (1). How, then, is this problem resolved? The answer is in Table 1; not wanting to consult it, it can be done as follows:

$-141 = 15 \times (-9) + (-6) = 15 \times (-9) + (-6) + 15 + (-15) = (-10) \times 15 + 9$ (6)
therefore, the quotient is $q = -10$ and the remainder $r = 9 < 15$.

In the case of dividing 141 by 15 , students generally understand that they should look for the largest integer that, multiplied by 15 , is as close as possible to 141 (that is, how many times 15 “fits” in 141), in this case, 9 . Then multiply that 9 by 15 , giving 135 , and then make the difference $141 - 135 = 6$, this being the remainder. When they are faced with the problem of division in which the first number is negative (-141 , for example), if it is not done as in (6), there is some difficulty in finding the largest number that multiplied by 15 is the smallest number as close as possible to -141 and which is smaller than -141 , that is, which has an absolute value greater than 141 . Thus,

$$\begin{array}{r} -141 \quad | \quad 15 \\ \underline{150} \quad -10 \\ 9 \end{array}$$

that is, 15 “fits” -10 times into -141 and 9 remains.

If there is a full understanding of the division algorithm (and not just the operation of the device), when applying it to other situations, it becomes more practical if the search for q and r is done as in (6) or, then, having actually incorporated such an idea, simply by looking at Table 1. When this does not happen, teachers in training generally do not pay attention to the fact that the remainder cannot be negative, according to the division algorithm itself in (4). Freshmen undergraduates, who have not yet studied the division algorithm, generally do not realize that they are facing an ordinary division process, with which they have had contact since

Elementary School, and which can be carried out through the device, even without knowing the division algorithm, just having understood the meaning of the division operation.

The motivation for exploring the division algorithm a little was the possibility of showing other ways of approaching such content, with some suggestions, which we believe to be didactic ways to contribute to the understanding of this algorithm on the part of Mathematics teachers in training. In this sense, an attempt was also made to expand knowledge about division and other processes of division of integers, making this teacher have a mathematical basis in relation to this operation. The moment was then taken advantage of to present a new algorithm, now based on repeated differences, suggesting its application in Question 03, as shown below:

Question 03: Another algorithm used to divide two integers is based on the use of repeated differences. To divide the integers D and d , say $D \geq d > 0$, we can apply repeated differences, proceeding as follows:

I. Make $D = D_1$;

II. Calculate $D_{i+1} = D_i - d$;

III. (a) If $D_{i+1} > 0$, then go back to ii), replacing i with $i + 1$;

(b) If $D_{i+1} \leq 0$, stop. The result will be:

$$D = id, \text{ if } D_{i+1} = 0, \text{ and}$$

$$D = (i - 1)d + a_i, \text{ if } D_{i+1} < 0.$$

a) Using this algorithm perform the divisions in i., ii., iii. and iv. of question 2, item d).

b) What is the relationship between this algorithm and the conventional division algorithm?

In item a) of this question, in division *i.* $D = 141$ and $d = 15$, they were expected to take the first step, that is, do $D_1 = 141$, and from there the next step, $D_2 = D_1 - d = 126$, and then all the steps necessary, until reaching a difference less than or equal to zero, in this case, $D_{11} = -9$. Finally, verify that the number of steps is equal to 10, and that $141 = (10 - 1) \times 15 + D_{10}$, that is, $141 = 9 \times 15 + 6$. In this case, in which both integers were positive, of the three students who were taking the Number Theory course and had already had contact with this algorithm, two were able to answer it, leaving some mathematical inaccuracies in the notation, but understanding and operating correctly. One of these three students, however, was unsuccessful, as he was unable to effectively identify who the D_i were along the steps necessary to arrive at the quotient and remainder. However, when the algorithm was applied to any two integers (divisions *ii.*, *iii.* and *iv.*), only Student 03 and Student 10 correctly operated this algorithm.

As for item b), no response was obtained, with the exception of Student 03, once again, who said:

The division algorithm based on successive differences uses the conventional division algorithm as a basis. The division based on successive differences algorithm is a variation of this conventional division method, where the divisor is subtracted not only from the current dividend, but from the previous one as well. This approach allows the algorithm to more quickly catch up with the rest. (Student 03)

Such a response is limited to stating that this algorithm is a derivation of the conventional algorithm, not establishing a relationship between its elements, such as, for example, that, when $D \geq d > 0$, the number of steps taken, i , until reaching a difference less than equal to zero, that is $D_{i+1} = 0$, corresponds to the quotient q in the division algorithm ($q = i$), while the remainder $r = D_{i+1} = 0$; when $D_{i+1} < 0$, we have $q = i - 1$ and $0 < r = D_i < d$. Furthermore, Student 03 made a not-quite-correct statement that the successive differences algorithm “allows the algorithm to more quickly approximate the remainder”. In fact, what this algorithm does is operate with subtractions, avoiding the typical idea of “how many times does d fit in D ?”; thus “simplifies” the process to a better known and simpler operation, which is the case of subtraction, but considerably increases the number of steps, the greater the division quotient.

Finally, the last question was stated as follows:

Question 04: When dividing 145 by 11, it is commonly done as follows:

- *Separate the digits 1 and 4, in that order, forming the number 14, and then divide 14 by 11.*
- *Then multiply 1 by 11, and subtract the result from 14.*
- *Now the remaining digit, 5, is lowered, placing it to the right of the remainder 3, forming the number 35, which is divided by 11.*
- *Multiply 3 by 11, and the result subtracts from 35.*
- *As there are no more digits to be lowered, the division ends. The quotient is therefore 13 and the remainder is 2, since $145 = 11 \times 13 + 2$ and $0 < 2 < 11$.*

With this in mind, explain why we proceed in this way to perform the division between two integers.

This question revealed a critical point in the understanding of the division operation on the part of teachers in training. None of the students answered this question, showing a gap in the formulation of the reasons for carrying out this step-by-step so familiar, even for themselves, in Basic Education. The only exception was again Student 03, who illustrated the resolution of

the account, largely repeating the procedure set out in the question, but without, in fact, justifying the reason for such actions. At least from those who were taking the discipline at the time of the survey or who had already taken it, a reasonable understanding of this procedure was expected.

The idea of separating the digits of the dividend arises from the fact that 145, for example, which is in positional notation, can be written in polynomial form

$$145 = 1 \times 10^2 + 4 \times 10^1 + 5 \times 10^0, \quad (7)$$

where 1 is the 3rd order digit (hundreds), 4 the 2nd order digit (tens) and 5 the 1st order digit (units). In this way, the digits of the dividend are separated. As the number formed by the digit 1 (one hundred) is such that $1 < 11$, it is not possible to divide it by 11; Thus, the next digit, 4, is considered, placing it to the right of 4, forming the number 14 and $14 > 11$. Then divide 14 (14 tens) by 11, finding for quotient 1 (1 ten) and remainder 3 (3 tens), $3 < 11$. Now, multiply 1 by 11 and, from 14, subtract 1×11 , giving 3. Then, lower the remaining digit 5, placing it to the right of 3, forming the number 35; being $35 > 11$, divide 35 by 11, multiply 3 by 11 and subtract the remainder from that division by 35. In this way, as there are no more digits to download, we have: quotient 13 and remainder 2, and $145 = 11 \times 13 + 2$, with $0 < 2 < 11$.

Another way that could be done is the following: when separating the digits of 145, given the impossibility of carrying out the division of 1 by 11, since, $1 < 11$, the next digit, 4, is considered, placing it at the right of 1, forming the number 14 (14 tens or 140 ones). As $145 = 14$ (tens) + 5 (units), we have that

$$\frac{14}{11} = \frac{11+3}{11} = \frac{11}{11} + \frac{3}{11} = 1 + \frac{3}{11}, \quad (8)$$

with $3 < 11$. Here, we divide 14 (tens) by 11, which gives 1 (tens), leaving 3 (tens), so that in $\frac{14}{11} = 1 + \frac{3}{11}$, the 1 (integer part in the division of 14 by 11) will be the first digit of the quotient (1 ten, therefore) and 3 (tens), the first remainder. To the right of 3, the first remainder, place the digit 5 (the last digit of 145 to be lowered), thus forming the number 35 (units), which will be divided by 11. Proceeding as in (8), we have

$$\frac{35}{11} = \frac{33+2}{11} = \frac{33}{11} + \frac{2}{11} = 3 + \frac{2}{11},$$

with $2 < 11$ and, in the sense of what was done in (8), here, the 3 in $\frac{35}{11} = 3 + \frac{2}{11}$ will be the second digit in the quotient of 145 by 11, and 2, the final remainder. Therefore, the quotient of 145 by 11 will be $10 + 3 = 13$ and the remainder $2 < 11$.

In summary, the answers obtained from the questionnaire showed that the investigated undergraduates still demonstrate a utilitarian knowledge of the division algorithm, that is, as a rule, they know how to operate it to find the results, but they do not understand the reason for the procedure performed, remaining in the mechanical understanding of how to do it, but without knowing the real meaning of what they do. It was also noted that the notions of the division operation are more consolidated in these undergraduates when dealing with positive integers, that is, natural numbers; when transported to negative integers, there is a misunderstanding of one of the fundamentals of the division algorithm, for example, in which the remainder can only be zero or greater than zero.

This corroborates our understanding that many undergraduates arrive at the initial training course with mistaken or inconsistent conceptions about the set of integers and the operation of division with these numbers. It is also shown that such misunderstandings remain, in general, unreflected during initial training; which means that, due to the lack of problematization and a more effective approach, the future teacher comes to complete the course maintaining such misconceptions (misconceptions), taking such knowledge (and non-knowledge) to their future practice in Basic Education.

In view of what has been discussed so far, considering that we have achieved our objectives, we proceed to the final considerations.

Final considerations

The discussion presented here certainly does not exhaust the debate on mathematical training in Number Theory for prospective Mathematics teachers. On the contrary, it underscores the importance of increasingly looking at this topic in a broad way, reflecting on what Mathematics students should learn and, specifically, scrutinizing the contents present in initial training and directing them to the training purpose of the course.

This is, after all, the great support of this writing: the mathematical topics covered in the degree should, to a greater or lesser extent, base the teaching practice of the future teacher, and, for that, it is necessary to attribute new perspectives to Scientific Mathematics, in conjunction with School Mathematics. In the words of Moreira and David (2005, p. 45), “[...] it is a question of thinking about the process of teacher education from the recognition of a tension – and not an identity – between school Mathematics education and the teaching of Elementary Academic Mathematics”.

It is important, in this sense, to scrutinize the licentiate’s curricula, with emphasis, to which this discussion belongs, in the Elementary Theory of Numbers. Although it is present in

virtually every initial training course for Mathematics teachers, the approach given to this discipline is still precarious and ineffective in terms of teacher preparation (Resende, 2007). Reviewing the curriculum is therefore important, but it is equally important that trainers reflect their practice in teaching this mathematical topic that underlies the work with numerical sets and operations, the basis of School Mathematics. The trainer cannot, after all, “[...] ignore, when working in the degree course, the pedagogical knowledge of the content, the historical and epistemological issues linked to the concepts with which he works” (Resende, 2007, p. 229).

It is pertinent to resize and redefine our training in what concerns the understanding of numbers and basic arithmetic, developing numerical and arithmetic thinking and refining our understanding of the relationships between sets and operations, including from tests and demonstrations. Which does not mean, in turn, that the classical-formalist, purely axiomatic approach to Number Theory, as found by Resende (2007), is sufficient for this purpose. On the contrary, we have seen that it has not been.

By way of conclusion, given the results presented by the questionnaire applied to teachers in training at a public university in Ceará, we note that basic ideas related to Euclidean division, and more specifically to the division algorithm, are not yet well settled and articulated in the undergraduate students studied. In initial training, we need to invest in articulating content knowledge – especially considering the subdomains indicated by Carrillo-Yañez et al. (2018): knowledge of topics, structure and mathematical practice – with pedagogical knowledge of content, integrating Scientific Mathematics and School Mathematics, in the terms of Moreira and David (2005), or, even, Mathematics to teach and Mathematics to teach, as postulated by Valente (2022), concerning the Theory of Numbers and the teaching of the division operation in Basic Education.

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