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Promoting the specialized knowledge of prospective mathematics teachers about the Euclidean division algorithm

Promovendo o conhecimento especializado de futuros professores de matemática sobre o algoritmo da divisão euclidiana

Promoviendo el conocimiento especializado de futuros profesores de matemáticas sobre el algoritmo de la división euclidiana

Promouvoir les connaissances spécialisées des futurs enseignants de mathématiques sur l'algorithme de division euclidienne

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Abstract

In this paper, we report the experience of a mathematics teacher educator –a mathematician– while teaching the Euclid's division algorithm theorem in a Number Theory course for prospective mathematics teachers. Considering that the knowledge of a mathematics teacher is specialized, from the perspective of the Mathematics Teachers' Specialized Knowledge model, we intended to identify which knowledge is mobilized when the teacher educator addresses this algebraic result. The professor conducted some activities to understand how the prospective teachers performed the division of integers before, during, and after knowing the theorem. Our analysis focuses on these different moments. Regarding the prospective teachers' knowledge, it was possible to observe, primarily, knowledge related to procedures involving algorithms. However, throughout the conducted activities, they established different connections with the Euclid's division algorithm theorem. About the teacher educator, we emphasize that his mathematical and pedagogical knowledge, combined with the goal of effectively preparing prospective mathematics teachers, can potentially promote specialized knowledge in the prospective teachers regarding the subject matter.

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Keywords: Specialized knowledge, Euclid's division algorithm theorem, Number theory, Mathematics teacher education, Mathematics teachers' specialized knowledge.

Resumen

En este artículo reportamos la experiencia de un formador de profesores, quien es matemático, al enseñar el Teorema del Algoritmo de la División Euclidiana en un curso de Teoría de Números para futuros profesores de Matemáticas. Considerando que el conocimiento del profesor que enseña Matemáticas es especializado, desde el punto de vista del modelo *Mathematics Teachers' Specialised Knowledge*, se pretende identificar qué conocimientos moviliza el formador y cuáles conocimientos evidencian los futuros profesores cuando el formador presenta este resultado algebraico. Algunas actividades fueron conducidas por el formador, en un intento de comprender cómo los estudiantes realizaban la división de números enteros antes, durante y después de conocer el teorema. Nuestro análisis se centra en estos diferentes momentos. En cuanto al conocimiento de los futuros profesores, fue posible observar principalmente conocimientos relacionados con los procedimientos que involucran el algoritmo. Sin embargo, a lo largo de las actividades realizadas, pudieron establecer diferentes conexiones involucrando el algoritmo de la división euclidiana. Sobre el formador, destacamos que sus conocimientos matemáticos y pedagógicos, combinados con el objetivo de formar efectivamente a los futuros profesores de Matemática, tienen el potencial de promover en los estudiantes un conocimiento especializado de la materia.

Palabras clave: Conocimiento especializado, Teorema del algoritmo de la división euclidiana, Teoría de números, Formación del profesorado de matemáticas, Conocimientos especializados de los profesores de matemáticas.

Résumé

Dans cet article, nous rapportons l'expérience d'un formateur d'enseignants, qui est mathématicien, lors de l'enseignement du Théorème de l'Algorithme de la Division Euclidienne dans un cours de Théorie des Nombres destiné aux futurs professeurs de mathématiques. En considérant que la connaissance du professeur enseignant les mathématiques est spécialisée, du point de vue du modèle *Mathematics Teachers' Specialised Knowledge*, nous cherchons à identifier quelles connaissances sont mobilisées par le formateur et quelles connaissances sont mises en évidence par les étudiants en licence lorsque le formateur aborde ce résultat algébrique. Plusieurs activités ont été menées par l'enseignant afin de comprendre comment les étudiants en licence effectuaient la division des nombres entiers avant, pendant et après avoir pris connaissance du théorème; notre analyse se concentre sur ces différents moments. En ce qui concerne la connaissance des étudiants en licence, on a pu observer principalement des connaissances liées aux procédures impliquant l'algorithme. Néanmoins, tout au long des activités réalisées, ils ont été capables d'établir différentes connexions impliquant l'algorithme de la division euclidienne. En ce qui concerne le formateur, nous soulignons que sa connaissance mathématique et pédagogique, associée à l'objectif de former efficacement les futurs professeurs de mathématiques, a le potentiel de promouvoir chez les étudiants en licence une connaissance spécialisée sur le sujet.

Mots-clés: Connaissances spécialisées, Théorème de l'algorithme de division euclidienne, Théorie des nombres, Formation de professeur de mathématiques, Connaissances spécialisées des enseignants de mathématiques.

Resumo

Neste artigo, relatamos a experiência de um formador de professores, que é matemático, ao ensinar o teorema do algoritmo da divisão Euclidiana em uma disciplina de teoria dos números para futuros professores de matemática. Considerando que o conhecimento do professor que ensina matemática é especializado, do ponto de vista do modelo *mathematics teachers' specialized knowledge*, pretendemos identificar quais conhecimentos são mobilizados pelo formador e evidenciados pelos licenciandos quando o formador aborda esse resultado algébrico. Algumas atividades foram conduzidas pelo docente na tentativa de compreender como os licenciandos realizavam a divisão de números inteiros antes, durante e após conhecer o teorema; nossa análise foca esses diferentes momentos. Nos licenciandos, foi possível observar prioritariamente conhecimentos relacionados a procedimentos envolvendo o algoritmo. Não obstante, ao longo das atividades realizadas, eles foram capazes de estabelecer diferentes conexões envolvendo o algoritmo da divisão euclidiana. Com relação ao formador, destacamos que seu conhecimento matemático e pedagógico, aliado ao objetivo de efetivamente formar futuros professores de matemática, tem o potencial para promover nos licenciandos um conhecimento especializado sobre o assunto.

Palavras-chave: Conhecimento especializado, Teorema do algoritmo da divisão euclidiana, Teoria dos números, Formação de professores de matemática, Conhecimentos especializados dos professores de matemática.

Promoting the specialized knowledge of future mathematics teachers about the Euclidean division algorithm

There is a consensus that quality education involves training good teachers, and several discussions are held to define what a "good" teacher is. A characteristic that seems fundamental for any teacher to perform their duties properly refers to knowing the content, that is, to deeply know the subject they should teach.

Content knowledge integrates professional teaching knowledge, defended by Shulman (1986, 1987) as a fundamental component in teachers' performance in general and, over time, expanded and discussed in specific areas, such as mathematics education. In this sense, authors such as Ball et al. (2008) and Carrillo et al. (2018) present specific proposals for the modeling of the knowledge of the teacher who teaches mathematics, constantly arguing that it is vast and in-dept, to involve knowledge of the content and pedagogical knowledge of the content.

In the scope of mathematics teacher education, initial education stands out as a unique moment for the development of teacher knowledge. Here, we will focus on a mathematics teacher eduction degree, exploring the potential of teaching Number Theory for the knowledge and practice of prospective teachers in basic education.

Because it deals with topics considered elementary, the course of Number Theory does not always have its role duly recognized in teacher education (Oliveira & Fonseca, 2017; Resende & Machado, 2012). Although the integer number content taught in school is considered uncomplicated, Number Theory is an area of high complexity in mathematics, and the homonymous course taught in mathematics teacher education degrees solidifies many elemental concepts through definitions and proofs. Many in-depth mathematical and pedagogical discussions can be carried out within the scope of this topic, for example, the Euclidean division, which is taught in the early years and is an integral factor in the school trajectory of all students, but the (not trivial) proof of such a result is only taught and discussed in the subject of Number Theory.

Thus, our focus in this text will be the teaching of Euclid division algorithm theorem (EDAT), which guarantees the existence and uniqueness of the quotient and the rest in the division between two integers, as follows: **Given two numbers $a, b \in \mathbb{Z}$, with $b \neq 0$, there are unique $q, r \in \mathbb{Z}$, such that $a = bq + r$ satisfying $0 \leq r < |b|$**. Based on the work with this result in a course focused on algebraic structures, we discuss the knowledge intended and evidenced by the prospective mathematics teachers and the one the teacher educator mobilized.

Literature review and theoretical framework

Numbers are introduced into the child's life since early childhood education, and their study is formalized throughout basic education. In early childhood education, according to the BNCC (Ministry of Education, 2018), numbers appear related, for example, to counting (objects, people, books), quantities and the recording of quantities using numerals.

In the early years of elementary school, specifically in the first three, the focus is on natural numbers, and the division is introduced from the notions of half and third part in the second year. From the third, the meanings of division (distribution in equal parts and measure) are introduced, related to the skill EF03MA08 "Solve and elaborate problems of division of one natural number by another (up to 10), with zero remainder and with non-zero remainder, with the meanings of equitable distribution and measure, through strategies and personal records" (Ministry of Education, 2018, pp. 286-287), as well as the meanings of half, third, fourth, fifth and tenth parts, associated with the skill EF03MA09 "Associate the quotient of a division with zero remainder of a natural number by 2, 3, 4, 5, and 10 to the ideas of half, third, fourth, fifth, and tenth parts" (Ministry of Education, 2018, pp. 286, 287).

In the final years, precisely in the seventh grade, according to the BNCC, the integers are presented. This document recommends the discussion of integers based on their uses, ordering, history, association with points on the number line and operations, including the division of integers. From then on, the subject no longer figures explicitly in the scope of basic education.

In higher education, in the mathematics teacher education degree, the divisibility is formalized, usually within the scope of number theory (Resende, 2007). An integer b is said to be divisible by another integer a if there is an integer c such that $b = ac$. In this context, the Euclid division algorithm is formally presented as the Euclid division algorithm theorem.

Number theory (NT) presupposes a favorable environment for developing important mathematical ideas about natural numbers and integers (Resende, 2007). However, although the knowledge of this subject is of great relevance for a broad understanding of mathematics and its processes, research focused on NT teaching, especially in the context of teacher education, is scarce (Bair & Rich, 2011; Oliveira & Fonseca, 2017).

Still, some investigations have been conducted to understand how the Number Theory course can contribute to teacher education in the mathematics teacher education degree (Almeida, 2020; Sinclair et al., 2003; Smith, 2002; Zazkis & Campbell, 1996a). We will approach some of such investigations below. According to Zazkis and Campbell (1996a), by including topics such as divisibility, prime numbers or linear congruences, which allow the student to revisit basic mathematical processes, the study of NT can lead them to reflect on Mathematical Knowledge.

This reflection, however, does not always happen spontaneously, and the educator is responsible for fostering it. According to Smith (2002), although many NT topics directly relate to basic mathematics education, most undergraduates cannot establish these relationships, eventually understanding NT topics totally disconnected from school mathematics. This is often the case with divisibility, which prospective teachers often treat as a trick or procedure to be performed and not as a relationship between integers (Sinclair et al., 2003).

Among the possibilities of connections between NT content and school mathematics, Almeida (2020) points out the

> divisibility and division, the formalization of results presented intuitively at school (number structure and its arithmetic properties, the study of prime number theory with mathematical rigor, the formalization of divisibility criteria presented at school), the study of set structures that are presented at school, modular arithmetic and finite set representation (cryptography). (p. 24)

Despite these possibilities, the literature points out various difficulties encountered by the prospective teacher in many contents present in NT, for example, in understanding divisibility (Brown et al., 2002; Zazkis et al., 2013), primality and the fundamental theorem of arithmetic (Oliveira & Fonseca, 2017; Zazkis & Campbell, 1996b), as well as in the properties of prime numbers (Zazkis & Liljedahl, 2004).

We understand that one of the ways to face such difficulties during the initial education of mathematics teachers is to work on NT concepts relating them to school mathematics. From this, we can seek to develop specialized knowledge on the subject in prospective teachers, which we will discuss in the next section.

The Mathematics Teachers' Specialised Knowledge

To address prospective mathematics teachers' knowledge, we will rely on the ideas of Carrillo et al. (2018), who argue for the existence of a specialized knowledge of the teacher who teaches mathematics. This specialized knowledge includes knowledge relevant specifically to mathematics teachers and teachers who teach mathematics, excluding knowledge necessary for other professionals who only use mathematics as a work tool –such as engineers, architects, and so many others.

Based on research, the authors developed a model called *Mathematics Teachers' Specialized Knowledge* (MTSK), presented in Figure 1. MTSK consists of three domains: *Mathematical Knowledge*, *Pedagogical Content Knowledge*, and teacher *Beliefs* about mathematics teaching and learning.

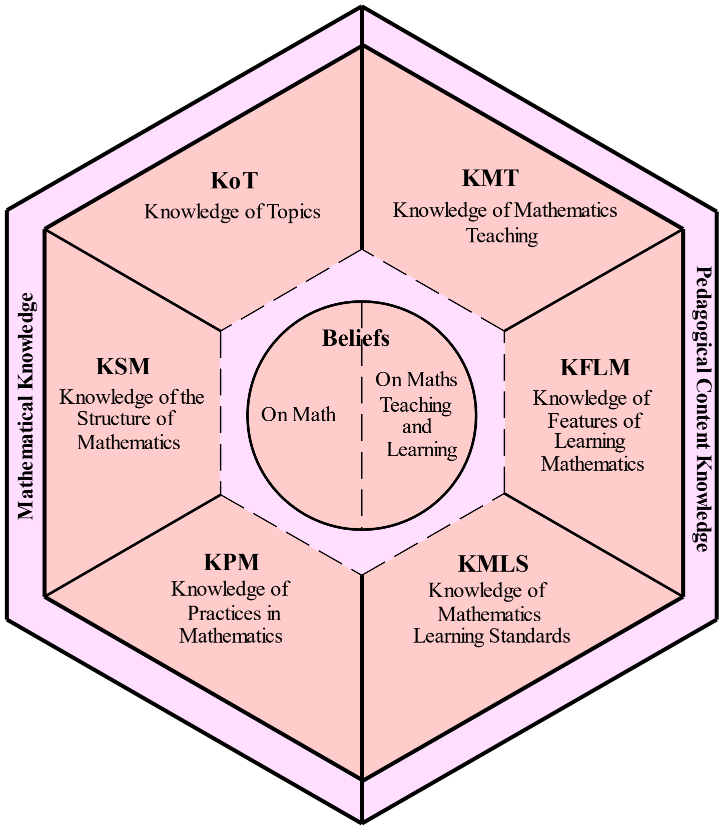

Figure 1.

Mathematics Teachers' Specialized Knowledge model (Carrillo et al., 2018, p. 241)

Situated on the left side of the model, *Mathematical Knowledge* is subdivided into three subdomains: *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KSM), and *Knowledge of Practices in Mathematics* (KPM).

KoT refers to what and how the teacher knows the topics they teach. It is related to knowledge of procedures ("How to do it? When to do it?"), definitions, properties and foundations, phenomenology and applications, and records of representation of the topic addressed. A knowledge related to EDAT, within the scope of KoT, is to know how to use the algorithm and the moment to interrupt successive divisions and subtractions (when the rest is obtained).

In the KSM, mathematical connections, which can be temporal or interconceptual, are addressed. Temporal-type connections relate the content covered to previous content (simplification connections) or later (complexification connections). Interconceptual connections are divided into auxiliaries, which concern the employment of one mathematical concept in another (for example, the understanding of function as an equation, in calculating its roots); and transversal, including knowledge about interrelated contents through an underlying concept (for example, the concepts of continuity, derivative, and definite integral are connected by an underlying idea, the notion of limit).

In turn, the KPM includes knowledge of mathematical creation and production, mathematical language and proofs. According to Delgado-Rebolledo and Zakaryan (2019), this subdomain includes ways to proceed, validate, explore, generate knowledge in mathematics, and communicate mathematics. As an example within the scope of the KPM, we cite knowing how to prove the existence and uniqueness theorem.

The *Pedagogical Content Knowledge* domain, situated on the right side of the model, includes three subdomains, namely *Knowledge of Mathematics Teaching* (KMT), *Knowledge of Features of Learning Mathematics* (KFLM) and *Knowledge of Mathematics Learning Standards* (KMLS). In the KMT, teachers' knowledge related to mathematics teaching is considered, such as material and virtual resources for teaching, strategies, techniques, tasks and examples for teaching mathematics, in addition to knowledge of theories, formal or personal, about mathematics teaching.

The KFLM encompasses knowledge related to the characteristics inherent in learning mathematics, focusing on mathematical content as a learning object. In this subdomain, knowledge of theories about mathematical learning, students' potential and difficulties in learning mathematics, ways in which students interact with mathematical content, and emotional aspects imbricated in mathematics learning are considered.

In turn, KMLS refers to knowledge about what the student should achieve at a given level, in conjunction with what they have studied previously and what they will study in the future. This subdomain includes expected learning outcomes, desired conceptual and procedural development levels, and topic sequencing.

Obviously, developing this type of knowledge during initial education, articulating mathematical and pedagogical knowledge, is a complex task for prospective teachers and demands the participation of the various professionals who work in the mathematics teacher education. To help undergraduates develop specialized knowledge, teacher educators must have their own specialized knowledge, which ideally covers the teacher's specialized knowledge, but must go further in terms of breadth and depth (Escudero-Ávila et al., 2021; Zopf, 2010), as we will discuss in the next section.

The Mathematics Teacher Educator's Knowledge

Teacher educators, according to Jaworski (2008), "are professionals who work with teachers and/or prospective teachers to develop and improve mathematics teaching" (p. 1). This perspective aligns with that presented by Contreras et al. (2017), who point out mathematicians, teacher educators, and mathematics professors who receive and guide undergraduate students in schools as mathematics teachers' educators.

Although the interest in the knowledge of the mathematics teacher educator stands out in recent research (Beswick & Goos, 2018), even those investigations that aim to examine the knowledge or expertise of these subjects generally do not discriminate the knowledge or expertise specific to the teacher educator, necessary for their professional activity. In addition, they do not indicate whether or how the knowledge of these professionals differs from the knowledge of the teachers trained by them (Coura & Passos, 2017).

The educators' knowledge can be based on the specialized knowledge intended to be promoted in prospective teachers (Carrillo et al., 2019). From this perspective, Escudero-Ávila et al. (2021) seek to delimit a specialized knowledge of the mathematics teacher educator, based on the specificities necessary for the work of this professional.

Thus, it is necessary, for example, that the teacher educators' Mathematical Knowledge encompasses teachers' Mathematical Knowledge but is not limited to it; teacher educators need an overview of Mathematical Knowledge, with an emphasis on connections and the depth of this knowledge. Escudero-Ávila et al. (2021) point out three differences between the Mathematical Knowledge of the teacher educator and that of the teacher:

- The teacher educator's knowledge becomes broader and deeper because it results from a growth process in which mathematics reaches greater complexity, allowing the teacher educator to establish more relationships between different concepts.
- The teacher educator attaches greater importance to the syntactic aspects of Mathematical Knowledge and understands, for example, the essence of proofs, the rigor of mathematical language, as well as the meaning of definitions and theorems.
- The teacher educator has a clearer understanding of the structuring ideas of mathematics, as well as connections that allow simplifying or increasing the complexity of a topic (Montes et al. 2016), being able to promote the construction of knowledge of in-service and prospective teachers.

From the point of view of pedagogical knowledge, Escudero-Ávila et al. (2021) study teacher educators' knowledge of the professional development of mathematics teachers, the teaching of content in initial education courses, and the curricular standards in different initial and/or continuous teacher education courses.

Teacher educators' knowledge of the professional development of mathematics teachers includes aspects on the characterization of this professional development, more likely difficulties in the specialization of the in-service and prospective teachers, more appropriate sequences or focuses for the construction of knowledge and what teachers know when entering training courses. Knowledge about teaching content in initial education courses involves knowing repertoires of activities to develop knowledge specific to the teaching activity, knowing the potential and limitations of tasks to be explored in undergraduate courses, different evaluation methodologies and the essential characteristics of each topic.

In turn, knowledge about curricular patterns in different teacher education courses includes not only knowing the curricular standard of the course in which they work as teacher educators but also the levels of education trained teachers will teach. From the perspective of Escudero-Ávila et al. (2021), the knowledge required o teacher educators is associated with the objectives of the training, that is, with the specialized knowledge that is intended to be promoted during the training.

In this article, we discuss the conduct of an activity used in the initial education of mathematics teachers to promote specialized knowledge about EDAT. The research question we intend to answer is: *What specialized knowledge is put into play when the teacher educator approaches the Euclid division algorithm theorem in a Number Theory course in a inicial mathematics teacher education?*

Methodology

In Brazilian universities, as in many other countries, mathematicians are usually responsible for the mathematical education of prospective teachers. Those professionals act as teacher educators, although they do not perceive themselves in this role (Leikin et al., 2018).

The information discussed here was obtained within the scope of a mathematics degree course at the State University of Western Paraná (Unioeste) in 2022. One of the authors, a mathematician, was the teacher educator responsible for the Algebraic Structure course – offered in the third year of the course– when he introduced the EDAT to the students. The other author, a mathematics teacher educator, collaborated with the analysis of the activities carried out.

The subject of Algebraic Structures of the course above is annual and consists of two parts, one dealing with number theory and the other with group theory and ring theory. The EDAT was discussed with students in the first part of the course.

The teacher educator conducted some activities aiming to understand how the undergraduates performed the division of integer numbers before, during, and after knowing the EDAT. Such activities will be described in detail in the next section.

The activities were not initially designed and developed to carry out an academic investigation. After its development, however, they were considered an interesting experience to be reported and an important source for a discussion about the specialized knowledge of the undergraduates and the teacher educator. Thus, the present investigation is a qualitative, descriptive research (Gil, 2002); that is, its main objective is to describe characteristics of a particular phenomenon –the teaching and learning of the EDAT in the mathematics teacher education degree, from the perspective of the MTSK.

Thus, the data sources consist of the teacher educator's class notes, photographs of the blackboard and the reproduction of calculations presented by the undergraduates. The class had, in total, nine undergraduates; of these, five were present at the first moment of carrying out the activities; hence, we selected their answers to examine in this article. The discussion about knowledge mobilized by undergraduates and the teacher educator is supported by the MTSK model (Carrillo et al., 2018) and the teacher educator's specialized knowledge, according to Escudero-Ávila et al. (2021), respectively.

Analysis and Discussion

For the reader to better understand the chronological order in which the activities were carried out, we chose to divide the analysis into moments. First, the teacher educator proposes that students perform some divisions with integer numbers before working on the EDAT. Then, the educator presents and proves the theorem, discussing the cases that arise from it. In a third moment, students are invited to perform new divisions, this time from the theorem discussion. At the end, two months after the third moment, we discussed the knowledge about the EDAT revealed by the students when performing the first written assessment of the subject.

Moment I: Divisions before EDAT is taught

In the previous class, the teacher presented and proved a motto, which is the particular version of the EDAT that considers the divisor as a positive integer: Let $a \in \mathbb{Z}$ and$b \in \mathbb{Z}^*$, then there are $q, r \in \mathbb{Z}$, such as $a = bq + r$ with$0 \leq r < b$. This motto includes Euclid's division taught in the early years, which considers a positive dividend and divisor and is usually proved before the discussion of the EDAT because it is initial and, therefore, a more simplified version of the theorem (KSM - complexification connections), and because the motto is used in the proof of the theorem (KPM - ways of proceeding in mathematics).

At the beginning of this class, the teacher comments that the motto proven in the previous class allows divisions whose dividend is negative and that, in fact, the division itself lies in the fact that it can write $a = bq + r$ with $0 \leq r < b$ (KoT - properties and foundations). In this sense, the professor proposed to the students to carry out, in the way they considered coherent, the following divisions: $123\ 15, 37$ by -4, -218 by 22 and -1328 by-116. Below, we reproduce the students' resolutions:

Student 1

123	15	37	-4	-218	22	-1328	-116
-120	8	-36	-9	+198	-9	1392	12
3		1		-20		64	

Student 2

123	15	37	-4	-218	22	-1328	-116
-120	8	-36	-9	198	-9	1276	11
3		1		20		52	

Student 3

123	15	37	-4	-218	22	-1328	-116
-120	8	-36	-9	+198	-9	+116	11
3		1		-20		-0168	
						+116	
						-052	

Student 4

123	15	37	-4	-218	22	-1328	-116
	8		-9		-9		11
3		1		20		52	

Student 5

123	15	37	-4	-218	22	-1328	-116
-120	8	-36	-9	+22	-10	+116	12
3		1		18		-168	
						+232	
						64	

Figure 2.

Responses of students at moment I

Analyzing the calculations presented, we note that all students proceeded correctly with the division of 123 by 15, obtaining quotient 8 and rest 3 (KoT – procedures, how to do the Euclidean division of two natural numbers), showing that they dominate the Euclid's division algorithm taught since the early years. The division of 37 by -4 was also adequately solved.

Most of them took 36 from 37 to obtain the remainder 1 (KoT – procedures, such as dividing a natural number by a negative integer), showing that, even with a negative divisor, they were able to adapt the algorithm usually applied to positive numbers (KSM – complexification connections).

In the third calculation, division of -218 by 22, in which the dividend is negative, all students could realize that, instead of decreasing a value of -218, they should add up, demonstrating mastery of the game of signs inherent in the division algorithm (KoT - procedures, because something is done this way), that is, they realized that the negative dividend and the positive divisor result in a negative quotient. However, only three students performed the calculation correctly.

Only student 5 obtained the quotient$v - 10$, so that, multiplied by 22, it resulted in -220, which, subtracted from -218, results in a positive sum. This shows that, in addition to knowing the algorithm, the student knows that the remainder must be greater than or equal to zero and less than the divisor, a theoretical consequence of the EDAT that is usually not scored in the teaching of Euclid division at school (KoT - procedures, characteristics of the result).

In the division of -1328 by -116, everyone reached a positive quotient, again demonstrating mastery of the (implicit) sign game in the EDAT. Students 1 and 5 obtained positive results, which shows that student 1 could rethink the algorithm in relation to the calculation made previously (KoT - procedures, result characteristics and KSM - complexification connections). Students 2 and 4 did not obtain a quotient and remainder compatible with the decomposition of the EDAT.

Student 4 did not feel the need to operate any amount with dividends, writing only the quotient and the remainder in long division format. This demonstrates that he obtained the values through trial and error using the identity $a = bq + r$. The other students also proceeded by trial and error, always considering the structure of the algorithm itself (KoT - procedures, how to do it). All were able to establish a parallel of the algorithm represented by long divisionss with the identity $a = bq + r$ of algebraic nature coming from the EDAT (KoT – representation records).

Students 3 and 5 divided -1328 by -116, just like taught in school (which uses the expression “bring down” a number), demonstrating that they did not only use trial and error but fully adapted the algorithm (KoT - procedures, because it is done this way, and KSM -

auxiliary connections, successive subtractions in the algorithm to obtain the Euclidean division), even with student 3 obtaining an incompatible remainder.

It is possible to observe that students, in general, are aware that the quotient must have a sign obtained according to the signs of the dividend and the divisor (KoT – procedures, characteristics of the result). In addition, most of them show knowledge of the connection between division and subtraction imbricated in the algorithm (KSM – auxiliary connections). Most students, however, did not seem to know the conditions imposed on the remainder, except for student 5.

We can also identify, at moment I, some elements of the teacher educator's pedagogical knowledge: he knows that the division algorithm with positive numbers is taught in the early years of elementary school and that, in the final years, the division of integers is introduced, worked based on the game of signs. With this in mind, the teacher educator chooses a teaching sequence that he considers most appropriate to address the EDAT, starting from an example of division with positive numbers and arriving at one with a negative dividend and divisor, only to enunciate the theorem in its general version later. Such knowledge is part of the teacher educator's knowledge about the professional development of mathematics teachers.

When proposing the task of moment I, the teacher educator aims for students to explore the algorithm usually used in positive numbers so that they understand and individually reach the version of the algorithm for negative numbers, which they did not yet know. Thus, the teacher educator seeks to develop in students a knowledge related to ways of exploring in mathematics, within the scope of the KPM. In this way, the professor also shows knowledge of the potentialities of the task explored (knowledge about teaching the content of initial education).

Moment II: Teaching the EDAT

Moment II begins immediately after moment I, when the teacher asks the class if performing the Euclidean division with negative numbers is possible. Students already expect a positive response due to the calculations proposed in the previous moment. The EDAT is presented to emphasize the hypotheses and theses of the theorem; it is qualified as a theorem of existence and uniqueness, making explicit its differences with the motto presented above. The teacher educator comments that this type of theorem (existence and uniqueness) is proved in two parts: existence, display appropriate quotient and remainder; and uniqueness, it is usually assumed that there are two elements and it is proven that they are the same (KPM - ways of validating).

After presenting the EDAT proof (which will not be explored in this article[3]), the professor exposes the trivial cases of the theorem: 1) in which the dividend is zero, in which case the remainder and the quotient are zero ($0 = 0b + 0$); and 2) in which the dividend absolute value is greater than zero and less than the divisor absolute value, in which case the quotient is zero and the remainder is the dividend itself ($a = 0b + a$) (KoT - foundations). Accordingly, four examples are presented considering the possibilities of divisor and dividend signs, and the resolutions of these are made by trial and error to satisfy the thesis that $a = bq + r$ with$0 \leq r < |b|$. Comparatively, the professor shows how to arrange this calculation in the canonical algorithm using the long division (KoT - representation records).

In each example presented below, the following requirements were considered in the resolution (KoT - foundations):

i) the remainder must be greater than or equal to zero and less than the absolute value of the divider;

ii) the sign rule between the dividend and the divisor.

We note that the mathematics teacher educator mobilizes mathematical and pedagogical knowledge by showing concern for the chosen examples and trivial cases (knowledge of the features of the professional development of mathematics teachers). They are examples of a simple nature in their formulation, but they confuse students since they usually rule out the possibility that the quotient is zero.

The teacher educator's intention to demystify the belief that the division algorithm is only used for positive numbers is evident, a belief that comes from the school period. Here, the teacher educator again shows Knowledge of the features of the professional development of mathematics teachers, who need to overcome this belief to build specialized knowledge about the Euclidean division algorithm. In all examples with negative numbers, the teacher educator notes the importance of looking for quotients compatible with the signs involved and how this impacts the algorithm layout (KoT - procedures, how to).

During moment II, the professor emphasizes that the Euclid's division algorithm is a device linked to a theorem with hypotheses and theses, especially regarding the remainder, which must be greater than or equal to zero and less than the divisor absolute value. In this way, the mathematics teacher educator makes it clear that the algorithm is only useful when one knows which numbers can be considered in the dividend and divisor and which theses

[3] An in-depth discussion of the specialized knowledge mobilized by a trainer when demonstrating the Euclidean Division Algorithm Theorem can be found in Almeida, Ribeiro and Fiorentini (2021).

allow the use of the algorithm to cease, demonstrating the concern to indicate the importance of the theoretical concepts (of the theorem) underlying the practical use of the algorithm (knowledge of teaching the content of initial mathematics teacher education programmes - most important characteristics of each topic).

In Figure 3, below, we present examples a and b. Example a was presented considering a positive divisor and dividend, it is the usual case and presumably known by the class since the early years.

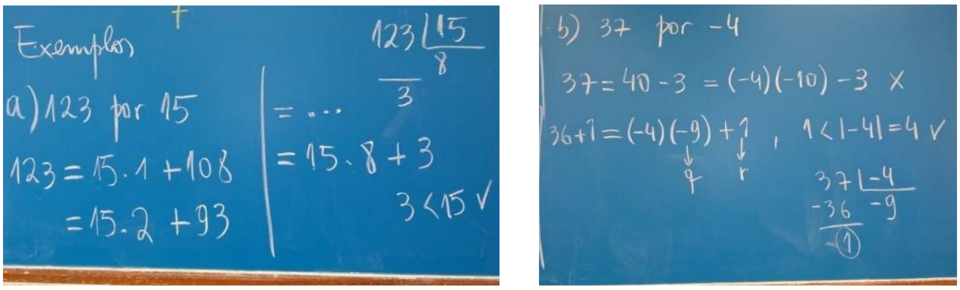

Figure 3.

Examples of division a) and b) proposed to undergraduates during moment II

In the division of 123 by 15, the teacher educator knows, according to the sign rule, that the quotient must be positive. In this way, it tests the values $1, 2, \ldots, 8$ until it obtains a positive remainder and less than $|15| = 15$, which is 3. Thus, in example a, the professor proceeds by trial and error, starting with q from 1 to 8 and observing which of these values fits the conditions of the EDAT. This method was frequently used by students during moment I.

In example b, the professor starts the Euclidean division of 37 by -4 pointing out that, by the sign rule, the quotient must be negative. The first attempt was to write $37 = 40 - 3 = (-4)(-10) - 3$, but -3 does not fit like the remainder in the thesis of the EDAT. The second attempt considered $37 = 36 + 1 = (-4)(-9) + 1$, in which, due to the uniqueness imposed by EDAT, the quotient is -9 and the remainder is$1 < |-4| = 4$.

Thus, the professor starts from the fact that $37 = 40 - 3$ and organizes this expression according to the equation $a = bq + r$ until obtaining q and r under the conditions of the EDAT, demonstrating to know more than one starting point to use the algorithm (KoT - procedures, how to)

Examples c and d are presented in Figure 4. Example c illustrates the case where the dividend is negative and the divisor is positive, which will result in a quotient that is also negative. In the table written by the professor (Figure 4), we observed the division of -218 by 22.

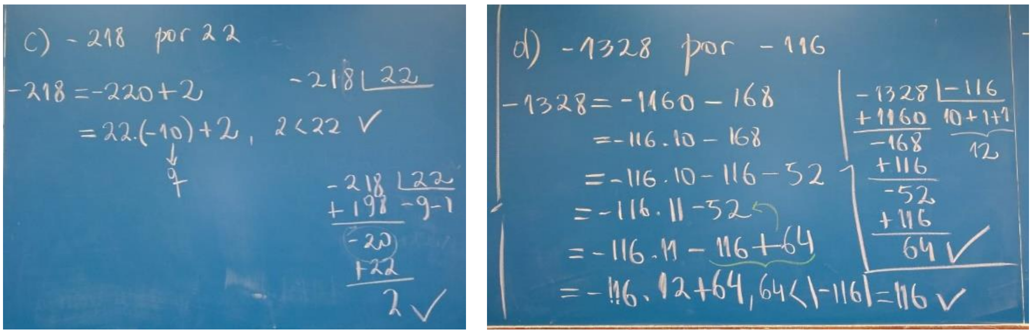

Figure 4.

Examples of division c *and* d *proposed to undergraduates during moment II*

In example *c*, the professor proceeds to the resolution like the previous one, purposely obtaining a candidate for remainder that is negative (-20), just as the students obtained at Moment I. At this point, he highlights the importance of adding the divisor once again to the account made below the dividend to get a positive remainder. Thus, he demonstrates knowing the typical mistakes that students make in this type of division, characterizing his knowledge of the features of the professional development of prospectives teachers.

Example *d* closes the cases considering dividend and divisor, both negative, thus obtaining a positive quotient. To perform the division of -1328 by -116, the professor chooses a trivial multiple of -116, in this case $-1160 = -116 \times 10$, and interacts with -168 until obtaining -1328. In the sequence, the mathematics teacher educator removes multiples -116 of -168 until obtaining a residue of 64, which fits into the EDAT as the remainder. He also demonstrates knowing how to group the numbers until he finds the quotient and the remainder –this is obtained through classroom experience since the proof of the EDAT does not give a path to obtain q and r.

In examples *c* and *d*, the professor points out the importance of realizing that the product of the quotient by the divisor is negative, which implies that such product should be added to the dividend and not decreased as usual. In examples *a* and *b*, this product is positive, which allows it to be subtracted in the usual way from the dividend.

In all examples, the professor shows, in parallel, how to proceed with the algorithm by long division (KoT - representation records), which demonstrates the concern to establish connections between the theoretical part and the practical part of the EDAT, with procedures done since the early years. The professor also emphasizes, in all calculations, the need to obtain a remainder r satisfying $0 \leq r < |b|$. This emphasis comes from their knowledge of the common difficulties of prospective teachers when working with the EDAT–a characteristic evidenced by them even at the moment III, after having seen the theorem.

The professor demonstrates in practice to know when to add or decrease a unit in the quotient to produce the appropriate remainder. Considering moment I, he reveals to be concerned with establishing connections with the content taught in the course and what is taught in school to support generalizations for the Euclidean division in integers. Thus, the teacher educator shows knowledge of the curriculum of the levels of education in which the prospective teachers will teach .

By proposing the initial activity with the divisions discussed in moment I, together with the examples of moment II, the mathematics teacher educator shows knowledge about the potential of this type of task in undergraduate courses, mobilizing knowledge of teaching the content of initial mathematics teacher education programs.

Moment III: Division after knowing EDAT

Moment III occurs at the beginning of the next class, the day after moments I and II. In it, the mathematics teacher educator proposes that the four students present in this class perform the Euclidean division of -615 by -73. Students 1, 2 and 4 proceeded according to the following reproduction:

-615	-73
657	9
42	

Figure 5.

Responses of students 1, 2, and 4 during moment III

Student 3 performed the calculation as follows:

-615	-73
+584	-8
-031	

Figure 6.

Student response 3 during moment III

At this time, students 1, 2, and 4 demonstrate that they have acquired the knowledge that the remainder must be greater than zero and less than the divisor absolute value (KoT - foundations). These students prefer to use only trial and error, testing values for the quotient q until the expression $(-73) \times q$ is less than $|-615| = 615$ (KoT - procedures, how to). Student 3, however, has not yet built such knowledge.

Moment IV: Written evaluation

Moment IV considered for the analysis occurs during the first written assessment of the subject. Among the questions proposed, two were EDAT-specific:

i) State Euclid division algorithm theorem.

ii) Under the conditions of EDAT, divide -1061 by -45.

The following are the answers given by the five students:

Student 1

i) If $a|b$, with $a \in \mathbb{Z}$ and$b \in \mathbb{N}^*, a = bq + r$, with $q \in \mathbb{Z}$ and$0 \leq r < b$.

ii) The student made successive multiplications of 45 by $2,3,4, ..., 24$ until obtaining:

-1061	-45
+1080	24
19	

Figure 7.

Student response 1 in question ii of moment IV

Student 2

i) $a = bq + r$ where $0 \leq r < |b|$. The EDAT ensures that the dividend is equal to the divisor times the quotient plus the remainder so that the remainder is less than or equal to 0 and less than the modulus of b, that is, the remainder is positive.

ii) The student multiplies -45 by 24 and obtains:

-1061	-45
+1080	24
19	

Figure 8.

Student 2's answer to question ii) during moment IV

Student 3

i) $a = bq + r, \quad 0 \leq r < |b|, \quad r = remainder, q = quocient, b = dividend, a = divisor.$

ii) The student multiplies 45 by 24 and subtracts this result from -1061, writing:

$$-1061 = (-45).24 + 19$$

Student 4

i) Given $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$, there are unique integers q and r such that $a = bq + r$, with $0 \leq r < |b|$.

ii) The student proceeds as follows:

-1061	-45
+1035	23+1
-26	
+45	
19	

Figure 9.

Student 4's answer to question ii during moment IV

Student 5

i) Being $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$, we have unique q r satisfying $a = bq + r$, being $0 \leq r < |b|$.

ii) Knowing that, by the EDAT, the division should be in the form $a = bq + r$, in this case, the division of -1061 by -45 can be written as $-1061 = (-45).24 + 19$.

-1061	-45
+1080	24
19	

Figure 10.

Student 5's answer to question ii during moment IV

It is possible to observe that all students divided -1061 correctly -45 (KoT - procedures), considering the remainder between 0 and 45. Thus, they revealed to know not only the algorithm but also its conditions (KoT - foundations). With the exception of student 4, the others arranged the decomposition $-1061 = 24 \times (-45) + 19$ of the EDAT into long divisions. Thus, they indicated relating the theorem to the usual way of expressing it (KoT - representation record).

All students obtained the quotient by trial and error (KoT - procedures), showing preference and mastery in the use of this technique. For this reason, it is possible to perceive that students prefer to verify the identity of the algorithm until they obtain identity instead of proceeding by disposition by long divisions.

In question i, we observed the reinforcement of some beliefs:

1) Student 1 believes the algorithm is valid only when considering the divisor as a natural number. It also evidences the belief that Euclidean division is an exact division when using notation $a|b$.

2) To enunciate a theorem, it is enough to display an associated equation or formula, in this case, $a = bq + r, 0 \leq r < |b|$, understanding that the hypotheses are secondary. We can see this in the answers of students 2 and 3.

Students 4 and 5 demonstrate that they know that the EDAT is a theorem of existence and uniqueness by showing that the quotient and the remainder are unique (KoT - foundations). Student 4 was the only one who reproduced the EDAT with his hypotheses and theses to enunciate the quotient and the remainder as integers. This is important because a theorem about numbers always delimits the set where they are contained. This student demonstrates that she knows how to enunciate the theorem logically (implication) and uses the usual and precise terminology and notation of mathematics (KoT - foundations). Except for student 1, who expresses a particular version of the EDAT, the others demonstrate that they know that the remainder is limited by the divisor absolute value (KoT - definitions, properties, and foundations).

An observable component of the mathematics teacher educator's knowledge during moment IV refers to the use of an assessment methodology, in this case the written test, which is part of their knowledge of the teaching of content in initial education courses. By focusing on the statement of the theorem, the teacher educator intends to evaluate whether the undergraduates could grasp the EDAT in its essence, considering the hypotheses and theses.

Table 1.

Prospective teacher's specialized knowledge indicators

Subdomains	**Categories**	**Indicators**	**Moment**
KoT	Definitions, Properties, and Foundations	Know that in Euclidean division the remainder must be greater than or equal to zero and less than the divisor absolute value	III and IV
		Know the EDAT as a theorem of existence and uniqueness	IV
		Know the statement of the EDAT	IV
	Procedures	How to do Euclidean division of two natural numbers in the division of 123 by 15	I
		How to divide a natural number by a negative integer in the division of 37 by -4	I
		How to use the sign game in the Euclidean division of -218 by 22	I
		Know the characteristics of the remainder in the EDAT to know the moment to stop the use of the algorithm in the division of -218 by 22	I
		How to the sign rule in Euclidean division of two negative integers	I
		How to use the algorithm through trial and error considering identity $a = bq + r$	I and IV
		How to use the algorithm through subtractions	I
		Knowing that the quotient is associated with the sign rule between the dividend and the divisor	I
		Know how to proceed with Euclidean division of -615 by -73 through trial and error	III
		Know how to proceed with the Euclidean division of -1061 by -45 through the identity $a = bq + r$	IV

		Know how to proceed with the Euclidean division of -1061 by -45 through trial and error	IV
	Registers of representation	Know the sign rule representation of the algorithm and relate it to the algebraic expression $a = bq + r$ of the EDAT	I and IV
		Know how to write the identity of the EDAT $-1061 = 24 \times (-45) + 19$ by the representation by the sign rule	IV
KSM	Connections based on increased complexity	Understand the division of negative integers from the division of positive integers and one positive by one negative	I
	Auxiliary connections	Know which successive subtractions are used in the algorithm	I
		Knowing that the Euclidean division is related to subtraction according to the EDAT	I

We can observe that the *KoT - procedures* manifest predominantly at moment I, while the *KoT - definitions, properties, and foundations* manifest more frequently at moment IV (evaluation). This fact shows that the students constituted knowledge about the foundations of EDAT, besides the procedures evidenced at moment I. It is also possible to observe that, after the class about the EDAT (moment II), the students abandoned the successive subtractions in the Euclidean division and started to use identity $a = bq + r$ as a guideline for the algorithm.

In Table 2, in turn, we compiled the knowledge mobilized by the teacher educator when addressing the EDAT.

Table 2.

Teacher educator's knowledge

Subdomains	**Categories**	**Indicators**	**Moment**
KoT	Definitions, Properties, and Foundations	Know that Euclidean division with positive divisor is performed when writing $a = bq + r$ with $0 \leq r < b$	I
		Know the trivial cases of EDAT ($0 = 0b + 0$e$a = 0b + a$)	II
		Know that in Euclidean division, the remainder must be greater than or equal to zero and less than the divisor absolute value	II
		Know the implicit sign rule in the EDAT	II
	Procedures	Know how the quotient sign affects the operation (addition or subtraction)	II

		performed on the dividend in the algorithm	
		Know how to use the algorithm through trial and error from identity $a = bq + r$	II
		Know how to approach the division of 37 by -4 and -218 by 22 through an initial difference/sum of round numbers, without using trial and error	II
		Know how to manipulate a numerical identity to explain the rest appropriate to the EDAT	II
	Registers of representation	Know how to represent the identity of the EDAT($a = bq + r$) in the long division layout of the algorithm	II
KSM	Connections based on increased complexity	Knowing that EDAT is a generalization of Euclidean division to positive divisor	I
	Auxiliary connections	Know which successive subtractions are used in the algorithm	II
		Know the trivial cases of the EDAT	II
KPM	Ways to proceed	Know that the motto is used in the proof of the EDAT	I
		Know how to prove theorems of existence and uniqueness	II
Knowledge of the features of the professional development of mathematics teachers	Teaching sequences	Knows a teaching sequence starting from divisions of positive numbers and gradually approaches the division to negative numbers, culminating in the EDAT	I
	Unconventional representations	Knows and addresses the trivial cases ($0 = 0b + 0$e$a = 0b + a$) of the EDAT to present unconventional representations of the algorithm	II
	Starting point when joining the training	Knows the starting point where students are in relation to the Euclidean division, in this case, the division of positive numbers	II
		You know that students come to initial education with the belief that the division algorithm only applies to natural numbers, which can cause difficulties in understanding the EDAT	II
	Typical Errors	Knows the typical errors of students in the execution of the algorithm, for example, considering negative numbers as the remainder	II

Knowledge of teaching the content of initial mathematics teacher education programmes	Task potentials	Knows and seeks to develop in students a knowledge related to ways of exploring (the Euclidean division) in mathematics	I
	Most important features of each topic	He knows that EDAT's hypotheses and theses are important to develop the knowledge (in students) of when and why the algorithm works and ceases.	II
	Different valuation methodologies	Knows the written test as an evaluation methodology	IV
Knowledge of the standards of mathematics teacher education programmes	Curriculum of the levels of education in which prospective teachers will work	He knows that the Euclid's division algorithm is introduced in the early years of elementary school and that the division of integers is addressed, in the final years, usually without connection to this algorithm, only based on the sign rule	II

The table above shows us that the knowledge evidenced by the teacher educator primarily during moment II, which was already expected, as this moment is the EDAT class. Still, we realize that the KPM is mobilized exclusively by the professor, such knowledge is typical of the teacher educator and is not evidenced in the knowledge of prospective teachers. The teacher educator demonstrates the three subdomains of pedagogical knowledge, with a predominance of knowledge of the features of the professional development of mathematics teachers, which shows the professors' concern with the pedagogical education of students regarding the theme and serves as an example to demystify the belief that teacher educators who teach Advanced Mathematics courses are concerned/involved only with mathematical content.

Final considerations

In this article, we seek to investigate knowledge mobilized and evidenced when a mathematics teacher educator approaches the Euclid division algorithm theorem in a Number Theory course in the initial education of mathematics teachers. The results point to knowledge of a diverse nature. In the case of undergraduates, there is a predominance of knowledge within the scope *of Knowledge of Topics*, especially in the *procedures* category, which is justified by the fact that they are working with the Euclid's division algorithm and by the fact that the teacher educator has proposed several examples of calculations using the algorithm. It is also noticeable that, from the sequence of activities posed by the teacher educator, the undergraduates were able to establish connections between the division of positive numbers and the division of negative numbers (connections based on increased complexity) and between division and subtraction in the use of the algorithm (auxiliary connections).

The Mathematical Knowledge mobilized by the teacher educator encompasses the knowledge of the teachers he trains. And it goes further, for example, when it evidences its *Knowledge of Practices in Mathematics,* knowing that the motto is used in the proof of the EDAT, indicating how to prove theorems of existence and uniqueness (ways of proceeding).

We highlight the pedagogical knowledge of the teacher educator within the EDAT's scope, which was also diversified, contemplating the three subdomains proposed by Escudero-Ávila et al. (2021). In the span of knowledge about the features of the professional development of mathematics teachers, we highlight the chosen teaching sequence, which starts from the division of positive numbers, helps undergraduates establish a connection between this and the division of negative numbers and then presents the EDAT. In this same subdomain, the mathematics teacher educator explicitly knows the starting point of the undergraduates when they enter the teacher education course: they know the algorithm of the Euclidean division for natural numbers and have the belief that the algorithm applies only to these numbers, which causes the typical error of considering negative numbers as possible remainders in the Euclidean division.

Regarding the knowledge of teaching the content of initial mathematics teacher education programmes, the teacher educator shows, for example, that he know the potential of the chosen task, having chosen to explore the Euclidean division gradually, through an introductory task, and then working with the theorem. He identifies the EDAT's most important features, namely the theorem's hypotheses and theses, and knows its importance for undergraduates to understand why the algorithm works and why it ceases.

In addition, knowledge of the standards of mathematics teacher education courses was also expressed by the mathematics teacher educator. This shows how and at what time of schooling the Euclid's division algorithm and the division of integers are introduced; that is, it knows the curriculum of the levels of education in which the prospective teachers will act.

In Almeida (2020), one of the limitations pointed out by the author in the investigation referred to the absence of research subjects, mathematicians, who identified with the role of the teacher educators. It is noticeable that, when intending to train prospective teachers, the objectives of the mathematics teacher educator, who is a mathematician, go beyond the transmission of Mathematical Knowledge to undergraduates, seeking, for example, that they establish connections between the mathematics contemplated in the discipline of Number Theory and the school.

In addition, EDAT's approach chosen by the mathematics teacher educator seeks to promote specialized knowledge on the subject in undergraduate students. This includes the discussion of particular cases of Euclidean division, the whys related to the functioning of the algorithm, different ways of solving (the long divisionmethod, trial and error), the conditions imposed by the theorem, unconventional representations of the algorithm, and the importance of hypotheses and theses in a theorem.

Although they have spent much time taking mathematics courses during their undergraduate studies, many working teachers consider that they have little relationship and relevance to their pedagogical practice (Zazkis & Leikin, 2010). We believe that analyzing and understanding the knowledge involved in these classes can, on the one hand, bring understanding about the various possibilities of approaching mathematical results in undergraduate courses and, on the other hand, help teacher educators reflect on the possibilities of their practices in the classroom and on the types of knowledge they want to help prospective teachers to build.

References

Almeida, M. V. R. (2020). *Conhecimento especializado sobre divisibilidade do formador de professores que ensina Teoria dos Números para estudantes de Licenciatura em Matemática* [Tese de doutorado em Ensino de Ciências e Matemática]. Universidade Estadual de Campinas. https://hdl.handle.net/20.500.12733/1640658

Almeida, M. V. R, Ribeiro, M., & Fiorentini, D. (2021). Mathematical specialized knowledge of a mathematics teacher educator for teaching divisibility. *PNA*, *15*(3), 187–210. https://doi.org/10.30827/pna.v15i3.15778

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Bair, S. L., & Rich, B. S. (2011). Characterizing the Development of Specialized Mathematical Content Knowledge for Teaching in Algebraic Reasoning and Number Theory. *Mathematical Thinking and Learning*, *13*(4), 292–321. https://doi.org/10.1080/10986065.2011.608345

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education, 21*(5), 417–427. https://doi.org/10.1007/s10857-018-9416-4

Brown, A., Thomas, K., & Tolias, G. (2002). Conceptions of divisibility: Success and understanding. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 41–82). Westport, CT: Ablex Publishing.

Carrillo, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., … Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge

(MTSK) model. *Research in Mathematics Education*, *20*(3), 236-253. https://doi.org/10.1080/14794802.2018.1479981

Carrillo, J., Montes, M., Codes, M., Contreras, R. C., & Climent, N. (2019). El conocimiento didáctico del contenido del formador de profesores de matemáticas: su construcción a partir del análisis del conocimiento especializado pretendido en el futuro profesor. In F. Imbernón, A. Shigunov Neto, I. Fortunato (Eds.), *Formação permanente de professores: experiências ibero-americanas* (pp. 324-341). São Paulo: Edições Hipótese.

Contreras, L. C., Montes, M., Muñoz-Catalán, M. C., & Joglar, N. (2017). Fundamentos teóricos para conformar un modelo de conocimiento especializado del formador de profesores de matemáticas. In J. Carrillo & L.C. Contreras (Eds.), *Avances, utilidades y retos del modelo MTSK. Actas de las III Jornadas del Seminario de Investigación de Didáctica de la Matemática de la Universidad de Huelva* (pp. 11–25). Huelva: CGSE.

Coura, F. C. F., & Passos, C. L. B. (2017). Estado do conhecimento sobre o formador de professores de Matemática no Brasil. *Zetetiké*, *25*(1), 7–26. https://doi.org/10.20396/zet.v25i1.8647556

Delgado-Rebolledo, R., & Zakaryan, D. (2019). Relationships Between the Knowledge of Practices in Mathematics and the Pedagogical Content Knowledge of a Mathematics Lecturer. *International Journal of Science and Mathematics Education*, *18*(1), 567–587. https://doi.org/10.1007/s10763-019-09977-0

Escudero-Ávila, D., Montes, M., & Contreras, L. C. (2021). What do Mathematics Teacher Educators need to know? Reflections emerging from the content of mathematics teacher education. In M. Goos, & K. Beswick (Eds.), *The learning and development of mathematics teacher educators: international perspectives and challenges* (pp. 23-40). Springer International. https://doi.org/10.1007/978-3-030-62408-8

Gil, A. C. (2002). *Como elaborar projetos de pesquisa*. São Paulo: Atlas.

Jaworski, B. (2008). Development of the mathematics teacher educator and its relation to teaching development. In B. Jaworski, & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 4, pp. 335–361). Rotterdam: Sense Publishers.

Leikin, R., Zazkis, R., & Meller, M. (2018). Research mathematicians as teacher educators: focusing on mathematics for secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *21*(5), p. 451-473. https://doi.org/10.1007/s10857-017-9388-9

Ministério da Educação. (2018). *Base Nacional Comum Curricular*. Brasília: Ministério da Educação.

Montes, M., Ribeiro, C., Carrillo, C., & Kilpatrick, J. (2016). Understanding mathematics from a higher standpoint as a teacher: an unpacked example. In *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 315–322). Szeged, Hungary.

Oliveira, G. P., & Fonseca, R. V. (2017). A teoria dos números na formação de professores de matemática: (In)compreensões acerca da primalidade e do teorema fundamental da Aritmética. *Ciência & Educação*, *23*(4), 881–898. https://doi.org/10.1590/1516-731320170040015

Resende, M. R. (2007). *Re-Significando a disciplina Teoria dos Números na formação do professor de Matemática na licenciatura* [Tese de doutorado em Educação

Matemática]. Pontifícia Universidade Católica de São Paulo. https://repositorio.pucsp.br/jspui/handle/handle/11207

Resende, M. R., & Machado, S. D. A. (2012). O ensino de matemática na licenciatura: a disciplina Teoria Elementar dos Números. *Educação Matemática Pesquisa*, *14*(2), 257–278. https://revistas.pucsp.br/index.php/emp/article/view/9077

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Sinclair, N., Zazkis, R., & Liljedahl, P. (2003). Number Worlds: Visual and Experimental Access to Elementary Number Theory Concepts. *International Journal of Computers for Mathematical Learning*, *8,* 235–263. https://doi.org/10.1023/B:IJCO.0000021780.01416.61

Smith, J. C. (2002). *Connecting undergraduate Number Theory to High School Algebra: A study of a course for prospective teachers*. Proceedings of the 2nd International Conference on the Teaching of Mathematics, Crete, Greece.

Zazkis, R., & Campbell, S. R. (1996a). Divisibility and Multiplicative Structure of Natural Numbers: Preservice Teachers' Understanding. *Journal for Research in Mathematics Education*, *27*(5), 540–563. https://doi.org/10.2307/749847

Zazkis, R., & Campbell, S. R. (1996b). Prime decomposition: understanding uniqueness. *Journal of Mathematical Behavior*, *15*(2), 207–218.

Zazkis, R., & Leikin, R. (2010). Advanced Mathematical Knowledge in Teaching Practice: Perceptions of Secondary Mathematics Teachers. *Mathematical Thinking and Learning*, *12*(4), 263–281. https://doi.org/10.1080/10986061003786349

Zazkis, R., & Liljedahl, P. (2004). Understanding primes: the role of representation. *Journal for Research in Mathematics Education*, *35*(3), 164–186. https://doi.org/10.2307/30034911

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson Play in Mathematics Education*. New York, NY: Springer New York. https://doi.org/10.1007/978-1-4614-3549-5

Zopf, D. (2010). *Mathematical Knowledge for teaching teachers: The mathematical work of and knowledge entailed by teacher education*. [Tese de Doutorado]. University of Michigan. http://deepblue.lib.umich.edu/bitstream/2027.42/77702/1/dzopf_1.pdf