ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2023v25i2p278-328

Study and research path: A device for research and professional formation

Percurso de estudo e pesquisa: um dispositivo de pesquisa e formação profissional

Recorrido de estudio e investigación: un dispositivo para la investigación y la formación profesional

Parcours d'étude et de recherche : un dispositif pour la recherche et la formation professionnelle

Rita Lobo Freitas [1]
Universidade de Estado da Bahia (UNEB)
Doutora em Educação Matemática
https://orcid.org/0000-0001-6914-1483

Saddo Ag Almouloud [2]
Universidade Federal do Para (UFBA)
Doutor em Matemática e Aplicações
https://orcid.org/0000-0002-8391-7054

Abstract

In this text, we present the structuring of a study and research path (SRP) carried out within the scope of the initial education of pre-service mathematics teachers attending the curricular supervised teaching practice at a public university in Brazil. One of the research objectives was the constitution of a theoretical-methodological device capable of promoting teachers' research and continuing education on plane analytic geometry (PAG) topics. The work built around the generative question of the SRP, "How to teach the analytic geometry of the point and the line?", brought the possibility of answering the general question of the research promoting a process of professional formation from a theoretical-methodological device with specific characteristics. The planning moments identified in the activities executed by the pre-service teachers were quite significant. However, we could not identify in the recordings the possible praxeologies thought for the teaching of the PAG. Overall, the SRP-PF developed proved to be a theoretical-methodological device with the potential for scientific research in the didactics of mathematics and the professional formation of pre-service teachers.

Keywords: Plane analytic geometry, Anthropological theory of the didactic, Study and research path, Professional formation, Theoretical-methodological device.

[1] rlobo@uneb.br
[2] saddoag@gmail.com

Resumo

Neste texto nós apresentamos a estruturação de um Percurso de Estudo e Pesquisa (PEP) realizado no âmbito da formação inicial de futuros professores de matemática, cursistas do estágio supervisionado em uma universidade pública no Brasil. Um dos objetivos da pesquisa se configurou na constituição de um dispositivo teórico-metodológico capaz de promover pesquisa e formação continuada de professores sobre tópicos de Geometria Analítica Plana. O trabalho construído em torno da questão geratriz do PEP, “como ensinar a geometria analítica do ponto e da reta?”, trouxe a possibilidade de responder à questão geral da pesquisa promovendo um processo de formação profissional, a partir de um dispositivo teórico-metodológico com certas características. Os momentos de planejamento identificados nas atividades realizadas pelos professores estagiários foram bastante significativos, apesar de não ser possível identificar nas gravações as possíveis praxeologias pensadas para o ensino de GAP. Globalmente, o PEP-FP desenvolvido se mostrou ser em um dispositivo teórico-metodológico que tem um potencial para a pesquisa científica em Didática da Matemática e para a formação profissional de futuros professores.

Palavras-chave: Geometria analítica plana**,** Teoria antropológica do didático, Percurso de estudo e pesquisa, Formação profissional, Dispositivo teórico-metodológico.

Resumen

En este trabajo presentamos la estructura de un Trayecto de Estudio e Investigación (PEP) realizado en el contexto de la formación inicial de futuros profesores de matemática, estudiantes de pasantía supervisada en una universidad pública de Brasil. Uno de los objetivos de la investigación fue la constitución de un dispositivo teórico y metodológico capaz de promover la investigación y la formación continua de profesores sobre temas de Geometría Analítica Plana. El trabajo construido en torno a la pregunta generadora del PEP "¿cómo enseñar la geometría analítica del punto y de la recta?", trajo la posibilidad de responder a la pregunta general de investigación promoviendo un proceso de formación profesional, a partir de un dispositivo teórico metodológico con determinadas características. Los momentos de planificación identificados en las actividades realizadas por los profesores en formación fueron bastante significativos, aunque no fue posible identificar en los registros las posibles praxeologías pensadas para la enseñanza de la GAP. En general el PEP-FP desarrollado, demostró ser en un dispositivo teórico y metodológico que tiene un potencial para la investigación científica en Didáctica de la Matemática y para la formación profesional de futuros profesores.

Palabras clave: Geometría analítica plana, Teoría antropológica de la didáctica, Recorrido de estudio e investigación, Formación profesional, Dispositivo teórico y metodológico.

Résumé

Dans ce texte nous présentons la structure d'un Parcours d'Étude et de Recherche (PER) réalisé dans le contexte de la formation initiale des futurs professeurs de mathématiques, étudiants en stage supervisé dans une université publique au Brésil. L'un des objectifs de la recherche présentée a été configuré dans la constitution d'un dispositif théorique-méthodologique capable de promouvoir la recherche et la formation continue des enseignants sur les thèmes de la Géométrie Analytique dans le plan (GAP). Le travail construit autour de la question génératrice du PER "comment enseigner la géométrie analytique du point et de la droite ?", a apporté la possibilité de répondre à la question générale de la recherche en promouvant un processus de formation professionnelle, à partir d'un dispositif méthodologique théorique avec certaines caractéristiques. Les moments de planification identifiés dans les activités réalisées par les enseignants stagiaires ont été assez significatifs, bien qu'il n'ait pas été possible d'identifier dans les enregistrements les possibles praxéologies pensées pour l'enseignement de la GAP. Dans l'ensemble, le PER-FP développé, c´est montré être un dispositif théorique et méthodologique qui a un potentiel pour la recherche scientifique en didactique des mathématiques et pour la formation professionnelle des futurs enseignants.

Mots-clés : Géométrie analytique plane, Théorie anthropologique de la didactique, Parcours d'étude et de recherche, Formation professionnelle, Dispositif théorique et méthodologique.

Study and research path: A device for research and professional formation

The studies developed around the anthropological theory of the didactic (ATD) in France and Spain since 1999 bring contributions that allow us to affirm that the theory has gone through transformations and may still change. More and more, such changes emerge to help analyze and explain the phenomena of mathematics teaching and learning. We include teachers' activities and functions in such phenomena when they propose to teach something to their students or even their schooling when they face challenges imposed by the task of teaching.

In this process, the didactic transposition that occurs within the classroom, in teachers' practice, is consciously or unconsciously influenced by teachers' praxeological equipment, which includes different knowledge inherent to the profession and built throughout education. Such knowledge includes didactic knowledge, which is, in our view, at the heart of the activity of didactic transposition of the mathematical knowledge at stake.

The anthropological focus of ATD is based on mathematical activity interpreted or modeled as a human activity like any other, in contrast only to the idea of a system of concepts or cognitive processes.

As per Sierra (2006), institutional mathematical activity understands that school mathematical activity is a particular case. The ATD proposes a mathematical knowledge model that describes school mathematics as a particular case of the didactic process. We agree with this author regarding didactics, within this scope, as being everything related to the study, production, and dissemination (or reproduction) of mathematical knowledge in different social institutions.

This perspective places the teaching and learning of school mathematics as a general aspect, the whole, and the didactic process as the particularity, the particular case. To Chevallard (2009a, p. 27), the didactics of the subjects "is the science of the conditions and restrictions of the diffusion of praxeologies in societal institutions". In this perspective, our study departed from a global view of mathematical knowledge to the particular case of the didactic problem of teaching and learning specific concepts of plane analytical geometry.

Our research is inserted in this theoretical context. We studied 28 pre-service teachers of a mathematics degree course in Bahia, Brazil, who were attending the curricular supervised teaching practice and voluntarily participating in a teacher continuing education proposal on plane analytic geometry (PAG). For this formative activity, we based the study and research path for professional training (SRP-PF) on Sierra (2006) and Sierra and Gascón (2018), with the theoretical support of the ATD.

Note that within the scope of the ATD, these authors' studies were configured as the first approaches of the SRP-PF as a possibility of teacher education (TE) in Spain. In our study, we use the expression "professional formation" precisely because of the additional theoretical contribution of Shulman (1986, 1987), Ball, Thames, and Phelps (2008), and Mishra and Koehler (2006). This contribution supported the conception of teaching knowledge as a constituent part of the praxeological equipment of the pre-service and in-service teacher, in this sense, characterizing the SRP as professional training, SRP-PF.

From a global point of view, the SRP-PF, developed with certain characteristics, specifically in the experimental phase, combined into a theoretical and methodological device that has potential for scientific research in the didactics of mathematics and the professional formation of pre-service teachers, and we add, as the possibility of applying this device in didactic microstrategies, that is, teaching strategies in classroom situations.

Fundamental theoretical aspects

Returning to the discussion about the process of evolution of the ATD, we understand that the various studies that have focused on this theory have also contributed to the consolidation, according to Chevallard (2009), of this didactic engineering: the *parcours d`étude et de recherche*, more popularly known as PER, in French, or *percurso de estudos e de pesquisa* (PEP), in Brazilian Portuguese, or even *recorrido de estudio e investigación* (REI), in Spanish. In our study, we adopted the English version, study and research path (SRP).

That said, we highlight some theoretical elements that, in our understanding, help explain the functioning of the SRP-PF/ATD device in the context of professional formation.[3]

Before the theoretical approach proposed in this text, it is worth highlighting the strategy, the way of writing the ATD, developed by Yves Chevallard in several texts. It is the use of mathematical symbols and algebraic representations to explain the theoretical structures presented in his theory. Chevallard (2020) emphasizes the importance of using and understanding this language and its functions to understand his theory better. In this sense, such language is also widely used in work on the SRP, and we will endeavor to justify and explain these symbologies.

Sierra and Gascón (2018) developed an SRP for teacher education on the numbering system with Spanish secondary school teachers, using a strategy for building a mathematical

[3]We kept the expression "formação profissional" [professional formation], in the version of the study conceived and presented at the 6th ATD International Congress, held from January 22 to 26, 2018, Autrans, France. SRP of professional formation rather than teacher education.

praxeology for teaching. This strategy consists, in general terms, of expanding a network of questions that arise as the experimentation process takes place. New questions that are part of the raison d'étre of the mathematical object of study that should be highlighted to the participants are included. Supported by an investigation developed by these authors, we construct and discuss what we call the *SRP-FP/ATD theoretical device*, which was the basis of our experimentation. The key notion of the SRP was the starting point of this investigation. The definition proposed by Chevallard (2009, p. 2, our translation) for the didactic system of engineering is as follows:

> X is a study collective (a class, a team of students, a team of researchers, a journalist, etc.), and Y is a team (usually shortened: Y can even be an empty set) of study aids and directors of study (professor, tutor, research advisor, editorial director, etc.). The purpose of building this didactic system is to study *Q*, that is, to try to provide an answer *R* that satisfies certain *a priori restrictions,* including putting it to the test by confronting the suitable *adidactic milieu x.* The summary of X's expected work under Y's guidance and supervision can be written as *S* (*X*, *Y*, *Q*) ➥ *R*.

In our investigation, we resumed the constituent elements of the didactic system *S* (*X*, *Y*, *Q*) ➥ *R*, where we define:

- X is the group of undergraduate degree students participating in the curricular supervised teaching practice, whom we call pre-service teachers. The professor responsible for the teaching practice could be included in this group, but in our research, this stakeholder is not a subject of analysis because he did not effectively participate in our experimental phase. However, we take into account some institutional restrictions that are related to the teaching practice professor's actions.
- Y is the set formulated by the first author of this chapter, who conducted and organized the study under the supervision of the second author. The leading investigator is the researcher, and while the teaching practice professor and the supervisor implement the study meetings and support the activities, the researcher is responsible for organizing and conducting the episodes.
- Q_0 is a generative issue, that is, it triggers the study and other issues;
- R is the answer to question Q_0, R must initially satisfy the set of constraints;

The didactic system *S* is the basis of the *study* organization. Under the ATD, the *didactic* is everything related to the study in development, encompassing the notions of teaching and learning typically used in the “pedagogical”[1] culture; the word “study” has a broad scope. The *study* is, therefore, everything that is carried out in a given institution aiming to give answers to questions or even carry out "problematic" tasks, if any. But the study activity should not be restricted or confined, as Sierra (2006) points out.

According to the author, the notion of study, in the case of mathematics, appears as an integrating notion that allows analyzing, from the same perspective, the work the mathematician carries out when investigating, the teacher when teaching mathematics, and the student when learning at school.

According to Sierra (2006), the study activity takes place in a community, which he calls study communities, when constructing new mathematics and teaching and learning known mathematics (situation of teachers and students). Such activity is carried out with the help of several study directors, the main researcher, and the professor, and is guided by a study program in the form of an inquiry program and curriculum. In this scheme, the professor appears as a director or one of the directors of a study community formed by them or their students.

ATD assumes that mathematical knowledge is constructed as a response to the study of problematic issues, thus appearing as a result, a product of a study process. This idea is strongly present in the development of the SRP.

In this sense, the SRP starts from a generative question Q_0 proposed within a didactic system $S(X;Y;Q)$, i.e., according to Chevallard (2009), the objective of building this didactic system is to study Q. Seeking to provide an answer R that satisfies certain *a priori* constraints includes testing by confronting the suitable *adidactic milieu*. Adidactic, because learners are unaware *of the professor's* intention. The expected summary of the work of X under the supervision of Y can be written, according to the author, as:$S\ (X, Y, Q) \rightarrow R$.

Chevallard (2009, p.1) calls this research domain *didactics of co-disciplinary research.* The author explains that this type of investigation rarely mobilizes a praxeological tool from a single discipline to produce the answer R; it usually precedes heterogenesis. It means that, with some exceptions, the research "works" with praxeological tools from several disciplines; therefore, it is a multidisciplinary approach.

The author also states that engaging in this investigation is equivalent to participating in the SRP motivated by this same investigation. To craft an answer$R$, it is necessary to gather and organize a working environment (medium) that contains old and new resources that X will use. These resources include some "ready-made" answers to Q_i, which are validated by the institution or other institutions. These answers will be those that supposedly received an institutional "seal" and are denoted by $R^{\lozenge}{}_i$, that is, they are answers that are justified and are part of a repertoire within the institution. The analysis of those answers will provide resources for the construction of the answer R, itself denoted as $R^{♥}$, by Chevallard (2009).

The author emphasizes that other works O from the culture will provide tools for the analysis of the answers $R^{◊}$ and the construction of the expected answer $R^{♥}$. The works of O will be, in part, results of various established disciplines, even if some of them come from disciplines that are not recognized because they are emerging or culturally deprecated.

In our view, within the scope of the theoretical device aimed at the education of mathematics teachers, the different elements of $O = \{O_1, O_2, \ldots O_n, O_{n+1} \ldots O_m\}$ may be those that emerge from the school culture, from the perspective of the teacher who teaches; for example, knowledge arising from technology and pedagogy, or even from the everyday knowledge of subjects, which do not always have the institutional stamp. It means to say that O_m may help as analytical tools for the answers $R^{◊}{}_1, R^{◊}{}_2, \ldots\ R^{◊}{}_n, \ldots$ of each question generated during the SRP, and all of this corroborates the expected answer $R^{♥}$.

According to Chevallard (2009), the detailed synthesis of the research work is part of what he called the Herbatian system, denoted in a condensed form by the representation: $(S(X; Y; Q) ➡ M)$ ➥$R^{♥}$, in which $S(X; Y; Q)$ represents the didactic system developed on the subjection of an environment M of conditions and restrictions. This system corroborates to arrive at the answer $R^{♥}$. Expanding the expression of its terms, we have:

$[S(X; Y; Q)$➡$\{ R^{◊}{}_1, R^{◊}{}_2, \ldots\ R^{◊}{}_n, O_{n+1}, \ldots O_m$➡ $M\}]$ ➥$R^{♥}$.

The notion of SRP, according to Chevallard (2009), will make it possible to include, in a broader context, a set of social practices of knowledge, more or less discrepant, whether scientific research or a police or journalistic investigation and others. In this sense, we include the processes of teaching and learning mathematics or other areas, such as physics and chemistry.

Within the scope of these social practices, mathematics education, as an area under construction, benefits from the didactic engineering methodology of the SRP, both in the scope of research and teaching and learning processes. The anthropological system of the ATD is at the heart of the constitution and analysis of the practices that are developed through the operation of the device *ATD/SRP-PF*, based on the mathematical praxeologies.

The author clarifies that

> [...] The school study is, however, what seems to serve modeling in terms of SRP the least; in fact, we can describe the traditional forms by saying that if it involves a survey

> on Q, this is done by the teacher and operates in another scene than the class (Chevallard, 2009, p. 2, our translation).

The SRP brings us the possibility of breaking with the traditional teaching model and ascending to another one, in which the student is responsible for their learning and resumes the process of scientific investigation. Apparently, it is in this sense that Chevallard (2009) explains in the sequence, that the *herbatien* scheme evokes a strong obligation of a democracy to fulfill, when each citizen or collective of citizens must be able to investigate any subject they like or are interested by using, in particular, the praxeological equipment that the basis of their school education provided them.

From our point of view, a supposed contradiction, in fact, is in favor of an emancipatory and democratic model, and, for this reason, it is difficult to implement the SRP in the school environment. However, several studies in France, Spain, and Brazil, for example, have managed to implement the SRP, which means that the difficulties are being overcome during the investigation process with this didactic engineering.

In the case of a study with prospective teachers, such as the *pre-service teachers*, the praxeological equipment, in the sense of Chevallard (2009), must have the knowledge acquired at school (basic education) and during initial teacher education (in the degree) incorporated into their repertoire. We hypothesize that its praxeological equipment can also be expanded, and the associated knowledge can be re-signified during the sessions (episodes) of the SRP. We are assuming that a SRP-PF will enable pre-service teachers to incorporate new knowledge and/or reframe the already existing in terms of teaching knowledge.

The SRP- PF, or professional formation SRP, indicates that we are including in the education of the *pre-service teachers* knowledge that, from our point of view, is basic for the teaching profession. Tardif (2012) states that professional knowledge is transmitted by institutions that form teachers in the education sciences. In this sense, the supervised teaching practice in the teaching degree should be one of those formative spaces.

We evoke specific knowledge necessary for the initial formation of the mathematics teacher that is much closer to knowledge related to the specific content of the subject, based on Shulman (1986; 1987), Ball, Thames, and Phelps (2008), Mishra and Koehler (2006), and Shulman (1987), called "general pedagogical content knowledge." At the same time, we seek to identify the place of pedagogical knowledge in the context of mathematics teacher education in Brazil. From our perspective, this body of knowledge is fundamental for bringing the general aspects of school activity and education as a whole, however, without covering aspects of the specificities of the teaching and learning processes specific to mathematics.

Shulman (1986) raises questions about the knowledge that the teacher mobilizes when acting in the teaching task. We believe these are pertinent questions for the in-service or the pre-service teacher.

Regarding the sources of teacher knowledge, for this research, we can state that the construct of teaching knowledge that corresponds to the knowledge of pre-service teachers sets off from their basic school to university education. They may have included in this knowledge those acquired in temporary teaching experiences (temporary work contracts in the public education system) in basic education classes.

Ball, Thames, and Phelps (2008) continue Shulman's study (1986) later through their own research. According to those authors, teaching knowledge constitutes domains of knowledge for teaching.

Such domains are divided into two categories: the first is related to knowledge or knowing of a subject, and the second, to the pedagogical knowledge of the content. The first category includes common content knowledge, horizon content knowledge, and specialized content knowledge. In the second category is knowledge of content and students, knowledge of content and curriculum, and, finally, content knowledge for teaching. The knowledge domains are summarized by the authors in Figure 1.

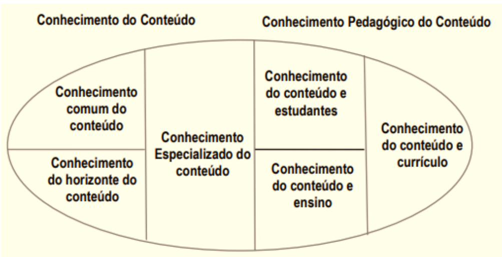


Figure 1.

Domains of knowledge for teaching (adapted from Ball, Thames, and Phelps, 2008, p. 404)

Another line of reasoning supported by Shulman's work appears from the idea of the intersection of knowledge in Mishra and Koehler's (2006) studies. These authors incorporate technological knowledge and its intersections with pedagogical and mathematical knowledge into the teacher's body of knowledge, for example, technological content knowledge. This same idea of “intersection knowledge” is evoked by Lima and Silva (2015), incorporating not only the pedagogical but the mathematical, technological, and didactic knowledge.

Although present in the ideas of other authors, this last category is stronger in our study. It is based on the concepts of didactics of mathematics from the French school. This intersection of knowledge is exemplified according to Silva and Lima (2015), who synthesize teaching knowledge into four blocks of knowledge: pedagogical knowledge (PK), content knowledge (CK), technological knowledge (TK), and didactic knowledge (DK). These authors also propose the intersection of these four blocks of knowledge, generating eleven more categories. Each of these eleven categories is a combination that involves at least two of the four presented (PK, CK, TK, and DK).

The DTPCK category, didactic technological pedagogical content knowledge, encompasses the four blocks of knowledge at the same time, which, we believe, would represent, *a priori*, a "more complete" knowledge in teacher education about a specific mathematical object to be taught.

With regard to the initial education of prospective teachers, in our study, we delimit, as a constituent part of the praxeological equipment of those prospective teachers, mathematical and didactic knowledge as being the center of professional education. This base can make it possible to add other important knowledge for such education, such as, for example, pedagogical and technological knowledge, and form new combinations of uncategorized knowings.

The teacher's pedagogical knowledge, we think, also integrates the knowledge of the theories of mathematics education and didactics of mathematics; however, in the initial education of the mathematics teacher in Brazil, mainly in mathematics degree courses in Bahia, the curriculum of the curricular components of education, such as, for example, the supervised teaching practice, still does not contemplate a consistent study of the theories of the didactics of mathematics. Even as a curricular component of the degree, it is often developed from a perspective of general didactics or general theories of education, a mistake in the curricular interpretation, in our analysis.

Basic elements of the ATD were present and developed in the repertoire of the pre-service teachers in the development of our SRP. Besides those elements, other aspects of the pedagogical knowledge went through the process of acquisition and/or re-signification by the individuals throughout the study.

Thus, for us, pedagogical knowledge *encompasses everything that refers to the theoretical knowledge of the didactics of mathematics, its elements, applications, or even everything that refers to the specific knowledge of the teaching and learning processes of mathematics, such as discussed in the scope of the didactics of mathematics*.

The mathematical knowledge at stake integrated knowledge related to plane analytical geometry (PAG) topics. In this sense, the work developed with the SRP was based on a general reference epistemological model (REM) for PAG, which we (re)delimited in an alternative reference epistemological model (AREM) for the study of PAG of the point, the line, and coordinate systems. This last model was configured with one of the several possibilities of research choice, and the REM included the plane analytical geometry, being, therefore, an organized source for new studies with PAG in the scope of the didactics of mathematics.

The REM (Figure 2) was built from the analysis of the epistemological dimension and it served as a reference in studies of the economic and ecological dimensions, as well as in the construction of the dominant epistemological model (DEM) and the alternative reference model (ARM) (support base of the SRP- PF).

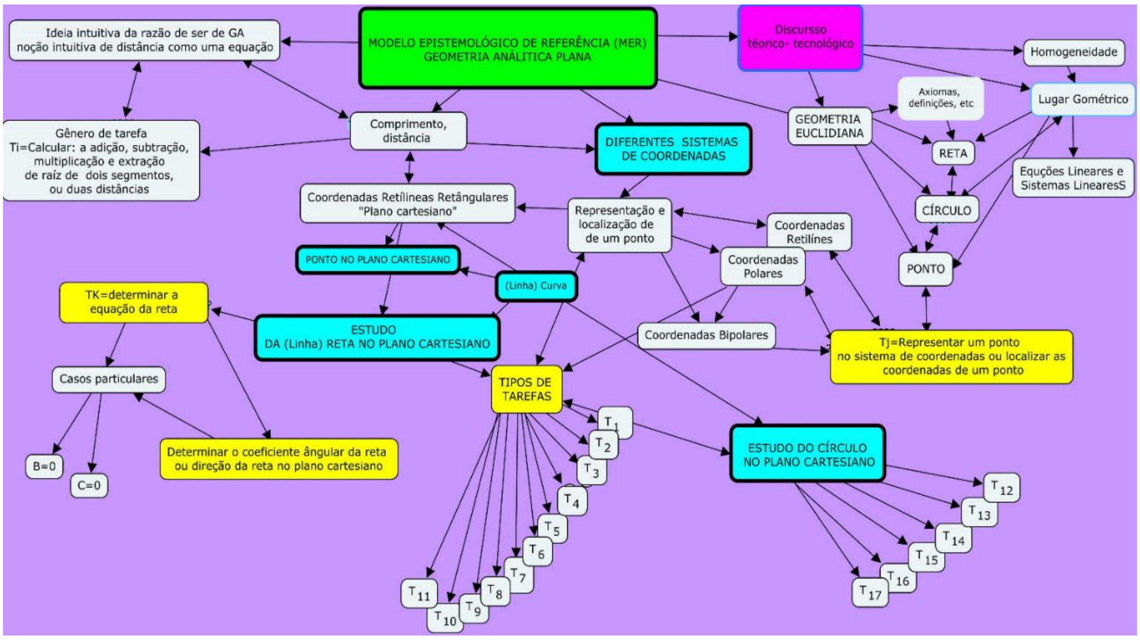


Figure 2.

Illustration of the reference epistemological model (REM) (Freitas, 2019, p. 161)

The abbreviations generically represented by T_i (1≤ i ≤17) indicate the tasks linked to some dimensions of the REM[4].

The study of the economic and ecological dimensions of the didactic problem, based on the REM, allowed identifying the dominant epistemological model (DEM) for PAG teaching in different institutions (primary, secondary, and university) (Figure 3).

[4]Cf. List of tasks in the appendix

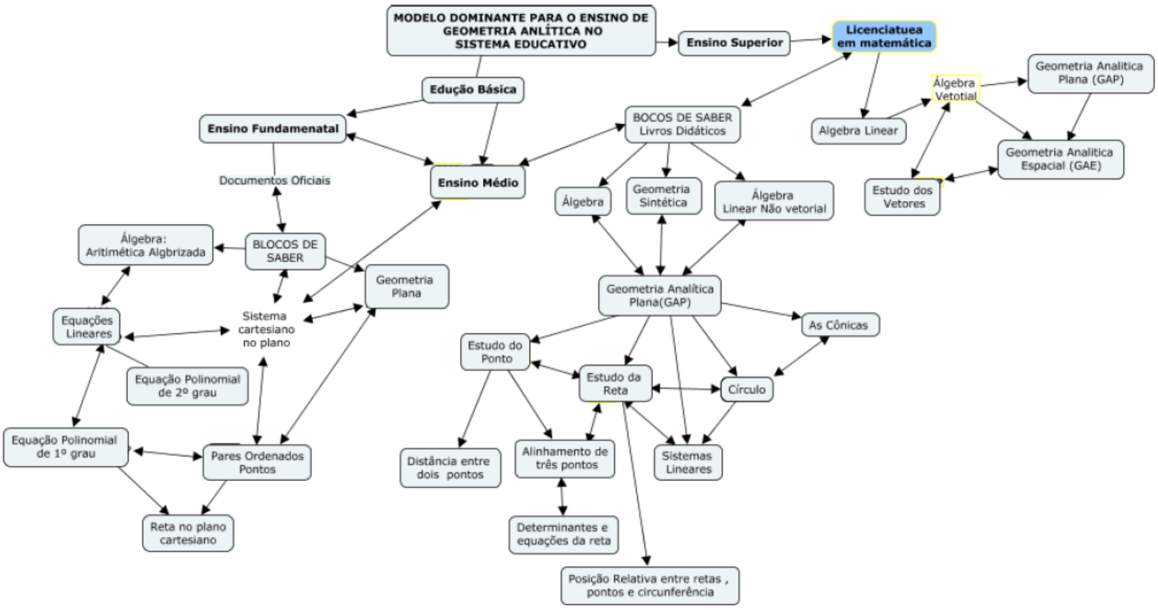


Figure 3.

Dominant epistemological model (DEM) in the Brazilian educational system (Freitas, 2019, p. 232)

In coherence with the REM and DEM constructed, and based on them, we created an alternative reference epistemological model (AREM) (Figure 4) of what it means to «teach and learn» mathematical knowledge in that field. The REM that was built served as support for developing the SRP- PF experimental phase.

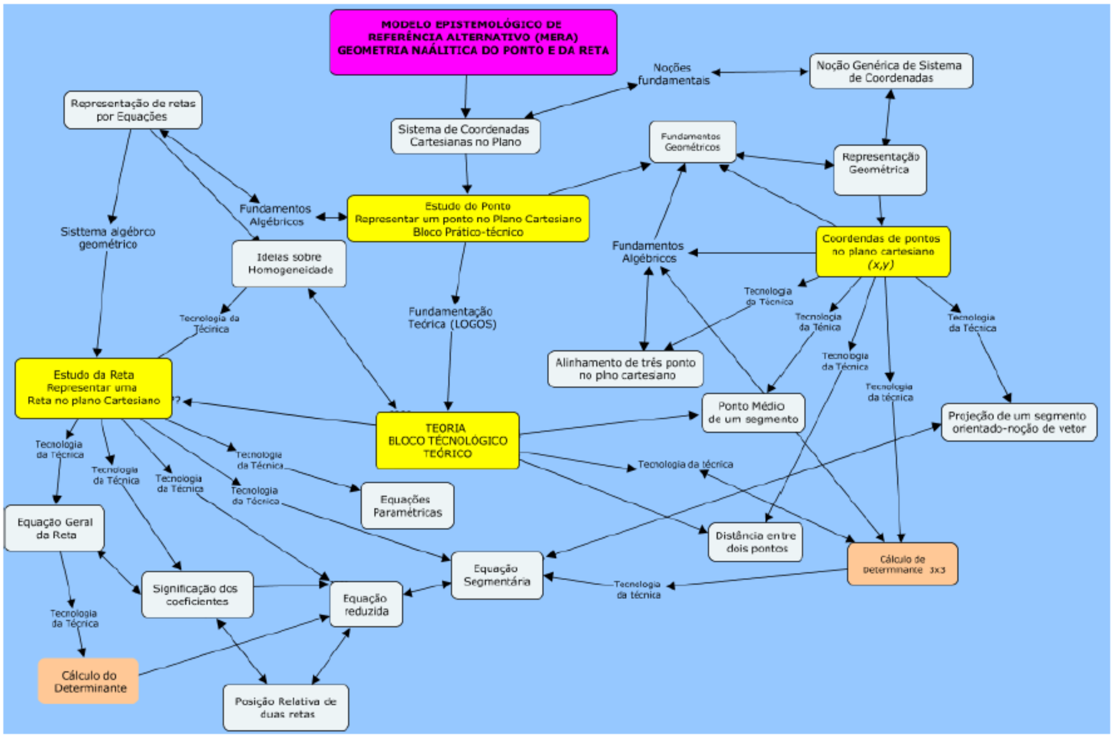


Figure 4.

Alternative praxeological reference model (APRM) (Freitas (2019, p. 237)

The theoretical aspects and ideas, as well as the considerations raised about mathematics teachers' education in Brazil, were fundamental for the constitution of the SRP since it is necessary to know the conditions and restrictions of diffusion of the knowledge inherent to teachers' education, as well as the *milieu* of work.

Didactic moments of the study episodes

To experiment and develop the SRP, we resorted to the assumptions of traditional didactic engineering, according to Artigue (1998), to realize a priori the design of the study episodes, safeguarding certain adaptations.

The organization of the study episodes had as internal foundations the study moments or didactic moments. According to Chevallard (1999), as in every praxeological organization, a didactic organization is articulated in types of tasks, usually cooperative, in technologies and theories. The questions that arise within the scope of this research are: How to describe this organization? What are the main types of tasks?

Chevallard (1999, p. 19) asserts that the term "didactic moments" refers to a temporal structure of the study process only in appearance. It means to say that the meaning given to this word refers to a dimension in a multidimensional space, a factor in a multifactorial process. The author also explains that good study management requires that each of the *didactic moments* be carried out at the appropriate time, as a *moment of study* it usually occurs several times in the form of a multiplicity of episodes distributed over time. The *didactic moments* are a reality of the functioning of the study and not a chronological reality, i.e., the order of the different *moments* is arbitrary. They are six moments, summarized according to Chevallard (1999).

The *first moment* corresponds to the first contact of the individuals with the mathematical organization, at least, with a type of task of this organization. The second moment is for the exploration of the type of tasks and development of the technique, the *exploratory moment*.

The third moment is the constitution of the *technological-theoretical* [q/Q] environment related to the technique τ_i. It is usually the moment of close interrelationship with the other moments. In the case of the first encounter with a type of task, there is usually a relationship with the previously developed technological-theoretical environment.

The fourth moment is the work *of the technique,* which should at the same time improve it, making it more efficient and reliable. The fifth moment is *institutionalization,* and aims to clarify what exactly the mathematical organization has developed, and distinguish which elements will definitely be part of the organization and which will not be integrated into it.

Finally, the sixth moment is *assessment*. It is a moment of reflection and is articulated with institutionalization (in some respects, it can be considered a sub-moment) in which what has been learned is evaluated. Mainly, it is the moment when one appreciates the control or mastery of the mathematical organization created, and still evaluates the mathematical organization and its validity.

Each of these moments can be carried out several times, not only because they proceed from episodes of time, but also because, for example, an episode of work on a technique can lead to retouching the established mathematical organization and, therefore, eventually carry out, or live, a new technological episode; in any case, consider another episode of *institutionalization.*

In general terms, the moments of study constitute a grid for the analysis of didactic processes, and in this sense, they are incorporated into the a priori and a posteriori analytical process of the development of the study process with its episodes.

The first step for the constitution of the study episodes is identifying a generative question. The initial question guides and motivates the study process, and it must be able to generate other questions that, at the end of the study process, produce a praxeology developed with the pre-service teachers (in the case of our research), based on the *AREM.*

According to Sierra (2006), the study process is driven by this generative question, although what is derived from it is not, a priori, fully determined, bearing in mind that many decisions will have to be made to carry out the SRP that this question implies. The determination of Q_0 goes through some questions related to the restrictions already identified in a pilot study (also carried out with pre-service teachers) and in the dominant models. Sierra (2006) promotes reflections on this choice.

According to this author, the study processes should establish a relatively long time, since mathematical work will require developing mathematical organization within the scope of several successive sessions to allow individuals to deepen and work in a useful and efficient way. Consequently, we consider that a study process must contain a broad mathematical object and one should not propose isolated study processes.

From this perspective, measuring the content in terms of study time was essential, given the institutional restrictions we found in the institution where the study was carried out. We delimited the study around the fundamental objects of AG (the study of the point and the line and the coordinate systems). The relatively sufficient time for developing the proposal was about four months.

As described in our alternative model *AREM* or simply M_A, the question we proposed was aimed to, first, develop mathematical organizations around the objects we defined for the study. We first formulated the question Q_0 = *How can we teach the analytic geometry of the point and the line?*

This question was the driving force behind the study of the analytical geometry of points and lines. We denote by the acronym GAP_{PR} the analytical geometry in the point and line plane and, to avoid repetitions in the text, we use the following terminologies and abbreviations:

- Q_0: initial question (generative question)
- OM_{sc}: mathematical organization of the coordinate system
- OM_P: mathematical organization of the point
- OM_R: mathematical organization of the line
- GAP_{PR}: mathematical organization analytic geometry of the point and the line

The engendered work of specific mathematical organizations generated from Q_0, will allow the construction of the desired mathematical organization GAP_{PR}, as illustrated:

$$Q_0 \Rightarrow OM_{sc} \Rightarrow OM_P \Rightarrow OM_R \Rightarrow GAP_{PR}.$$

We rely on Chevallard (2009) to state that the (mathematical) works developed from this type of question in a SRP may be characterized by an intramathematical infrastructure. According to the author, it is like some mathematical works, which are designed to allow the work of engineers, chemists, or biologists. The structure of this work led to the study, on the one hand, of many problems, most *intramathematical*. On the other hand, the research device must bring up other problems related to teaching processes and of learning a mathematical object, a technique, or a task about that object. For example, how to teach the equation of the line? These problemsgenerate questions about the teaching process, the teacher's task, the didactic transposition.

Let us give another example: during the development of a mathematical task, the research device should bring up questions for pre-service teachers about how to develop the same task for their future high school students. It should lead them to question: Is the task suitable? Why? What is the best technique? What approach to technique facilitates student learning? What theory does the student need to know to develop the technique? What learning difficulties can be generated from the choice of a type of didactic transposition?

In fact, such issues come from (didactic) problems that raise in pre-service teachers the questioning of didactic transposition. We call these problems *intradidactic,* because they are within the didactic dynamics of teaching. About the didactic problem posed as the generating question of the study Q_0, (how to teach AG...), its development within the framework of the research device will make the knowledge around PAG aimed at the didactic transposition in high school emerge and (re)signify.

It is the research device itself, as it is engendered, that brings to life, within the scope of the work carried out with mathematical and didactic organizations, teaching knowledge: knowledge of the content, in the case of the PAG, didactic knowledge of the content, technological didactic knowledge of the content, didactic and pedagogical knowledge of the content and, finally, the didactic technological pedagogical knowledge of the content.

We chose to use a more synthetic representation of these categories, which we chose, first, from a symbolic representation, or an abbreviation that respectively corresponds to each category of knowledge: knowledge of the PAG $=C_{GAP}$, pedagogical knowledge of the PAG $=CD_{GAP}$, pedagogical-didactic knowledge of the PAG = CDP_{GAP} and technological pedagogical didactic knowledge of the PAG $=CDPT_{GAP}$.

Taking into account this set of knowledge, defined in the research device as categories composed in the subject's cognitive instance, we assume that they are knowledge that is in accordance with the relations between the *subjects* and the *objects*, according to Chevallard (2007a).

It means that there is a *personal relationship*, $R(x, o)$, between the teacher (or prospective teacher) and an object (or set of AG objects, for example), and an *institutional relationship* of the subject within an institution $R_I(p, o)$ with an object, which therefore composes the *teacher's praxeological equipment* $EP(x)$ and relations relating to the affairs of the institution I in position p inside it $EP_I\,(p)$.. This articulation is supported by Chevallard (2007a, p.11) when he states that:

> There was talk of personal and institutional relationships *R* (*x*, *o*) and *RI*(*p*, *o*) to an object *o*. One can now speak of a person's set of praxeologies$x$, what we call (gentlemanly, the praxeological equipment x, and what I denote *EP* (*x*), and, evidently, those relating to the affairs of an institution *I* in position p inside it, *EPI* (*p*). The relationship of *x to o* is then somehow the "cut" of *EP* (*x*) by the object *o.* It will be constituted from praxeologies that the *EP* (*x*) activates *o* in one way or another –for example, on the technical plane, in the technological register, etc. The praxeological theory takes over and thus extends the theory of relations, to which one can always return if necessary. (our translation)

2

From the identified theoretical elements, with the contribution of the ATD, we propose articulating the elements of professional teacher formation, in the case of our research, with the defined a priori categories. So, let us go to the a priori design of the experimentation.

The traditional a priori analysis proposes a precise detailing, the most accurate as possible, of the set of behaviors that individuals may have a priori, before the knowledge at stake, through situations. In the case of the SRP- PF, we designate the a priori design, as it is more or less open, even though the traditional a priori analysis means a more or less closed SRP. In other words, the search mechanism proposed in our study needs some level of control actions, unlike the initial meaning given by Chevallard (2009) for the completely open SRP. For each study episode, we predicted a time of around three class hours of work sessions with the pre-service teachers.

It is noteworthy that, although the SRP is not pre-established, the a priori analysis aims to study the generating power ofQ_0, by an open and comprehensive process (*chronogenesis)*, by which questions are engendered from the initial question (Q_0), then the intermediate answers, and so the cycle starts again with new questions until we reach an answer to the generating question (Q_0) ($Q_0 \curvearrowright R^{\heartsuit}$)*,* as we will illustrate later in the text.

We illustrate each study episode through question maps and/or schemes, which aim to identify each study stage. The questions generated from Q_0 they will produce intermediate answers, raising new questions that feed the inquiry process. The dialectic of questions and answers contributes to self-feeding the process, designated by Chevallard (2009) of *chronogenesis.*

The answers obtained need to be tested with empirical data and available knowledge to be integrated into the environment. This integration leads to new questions Q_i, *(mesogenesis*). The author calls this dialectic "*medium-media*". The medium evolves during the inquiry process, and the dynamics are described in terms of dialectics (Chevallard, 2003). A didactic contract of responsibility sharing permeates the work dynamics between the subjects, that is, the subjects involved in the inquiry process share the responsibility through planning, raising questions, organizing and prioritizing reports and validating them, reaching an agreement to give a final answer and defend it (*topogenesis*).

The notion of media (Chevallard, 2007b) means any representation system of a part of the natural or social world addressed to a specific audience, for example, a course of a mathematics teacher, a chemistry treatise, a site on the internet, a newspaper from a television presenter, etc.

Experimentation: description and a priori and a posteriori analysis of the study episodes

In the experimentation stage of the research, entitled "*Didactic modeling workshop on PAG,"* 28 pre-service teachers initially participated. During the course of the project, 14 continued until the end of the workshop.

The main objective of the project as a research procedure was to match the answer to the general research question: *What knowledge can pre-service teachers working with plane analytic geometry acquire with the help of a study and research path for teacher education (SRP-FP)? What benefits can they obtain to project (keep) this knowledge in high school?*

The engineering of the SRP-PF, based on its assumptions, contributed to the construction of a set of answers to the research question raised, which should not be confused with the generating question of the SRP Q_0: *How can we teach the analytic geometry of the point and the line?*

We selected some study episodes, although just a part of the experimentation stage, considering that, in all, there were thirteen work sessions. Initially, we had planned ten sessions. In this sense, the a priori and a posteriori description and analyses were a little extensive, due to the necessary detail as research data. Thus, we will bring part of the experimentation to comply with our object: to explain how the SRP- PF/ATD device operates.

In the first episode of study, students should have their first contact with mathematical organization (MO) and didactic organization (DO) around initial questions that would be proposed: Q_0: *How to teach the analytic geometry of the point and the line?* Q_1^D*: Whom to teach AG?* Q_2^D*: Where to teach AG?* Q_1^M*: What is AG?*Q_2^M*: In general, what differentiates AGfrom other geometries?*Q_3^M: What does AG study? Q_1^{DM}*: What content of AG should I teach in basic education?*

We predicted firstly that students could take two paths in the construction of the study: questioning within a purely didactic-pedagogical dimension or in another didactic-mathematical dimension, but still in separate ways. The proposed initial questions would be simple, and would guide the beginning of the study process. When choosing the questions, we took into account the two dimensions: mathematics (M) and didactics (D).

This first episode had to fulfill two functions: to hold the first meeting with the *reason for being* (*raison d'être*) of mathematical organization, plane analytical geometry of point and line (GAP_{PR}) that was intended to be reconstructed with some of the questions that the MO should answer, and at the same time, experience a first encounter with the didactic organization (DO) that would structure the study process. The first aspect we highlight is the interrelationship between the mathematician, the activity to be carried out and the didactic, how pre-service teachers propose to organize such activity.

As part of the study, as first planned, the students were given the task of addressing the initial question and others that could arise, based on their prior knowledge, in which the following representations were adopted: Q_i^D corresponds to questions of a didactic nature (D), i is the variation of the different questions, Q are questions that would be progressively being generated ($i = 0,1,2,3$) at every study session. Following this idea in Q_i^M, they are questions of a mathematical or intramathematical nature (M), that is, they evoke or mobilize this knowledge. In case of Q_i^D questions of a didactic nature, i.e., they evoke or mobilize this type of knowledge, and finally, Q_i^{DM} evokes mathematical and didactic knowledge in an articulated way (DM).

We projected that students would seek help from digital media, such as computers, cell phones, or *tablets* at hand. They could also seek traditional media such as books in the library or help from the researcher (who shares the role of teacher and study conductor). This dialectic *milieu-media* would allow the subjects to find intermediate answers, generated by Q_0, for the first session, $R^{\lozenge}{}_A = \{ R_1^D;R_2^D;\ R_1^M;\ R_2^M;R_3^M;\ Q_1^{DM}\}$. The acronym $R^{\lozenge}{}_A$ corresponds to the set of intermediate answers to the set of questions in session (A), respectively given by R_i^D answers of a didactic nature (D), and i varies as the questions posed for the session (A) vary. These nomenclatures were progressively adapted to section A, B, C, etc.

We reinforce what we have already highlighted: the SRP cannot be predetermined, however, we anticipated likely directions or paths the study could take the students, considering that the initial questions are not part of a difficult *milieu*.

We assume, therefore, that the students knew the basic education curriculum or at least knew where to find it. They should search in those documents or straight away on the internet for other sources on the topic, or they could ask the researcher for help to indicate research sources, such as textbooks and others, to provide the answers R_1^D, R_2^D andR_3^{DM}. The context of these answers clearly involves an interrelationship between the mathematical, around the objects, and the didactic, more precisely the elements of the curriculum. Studying the issues should lead pre-service teachers to focus on the curriculum around AG. We emphasize that regarding teaching knowledge, the knowledge of content curriculum is defined in the framework of Ball, Thames, and Phelps (2008). However, for our research purposes, we included this definition in the category of mathematical didactic pedagogical knowledge ($CDPM_{GAP}$).

The answers (R_1^D and R_2^D) could be summarized, in general terms, as follows: the basic education curriculum provides for the teaching of notions of plane analytic geometry in the 7th and 8th grades, PAG in the 3rd year of high school, and AG is also taught in courses the exact sciences, such as mathematics, engineering, physics, etc.

The questions Q_1^M and Q_2^M, having as answers aR_1^M and R_2^M, could present the first notions about AG. Such notions could be found on the internet, or in other media, as we have already explained. The pre-service teachers could also bring a historical perspective of AG (works by Fermat and Descartes) as answers; or a perspective of the AG subjects they took during the degree, or that of the high school textbook, etc. They may identify as an answerR_2^M (to the question Q_2^M) the differentiation of geometries from the incorporation of a coordinate system. In any case, the objects, point, line, coordinate system, and oriented segment, among others, such as conics and vectors a priori would appear in the set of answers R_3^M.

Although it seems to have a straightforward answer: "point and line" -since it is evident in the initial question when submitted to the research device- question Q_1^{DM} could make emerge, in the discoveries of the pre-service teachers, the MO and the DO around the objects point and line. The coordinate system could appear as an object, through which a point and line can be studied and explored.

Firstly, the functioning of the device should allow the pre-service teachers to validate, through the evaluation in small groups and the collective, the intermediate answers. In this way, they would (re)construct, and mobilize intermediate answers $R^{\lozenge}{}_{ij}$ to integrate them into the *milieu*, and thus continue the study process with a view to finding $R^{\heartsuit}$.

For the collective socialization of constructions, the pre-service teachers could use all available resources, both technological, such as datashow, and mental maps etc. Such guidelines will be provided throughout the work. Each group would expose their answers to the entire collective involved. In this process, the messages sent are analysed, evaluated, and classified, justifying the advantages and disadvantages, and the professor-researcher can summarize the results obtained together with the students.

The moments of collective socializations in this first episode should happen in two steps: one step for the answers to R_1^D; R_2^D; R_1^M; R_2^M and a second moment for the collective to discuss how it would build the answer to Q_1^{DM}. At the end of this socialization, the pre-service teachers would be instructed on the work report developed in this episode.

The report could be built digitally and posted on the virtual learning environment (VLE) of the *Moodle*. What was expected is that it would be a collective construction of the class, which would help in the final report of the workshop. To this end, students could be articulated with the responsibility of sharing tasks. In this sense, organize themselves to continue the study in order to answer the question Q_1^{DM} that would guide the next episode. We summarized the a priori path as a possible development for $Q_0{}^{(\mathbf{A})}$, with A indicating the study of the first episode: defining analytical geometry, the first round of experimental work, schematized in Figure 5.

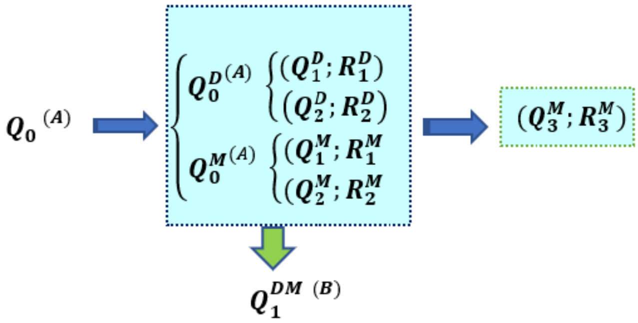


Figure 5.

Chronogenesis of "A", possible development $Q_0{}^{(A)}$ (Freitas, 2019, p. 263)

Analyzing a posteriori the first episode, most of the planned actions were developed: first, we present the initial questionQ_0: *How to teach the analytic geometry of the point and the line?* After presenting the question, the students began to discuss in groups how to answer this question. Seven groups, composed of four students each, participated in the work. Unanimously, all groups responded that it would be necessary to deepen and develop some type of research to answer that question.

We explained that question Q_0 would guide all subsequent work sessions, which would be developed along the formative path planned for the workshop, and that the answer to the question would be built along the path. We discussed the importance of studying and doing scientific work guided by questioning, or questions in the sense of the ATD, of the paradigm of questioning the world. Soon after, we proposed other questions based on the elements that make up Q_0. In this sense, we suggested three more questions aimed at elucidating what would be behind Q_0: Whom to teach AG? Where to teach AG? What is AG?

The students resumed the discussion work in the group, now around the new proposed questions. We highlight the answers of four of the seven groups, considering that, of the seven, only four handed out the written report of that moment. Each group should provide the audio of the team's work, via the WhatsApp mobile application, and the group's written report of the conclusions or reflections. For that, each group had an audio monitor and a reporter for the written part. The idea was that at each work session of the functions were exchanged, so that all group members could collaborate in the production of material (and data).

Regarding each group's production, besides the general description of the work with the subjects' speeches, we highlighted the information we deemed significant for the research and the composition of the analyses, leaving the other data available for future research. As the groups were not composed of the same students in each session, they were reorganized according to the participants' attendance. For the evaluation of the SRP, we take into account the production of the participating collective as a whole.

Later, we highlighted the work of Group A, which started the discussion by questioning colleagues about AG in higher education, postgraduation and basic education. They searched the internet for answers, especially what AG is, and discussed about the difference between AG and Euclidean geometry, as predicted. One part of the group defended that: what differentiates the two geometries is the presence of algebra, and the other part of the group defended that it is the emergence of Cartesian geometry. The group uses internet research to identify what AG is, and in this research they found the following statements: “connection between geometry and algebra; junction of Euclidean geometry with algebra”.

Regarding the construction of the answers, this group stated that AG should be taught in basic and higher education, in public and private education. One of the members of the group stated that in the private network this teaching is more in-depth. Another member of the group thought that the discussion would not be relevant to the proposed question, as it makes a comparison between public and private education.

Next, they sought to construct the answer to the second question: What is AG? One of the group members reports that he found that AG is the connection between geometry and algebra. The student stated: *With my students, this was the explanation I gave for what AG is*. Then another member of the group stated that it was the junction of Cartesian geometry and algebra, and they asked each other: *Is geometry a branch of mathematics?* Another student replied that yes, that geometry, as well as algebra, is a branch of mathematics. The group, as expected, sought a document that would support the answers, for example, why AG should be taught in the third year of high school. He stated, therefore, that AG is taught in the third year according to the textbook and the National Curriculum Parameters (PCN). One of the colleagues asked:

> Why analytics? What is analytics? Someone in the group replies: analytics comes from analysis... Detailed; now, detailing, what is it? Cartesian plane? Someone replies: analytics comes from analysis, analysis of the Cartesian plane through pole algebra: the Cartesian pole? Analytical geometry is what? The branch of geometry that establishes the relationship between algebra and geometry...

In the context of this study, as we predicted, the moment of the first encounter with the raison d'être of AG occured, more especially, with the questions about what this geometry is. The pre-service teachers also mobilized their knowledge about the school curriculum, in order to identify the appropriate stage to teach analytical geometry, and one of the groups consulted the BNCC, and identified some issues related to AG that are still studied in elementary school and in high school, for example.

> - And we found that, in the 6th grade, students acquire basic knowledge of the Cartesian plane - what the Cartesian plane is, what ordered pairs in the Cartesian plane are, what quadrants in the Cartesian plane are - and [the issue] will be deepened a little more in the 8th grade with the demarcation of ordered pairs on the plane, knowledge of geographic maps, and Cartesian coordinates. But it is in the 3rd grade of high school that analytical geometry takes shape, with the study of the point, the distance between two points, the midpoint of a line segment, the condition of alignment of three points, the study of the line, the slope of a line, angular coefficient of a line, the fundamental equation of a line, relative positions of two lines in the plane, distance from a point to a line, area of a triangular region, the study of the circumference, the study of conics, parabola, ellipse, and hyperbola (Transcript of Group B).

Globally, the answers presented by each group were similar, but only group B identified the subjects studied in AG in detail. All groups stated that there is a "connection" between geometry and algebra in AG, and Group C related the orthogonal Cartesian system as the coordinate plane.

> Analytical geometry is the part of mathematics that involves geometry through an algebraic approach, based on the coordinate plane. Thus, equations related to a locus in the coordinate plane (GROUP C) were constructed.

After the group work time, we proposed a sharing and socialization of the answers. We called it sharing, because all the constructions could be shared between the groups to socialize the collective findings.

In episode 2, we resumed the discussion of the previous episode regarding the question $Q_1^{DM\,(B)}$: *What AG content should I teach in basic education?* We assumed that students would bring some suggestions for the answers; however, given the restrictions regarding mathematical knowledge that we identified in the pilot study carried out, the most likely would be that they would present study topics such as *the Cartesian coordinate system, the point and the line*, without exploring and presenting mathematical organizations. Or they could also present, based on textbooks, an MO, for the representation of a point in a Cartesian coordinate system, alignment between three points, the definition and equation of distance, the general equation of the line, that is, those elements present in school textbooks. A third possibility was for the pre-service teachers to construct a lesson plan with the stages of each step and didactic moments, in the general sense of the term, with or without the mathematical organizations that would be taught initially to their future students.

From the point of view of the training proposal that we aim at in this SRP-PF/ATD research device, the conducting of the study by the research professor should allow, within the scope of the search for the answer, to develop and introduce elements that would enable (re)composing the praxeological equipment of the subjects, the pre-service teachers, reconstructing the knowledge at stake from an alternative perspective to the current (dominant) epistemological models.

In this sense, the "questions" that would guide the study needed to be incorporated into these elements of recomposition. Such elements are not only the mathematical objects under study, they are also theoretical elements of the ATD, which, in this case, are still constituted as part of what we designate as didactic knowledge, for example, the notions of *task, technique*, *technology* and *theory*, or even elements such as the construct of the "*register of semiotic representation" (*DUVAL, 1995), the difference between the *notion of the object and its representation.*

We highlighted, still from a didactic, mathematical, and pedagogical point of view, the importance of the exploratory moment around the genesis of coordinate systems, as they are the basis of AG.

In this sense, an exploratory study of *techniques* used to deduce, organize, or produce the coordinate systems that, in this case, are closest to a *technology* (from a technological-theoretical block), bring out the raison d'être of these systems, even without dealing with the problem of Papus, developed by Descartes, according to Ian Maire (1637).

Figure 3 illustrates the chronogenesis of the development of sessions B and C, showing the functioning of the device engendered by the tasks within the scope of the generative questions referring to the mathematical organizations of the coordinate systems.

Thus, the questions that unfold from $Q_1^{DM\,(B)}$ (Figure 3) are based on coordinate systems, as generators of the study of the point. Therefore, $Q_1^{DM\,(B)}$ should generate a question ($Q_{1,1}^{DM\,(B)}$), which, in turn, generates a provisional answer $R_{1,1}^{DM\,(B)}$, which will generate a new question$Q_2^{DM\,(B)}$, which, in turn, will generate a new answer $R_2^{DM\,(B)}$. The cycle starts again with new questions $Q_{2,1}^{DM\,(B)}$, $Q_{2,2}^{DM\,(B)}$, $Q_{2,3}^{DM\,(B)}$, that will generate new answers, respectively $R_{2,1}^{DM\,(B)}$, $R_{2,2}^{DM\,(B)}$ and $R_{2,3}^{DM\,(B)}$, and the cycle starts again, in a new study session, i.e., this functioning of the device is cyclical, it imposes itself for the construction of the following stages of the study, as described in the question map, according to Figure 6.

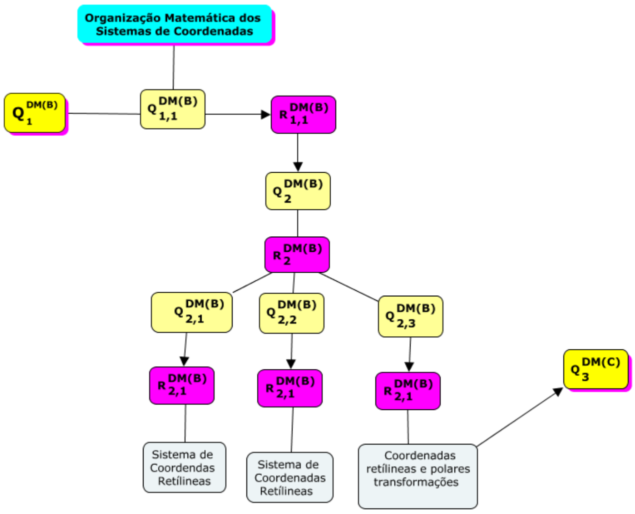


Figure 6.

Map of SRP-PF/ATD device issues, session B and C (Freitas, 2019, p.266)

Every question $Q_{i,j}$ and their respective answers $R_{i,j}$ involve elements of the mathematical and didactic organization under study, engendered to give rise to new stages. The answers are classified as provisional compared to the final one, which will be the set of answers from each study session or episode.

In this second episode, we propose the study from an excerpt of the historical document analyzed in the REM, regarding the different coordinate systems. The objective would be to identify the main differences between the coordinate systems and their generic idea as a mathematical organization, confronting them with the praxeological equipment of the pre-service teachers about this theme. The second objective would be to experience the moment of the first meeting and the exploratory moment of the MO, "coordinate systems." The study of the coordinate systems inevitably includes the representation of the point, then the line. In this sense, the next stages of the study should, first, go in that direction.

The study of the clipping of this historical document would explore the techniques for representation and location of a point, in the rectilinear and polar system. In this sense, this second study and research activity (itAER_{Q2}) would have two stages: the first is studying the document, students' first action; the second, exploring the document, focusing on possible developed techniques. Here we begin to introduce the basic notions of the ATD, mainly the idea of praxeology, the notions of *task*, *technique* and *technology*. In this activity, the notion of technique would be handled; at other times, it would be resumed and the other notions introduced to the other stages of the study.

For the second part of the exploratory study of the document, students would be given tasks that are also study questions $Q_{2,1}^{DM\,(B)}$: Which technique is used to derive the rectilinear coordinate system? $Q_{2,2}^{DM\,(B)}$: Which technique is used to deduce the rectangular rectilinear coordinate system? $Q_{2,3}^{DM\,(B)}$: How to represent the same point in the rectilinear and polar coordinate system?

At first, we expected the subjects to reconstruct the route and the technique used by the authors of the work (Briot & Bouquet, 1860), justifying each passage.

As we have already pointed out, what we are calling "technique" in the tasks described by questions $Q_{2,1}^{DM\,(B)}$, $Q_{2,2}^{DM\,(B)}$, and $Q_{2,3}^{DM\,(B)}$, is the technological-theoretical discourse used by the cited authors in the constitution of the coordinate systems.

It is worth noting that when conducting the study, the researcher would provide some commands, that is, the students could retrace the path, the mathematical route developed by the author, undertaking a new technique as long as they justified each stage of construction. The answers $R_{2,1}^{DM\,(B)}$, $R_{2,2}^{DM\,(B)}$, and $R_{2,3}^{DM\,(B)}$ for the proposed questions should induce the continuity of the study around the representation of points through rectangular rectilinear coordinates. At the end of this second part, students would validate and evaluate what they have built in the moment of group socialization and collective discussion.

After that, we point out that it was necessary to resume the initial questions of the study from the previous session for socialization and sharing of answers, considering that: what we think in the a priori analysis was not achieved in the first study episode.

The groups shared their proposals, explaining the written report, and then met to discuss the last question of the previous session: *In general, how does AG differ from other geometries?*

After group discussions and sharing of answers, students showed that they had expanded a little more their view of the raison d'être of AG when they related the coordinate system as an integral part of this geometry. Whe sharing, the students claimes that despite having already attended two analytical geometry courses, at no time did they reflect on the raison d'être of AG. In other words, what is AG?

The moment of sharing also showed that the answers given -such as, for example, connection between geometry and algebra, geometry of the coordinate system- did not represent a final answer as, according to the respondents' speeches, more information should be sought for a conclusive answer.

We emphasize that the entire study and research process, based on the proposed questions, was guided by professional teaching knowledge, regarding knowledge of the curriculum. The students showed that they knew which sources they should seek for the basic education curriculum, in this case, the National Curricular Parameters (PCN) and the National Common Curricular Base (BNCC), and, from them, indicate the suitable teaching stage for the AG studies. However, they held long discussions about the actual teaching stage in which students begin to see some notion of AG. While some argued that it was only in high school, others defended that it began with the Cartesian plan in elementary school. Regarding the study guided by the questions: $Q_{2,1}^{DM\,(B)}$ = Which technique is used to derive the rectilinear coordinate system? $Q_{2,2}^{DM\,(B)}$= Which technique is used to deduce the rectangular rectilinear coordinate system? $Q_{2,3}^{DM\,(B)}$ = How do you represent the same point in the rectilinear and polar coordinate systems? –the three groups gathered around the study, discussing the proposed material among themselves. A question was raised, inquiring about the definition of the term *technique*. The professor-researcher intervened at that moment to clarify the concept of technique according to the ATD and explain the importance of identifying it in a mathematical task. Furthermore, the notions of *task*, *technique* and other theoretical elements of the ATD, *technology, and theory* were communicated. Then the participants resumed the discussion in groups for finalization.

The third episode concluded the second episode. The students again came together to study and discuss the background material. Thus, we do not apply the proposal designed for the third episode at this time.

Before all else, we predicted that the students would look for textbooks, but this did not happen, as they looked for complementary references on the internet. However, because the text was translated from part of past originals, students claimed that they still did not know the approach and that they did not find something similar. They said that the study was something new for them and they unfamiliar with this historical perspective, even linked to the AG components they had already studied.

The groups, assisted by the support material, delved into the task of understanding the conceptions of the different coordinate systems –rectilinear, rectangular rectilinear, polar, and bipolar– and the generic idea for any coordinate system. Students also correlated the current Cartesian system and the rectilinear and rectangular rectilinear system.

During the group discussions, we identified that, during the study process, students tried to redo the technical steps of each coordinate system, validating the process developed by the authors Briot and Bouquet (1860) from the provided text. In some moments, they asked the professor-researcher to help them solve doubts, i.e., validate or not what they had understood about the mathematical, geometric procedures that underlie the representation, for example, of a point. As we anticipated, it was the exploratory moment during the study and discussion of the material provided.

We add that, according to each group participants' speeches while working, we observed that the underlying geometric notions of the techniques used to organize the coordinate systems were mobilized *during the* discussions, constituting the *theoretical technological environment*. They moblized notions of perpendicularity and projection of segments in the plane, of orthogonality, to explain the technique of the Cartesian coordinate system (rectangular straight lines), for example.

To advance in the discussion of the support material in a more dynamic way, the professor-researcher proposed a collective discussion of the material, something not anticipated but which proved to be necessary in view of students' difficulties in advancing in understanding the text and the contents contained therein.

During this moment of collective study, some participants asked some questions, for example, to know the difference between AG and other geometries, Euclidean geometry, for example. In this collective discussion, the difference between linear distance and the distance between two points in a Cartesian plane (rectangular straight line) was also addressed. One of the students stated that the difference is due to the coordinate system.

The researcher-professor institutionalized how the Cartesian coordinate system works (rectangular straight line) and the convention of positive and negative signs, the right and left of the origin, respectively, and addressed the historical importance of AG and its emergence as we know it today. The educator also pointed out the relevance of Gáspar Monge's studies and quickly addressed the appearance of the first textbooks of analytical geometry in France.

This one *moment of institutionalization* was not anticipated; however, during the operation of the device, by asking questions, the students themselves evidenced the need for *feedback*, i.e., an institutional confirmation of what they understood from the discussion.

After institutionalization, the collective debate on the polar coordinate system went on, and step by step, the collective carried out the reading and brought elements to the discussion.

The researcher teacher made “micro” moments of institutionalization about the coordinate systems presented by the groups. In this part of the episode, at each moment of each groups' sharing, questions asked, and issues debated by the pre-service teachers during the discussions were again raised by them in the collective discussion, and the researcher-professor institucionalize the matter. One of the institutionalized theoretical points was the point of the positive and negative sign adopted in the Cartesian (orthogonal) system. The researcher-professor explained the issue of adopting a notation, giving the example of the optical system, which would be studied later.

During institutionalization, whenever possible, the researcher-professor carried out the ecological approach, pointing out that, in the case, for example, of the Cartesian system, the notation adopted in the 19th century –signs of the axes– remains until the present day. But terminologies, such as the very name “analytical geometry”, did not exist in the 17th and 18th centuries. The types of systems, such as bipolar, polar, etc., do not appear in current basic education, i.e., they have disappeared from that *habitat*, currently being studied in higher education, in some undergraduation courses. The students' reports showed that they did not know the bipolar system nor the idea of the plurality of coordinate systems.

We conclude that the moments of institutionalization, supported by our epistemological model of alternative reference, began to impact the progress of the research and formative process, to circumvent the *conditions* of the individuals' mathematical and didactic education. One of the pre-service teachers indicated that, in high school, the polar coordinates are studied through complex numbers. The professor points out the importance of linking physics and mathematics studies, for example, in vectors, which should be approached in high school in mathematics, and not as it happens today when the topic is dealt with in physics but disconnected from mathematics.

Continuing the collaborative study, the researcher-professor asked what differentiates the rectangular rectilinear coordinate system from the rectilinear coordinate system. The collective general answer was the orthogonality of the coordinate axes.

Then, the students began to discuss the question $Q_{2,1}^{DM\,(B)}$. The audios of the groups' work, in general lines, revealed that students reconstructed the technique to represent a point in the rectilinear coordinate system using the same technique of parallel lines, of the rectangular rectilinear system, known as the Cartesian system.

Finally, in the next session, the students themselves explained how the systems work, without the professor's institucionalization. We consider this work as a *exploratory moment* and *of the work of the technique*, which was essential for the sequence of the study. It was also characterized as the moment of constitution of the technological-theoretical environment, which underlies the raison d'être of the coordinate systems.

The researcher-professor resumes the questions generated in this stage of the study, asking two groups to resume the collective discussion presented on the polar and bipolar coordinate system. The first group explains the polar system according to the support material, and a second group explains the bipolar system by comparing it with the previous one. Although some students stated that they had never seen the bipolar system, they could explain how this coordinate system works and ended up generalizing the characteristics of the coordinate system proposed in the study material.

The goals drawn at first were achieved; we mention, for example, identifying the main differences and similarities between the several coordinate systems and the common characteristics. There are indications that these aspects constituted a theoretical-technological basis that underlies AG's raison d'être. This became clearer as the study progressed.

We highlight the explanation of one of the groups in the collective, as each one presented their answers to the questions demanded. The first question returned to the technique used to deduce the rectilinear coordinate system. For this question, one of the groups presented it on the board to the colleagues, according to the constructions written below:

> The technique used to find the coordinates of a point in this system consists of drawing straight lines passing through the point where we want to find its coordinates, these (lines) being parallel to one of the axes and concurrent with the other simultaneously; the points of intersection of the lines with the axes are the coordinates. (Production Group A)
> We used prior knowledge of point, line, position between lines, space, intersection, angles. Based on the concept of Cartesian coordinates, we drew a plane figure and, by outlining two parallel lines, one to the x-axis and the other to the y-axis, the intersection of the lines will be the point situated on the plane, i.e., that will be our Cartesian coordinate, considering that x and y axes are not orthogonal. (Production Group B)
> Draw lines parallel to the y-axis and to the x-axis, which determine points on the plane (Production Group C).

Episode 4 was initially thought as Episode 3, however, as the first two episodes were exceeded, something perfectly expected in the development of the SRP, the planning was reorganized as the experimental stage advanced.

In this episode, the professor-researcher should resume the discussions of the previous episode to continue the study. The question that arose, from an epistemological point of view, is related to the consequences of the answers given in the previous session. The representation of points employing rectangular rectilinear coordinates, which students would at first call Cartesian coordinates in the plane, as this system is called today, would expand the study so that one can understand that other objects can be built in this system from two points, three points, straight line segments, vectors, regular polygons, and curves, etc.

Would the length of segments, or distance between two points, be originally the basis for calculation in tasks involving this topic whose guiding question of the study would run through $Q_3^{DM\,(C)}$ = *What should I teach from the location of a point*? The answer to this question should consolidate the concept of line segment, distance between two points, and aligned points, in addition to consolidating the OM_P. It is the *exploratory moment of the technique* and constitution of the *technological-theoretical* environment around the point study. Also, it would be introductory to the study of the line. We hoped that the development of this session, engendered with an exploratory study activity, could consolidate OM_P. In that sense, the question $Q_3^{DM\,(C)}$, which generated this session, would bring other questions, according to the scheme in Figure 3, whose answers could create conditions to reach the final objective. Intermediate questions would be the "background" of this session. $Q_3^{DM\,(C)}$: *What should I teach from the location of a point*?$Q_{31}^{DM\,(C)}$: What are the coordinates of a point? of two points? Etc.? $Q_{32}^{DM\,(C)}$: How to teach how to determine the distance between two points? $Q_{33}^{DM\,(D)}$: How to teach aligned points? $Q_4^{DM\,(D)}$: How to teach the object "straight line"? The development would be engendered initially as illustrated in the map of issues in Figure 7.

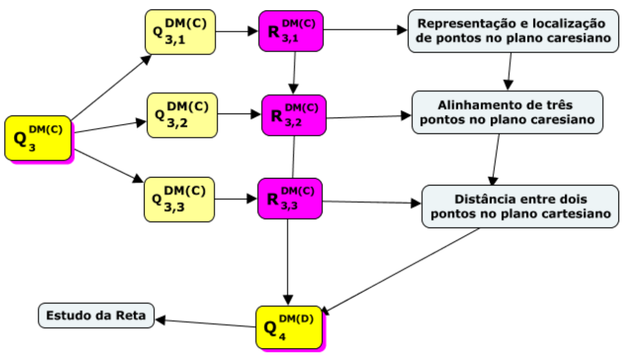


Figure 7.

Map of questions for episodes three and four (Freitas, 2019, p. 273)

As we can see, according to Figure 7, the last question arises from the global development of the didactic-mathematical structure around the mathematical organization of the study of the point in the Cartesian plane, i.e., in the development of the study activities, the object "line" appears at different times, thus necessitating questioning and studying it. In this sense, the question about the object "line" indicates the direction of the study.

In the development of Episodes 4 and 5, we propose problem situations of modeling the context of human vision optics. The goal was to establish relations between the PAG elements by exploring the so-called "Cartesian coordinate system", or Cartesian plane (rectangular coordinate system), the location of points, the determination of the algebraic expression that allows calculating the distance between two points, and the alignment condition between points. To develop a modeling situation around the optical system of the human eye, we relied on the study described in the support material, which dealt with basic notions of optics and the functioning of the human eye.

Table 1.

AEP 1 study and research activity, introduction to the topic (Freitas, 2019, p.272)

According to the media, several *websites* that deal with vision health, including doctors, have warned about the increase in myopia problems in children, adolescents, and adults due to excessive use and exposure to electronic devices, such as smartphones, computers, tablets, etc. In addition, excessive use and constant proximity of electronic devices to the eyeball can cause or increase vision problems, especially myopia. This is because electronic devices close to the eyes contribute to an excessive effort of ocular accommodation. This is because the muscle of our eyes, which works as a kind of zoom to capture the image, needs to do this work repetitively, which can trigger myopia[5] or worsen it. Given this context, based on the study material, carry out an investigation with your group of colleagues based on the following guiding questions. a) Consider that each subject has a "close point" (P_p)and a remote point (P_{R}) located horizontally from the retina (eyeball). What is the "close point" of each colleague? And the remote point? Represent them geometrically. b) Given the identified results, can you indicate which of your colleagues have vision problems?

To solve this exploratory activity (or study and research activity -SRA), students should start by studying the support material, reading it, and doing the experiment with their colleagues. And then register what they could see. They should first use a standard ruler.

From then on, we proposed exploratory Activity 2 (Table 2), which aimed to bring the real-world situation to a geometric and algebraic analysis, with the contribution of the study of spherical lenses.

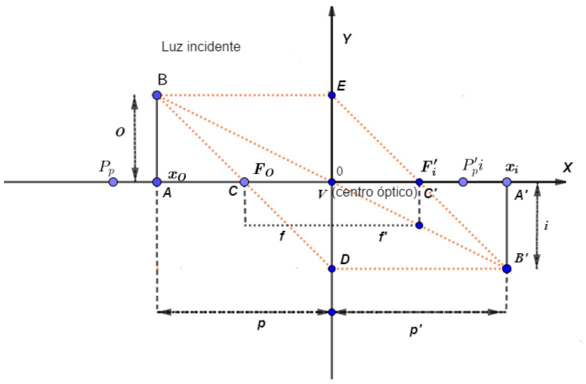
[5] Fonte: Disponível em: http://visaoparaofuturo.com.br/excesso-de-celular-pode-causar-miopia-saiba-mais/ .Accessed on: 02/19/2019

Table 1.

Study and research activity: spherical lenses (Adapted from Freitas, 20159 p. 312)

Let be an object of length O, situated at a distance p from the vertex of the ametropic optical center, according to Figure 84, which applies a converging correction lens. In this system model, several points corresponding to a minimal optical system (reduced eye) are represented in the plane. Analyze the figure and answer the following questions, justifying and geometrically representing your answers when necessary.

Geometric representation of the reduced optical system



A) How do we define $p, p', f\ e\ f'$ from the geometric representation?
B) Analyze the triangles VAB and VA'B'; ABC and VDC, and establish the mathematical relationships between i, o, p and p' and $p, p' e\ f$.
C) Determine the length of the segment VB and V'B' and justify (discuss and prove) its construction.

Exploratory task 2 aimed to enable students to conduct an analytical study of the phenomenon from the perspective of analytical geometry. This analytical study should allow students to realize that the calculation of spherical lenses is based on the concept of distance between two points and aligned points in a Cartesian plane. In addition, maybe, throughout the development of the tasks proposed in the activity, they could identify a technological-theoretical discourse that supports the mathematical relationships established between physical quantities. For this study and research activity, theoretical support from the field of physics and studies of the human eye is required. In this sense, it was essential to bring basic elements from these studies, hence the importance of the support material used. Activity 2 should be answered based on the figure in Chart 2. The SR activities to be performed are described in this Table 2.

The proposed tasks consisted of defining mathematical relationships based on geometric concepts -which function as a technological-theoretical discourse- to substantiate the existence of such relationships. This is the moment of the constitution of the environment *technological-theoretical*. Nevertheless, it is also characterized as a moment of the *first encounter* with the mathematical organization of the optical system and the exploratory moment. The optics context is, therefore, a rich didactic variable that enhances the teaching moments that will enhance the functioning of the device (PEP-PF/TAD).

From the study of specific elements of the human optical system contained in the study material, it was possible to conclude that the mathematical relations defined from the geometric representation of the physical phenomenon of image formation and the use of corrective spherical lenses have, therefore, the plane analytical geometry as a background to mathematically justify the relationships, that is, the PAG is the mathematical *raison d'être* of the image formation phenomenon. In this sense, we propose to carry out a didactic modeling of this phenomenon, in terms of PAG knowledge, at the service of the mathematical and didactic education of pre-service teachers.

The mathematical relationships that could arise were several, from the basic ones developed in the complementary study on optics, to the Gauss equation and the linear increase, the deduction of the distance formula between two points in the plane, and the relationships between the alignment of three points (collinear points). Such constructions, except for Gauss's framework, had already been written and described in the institutional study carried out in the construction of the dominant epistemological model. However, they appear in experimentation as the basis of the problem and should be evoked by pre-service teachers, but in another context. We analyzed the possible answers that would be provided by the participants, *the moment of working with the technique*, in which students could also, from the *theoretical technological* environment around triangle similarity and proportionality, deduce alternative techniques for the proposed task.

The answer to the letter (A) by definition of the optical system, of the reduced eye, f represents the focal length of the object, that is, the linear distance ($f - 0 = f$) from the apex of the optical center, located at the origin of the coordinate axes, to the focus of the object. Similarly, f' represents the focal length of the object image, that is, the linear distance ($f' - 0 = f'$) from the apex of the optical center to the focus of the object image.

The answer to the letter (B) corresponds to the deduction of the linear increase equation and Gauss reference equation. Taking proportion $\frac{A'B'}{AB} = \frac{VA'}{VA}$ from the similar triangles VAB and VA'B' and substituting second values shown in the figure, we have $-\frac{i}{o} = \frac{p'}{p}$ (I), the linear increase equation, which, exactly, establishes relations between i, o, p, and p'.

Triangles ABC and VDC are also similar, and we have $\frac{AB}{AC} = \frac{VD}{VC} \Rightarrow \frac{o}{p-f} = \frac{-i}{f} \Rightarrow f.o = -i(p-f) \Rightarrow \frac{f}{p-f} = \frac{-i}{o}$(II). We can deduce the Gauss equation, which is important in optics, from the equations found. We expected a priori that students discover this construction through research without being asked to do the task. Therefore, if we substitute (I) into (II), we get $\frac{p\prime}{p} = \frac{f}{p-f} \Rightarrow \; p'(p-f) = p.f \; \Rightarrow p.p' - p'f = p.f \Rightarrow \; p.p' = p'f + p.f$; Multiplying two members of the last equation by $\frac{1}{p.p'.f}$, we will get: $\frac{p.p\prime}{p.p\prime f} = \frac{p'.f}{p.p'.f} + \frac{p.f}{p.p'.f}$, which results in: $\frac{1}{f} = \frac{1}{p} + \frac{1}{p\prime}$, Gauss equation. This construction is easily found on the internet at *websites* dealing with the study of spherical lenses, as described in Appendix 1.

Regarding the letter (C), we have that: the line segment VB is the hypotenuse of the right triangle AVB, whose legs are AB and AV. Using the Pythagorean theorem: $|VB|^2 = |AV|^2 + |AB|^2$. Here we have a horizontal segment delimited by two points, leg AV (coordinates of two points in x), and a vertical segment delimited by two points, A and B (coordinates of two points in y). Therefore, the sides of the triangle have, respectively, the following lengths: AV = $(x_A - x_v)$ and AB= $(y_A - y_B)$, therefore, $|VB|^2 = |AV|^2 + |AB|^2 \Rightarrow d(VB) = \sqrt{(x_A - x_v)^2 + (y_A - y_B)^2}, \Rightarrow$

$$\Rightarrow d(VB) = \sqrt{(x_o - 0)^2 + (0 - y_B)^2} \Rightarrow d(VB) = \sqrt{{x_o}^2 + {y_B}^2}$$

A priori, students could do $d(VB) = \sqrt{o^2 + p^2}$. Analogously, we will have A'V= $(x_i - x_v)$ and the AB= $(y_A - y_B)$. Where: $|VB'|^2 = |A'V|^2 + |A'B'|^2$, and $d(VB) = \sqrt{(x_i - 0)^2 + (0 - y_B)^2}$. Students could also do $d(VB') = \sqrt{i^2 + p'^2}$ first.

We expected to activate the notions of a line segment, an oriented segment, and a line in the plane, also mobilized by the participants in discussions about the figure. The resolution and discussion of questions in the group, mathematical constructions, and the deduction of equations constitute the moment of working with the technique. Therefore, the task aims to promote students' work with the technique; that is, from the constructions carried out, they must be able to synthesize and structure the practical-technical block with the technological-theoretical one, aiming to make the technique more efficient and reliable.

The full resolution of the problem was based on the concept of distance between two points, based on the deduction of the formulas used. We accepted that students undertook an alternative technique, which might or might not be the same as we proposed. However, whatever the technique, it would permeate the theoretical-technological discourse we presented.

The notion of locating points on the plane was fundamental for the interpretation of the figure in Table 2. In this sense, the steps should be as follows: students would represent auxiliary figures (right triangles) to deduct conjugate distances.

In group work, students should aim to present a technique to solve the problem, seeking different research sources, including the support material. After the groups' socialization, the researcher-professor would propose an institutionalization, which could contemplate the difficulties arising during group work.

The evaluation should be articulated with the institutionalization, in which the students, together with the professor-advisor, would discuss the mathematical organizations that had been built until then, evaluating what was built and the limitations of the developed technique. The institutionalization process should clarify what was elaborated in terms of mathematical organization and promote a reflection on what was learned, in addition to allowing students to articulate themselves to organize written production as a partial answer to the initial question that generated the path. To this end, the activity report, in addition to the productions, must have as guiding questions the ones initially proposed in the development of the device.

The notions deepened in this session, such as the study of the straight line, should be resumed in the following episodes of the study, with students being able to seek other sources of study even to validate their mathematical constructions.

The activities took place according to the a priori design. We highlight some answers given by the participants:

Group A started to develop the experiment by having a colleague initially as a volunteer. She explained what happens when a specific object or image is brought closer to or farther from the eye based on the experiment, and then on the explanation provided in the support material. The team relied much more on the study material than the experiment itself and made deductions about the distance and angle of the field of view. We illustrate it with an excerpt from the audio transcription of the group's work (F1 and F2 are female speakers).

> F1: - *When I, ah... the distance was less than the near point...*
> *F1: - No. When we take the object and bring it closer to the eye, this angle immediately decreases, right?*
> *F2: - Yes. Yes.*
> *F1: - When this angle decreases, can we say that our field actually decreases? Isn't it? That is, we... it does not decrease the field, it increases the field of view, but the visual acuity decreases, you know, of the object, of the vision that we have of this object. Why does it happen...? This happens because also, where the object was... before we were*

seeing it more clearly, it was after the near point, but at that moment now it is before the near point. So, every time we have an object before the near point, the acuity we have... the angle in relation to it is greater, right?
F2: - Yeah. (Audio transcript, Group A)

Apparently, after this initial moment of work, the group members did not identify the distance from the object as a moderating factor for greater or lesser accurate vision. Next, they discussed when converging and diverging lenses should be used, based on the support material, and then related the distance of a clear vision with one lens or another.

That moment of work was the first encounter with the intuitive notion of distance, as we still did not have an explicit mathematical organization regarding this notion. This organization was developed in the following didactic moment. Initially, we did not measure precisely the didactic time for the planned activities on the optical system or all the study episodes. The study of the optical system and the proposed activities were re-dimensioned into three sessions, or episodes, instead of a single one.

The line as a geometric locus (Episodes 5 and 6)

This stage of the study was marked by the study of the line. In this case, some clarification is needed. From the pilot study carried out at the beginning of the research, several restrictions from the point of view of the mathematical education of the pre-service teachers were identified. Within the framework of these restrictions, there is an aspect that is fundamental in the study of the AG: it is the connection between the geometric and the algebraic, for example, identifying geometric properties and representing them through equations. This connection can be facilitated -or not- with the use of technological interfaces that allow the individuals, through them, to manipulate the representations of objects. In this sense, the geometry dynamic software Geogebra can be configured as part of the *milieu* in the development of the proposed tasks. It is possible that the incorporation of the use of this type of technology promotes the economic aspect of the posed didactic problem: teaching AG.

Another important aspect we must mention is the use of proof and demonstration, which has been a difficult restriction in field research carried out with teachers, and which our pilot study also revealed. Allied with all this, our SRP-PF, now under development, must have two general perspectives globally: to develop the mathematical knowledge present in the study of objects, and to develop the didactic knowledge present in that same study, thus circumventing the didactic problem of teaching AG.

Last but not least, there is the ecological aspect in studying the AG, which, as we saw in the study of the reference and dominant epistemological models, reveals several gaps in the development of knowledge related to the PAG, given the disappearance of some didactic approaches, from the technological-theoretical block in mathematical organizations, and the disappearance of tasks that include demonstrations, which enhance the development of theory. An example is the task of showing that three points are aligned without using the determinant technique or the technological-theoretical discourse based on synthetic geometry, not linear algebra. All this exposed, we organized a map of questions that guided the study episodes, according to Figure 8.

Figure 8.

Map of questions from episodes E to J (Adapted from Freitas, 2019, p.283)

The map aimed to elucidate which questions may arise when we start from the question $Q_4^{DM\,(C)}$ (How to teach "line"?), considering the alternative model, which considers the relevant aspects of each epistemological model, taking into account the ecological, didactic, and economic aspects. From the question “How to teach 'line'?”, are we asking what the geometric locus of the line is? The answer to this question will generate several other questions outlined in two mathematical aspects: the geometric and the algebraic. These two aspects mobilize the didactic question of the notion of the object and its representation (geometric, algebraic, or others).

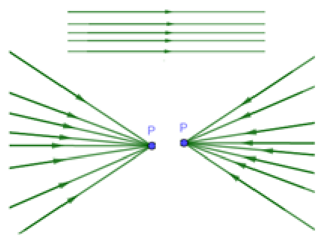
It is in this way that the study episodes were first outlined. The study path and the (re)construction of mathematical and didactic organizations regarding the line should allow individuals to answer the didactic question: teaching straight lines through the intermediate questions of the study. We emphasize that the engendered questions, represented on the map in Figure 8, are defined a priori.

Each theme listed in the light blue blocks refers to the specific element of the mathematical organization being developed and studied.

The following questions are inserted in each work session, based on the study of the geometric locus "line," describing, through the map, its relations with the provisional answers and the general themes of the didactic organization: $Q_{4,1}^{DM\,(E)}$: What is the geometric locus "line" in the plane? $Q_{4,2}^{DM\,(E)}$: Which mathematical relation(s) involves a line in the plane? $Q_{4,3}^{DM\,(F)}$: What is the direction of a line in the plane? $Q_{4,4}^{DM\,(F)}$: How to teach the general equation of a given line? $Q_{4,5}^{DM\,(G)}$: What mathematical relationships can be established between points and lines in the plane?$Q_{4,6}^{DM\,(H)}$: What are the relative positions between two lines in the plane? Concerning questions $Q_{4,1}^{DM\,(E)}$ and $Q_{4,2}^{DM\,(E)}$, students are globally expected to be able to deduce the general equation of a given line.

Como vimos as sessões anteriores para que um observador enxergue um corpo, seus olhos devem receber a luz que esse corpo emite. Por exemplo, uma lâmpada, para representar a luz se propagando e atingindo os olhos do observador, utiliza-se linhas orientadas que fornecem a direção e o sentido de propagação da luz. Tai linhas são denominadas raios de luz, que se representa segundo a figura 85. Um feixe de luz é cilíndrico, quando seus raios são paralelos, é cônico quando todos os raios de luz têm direções que passam por um mesmo ponto P, sendo neste caso, convergente ou divergente. O ponto P é o vértice do feixe. Quando o feixe é composto por retas paralelas, o vértice está no infinito.
Tendo por base a figura, reconstrua no Geogebra esses tipos de imagem e seguida responda as questões.

Representação gráfica de feixes de luz



Adaptado de Sampaio e Calçada (1998, p. 2)

A) Quais entes geométricos podem ser associados os raios luminosos?
B) E se prolongarmos esses raios indefinidamente em um plano? o que acontece? simule outras situações.

Figure 9.

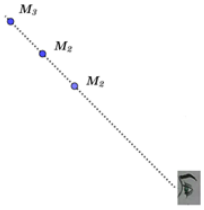
Exploratory activity (Adapted from Freitas, 2019, p.285)

In this episode, we proposed the continuity of the interdisciplinary activities on optics with the formation of images, carried out in previous sessions (spherical lenses, human eye lenses, ametropias, etc. Moments of the first encounter with OM_R (mathematical organization of the line). In this sense, we suggest some exploratory activities, whose objective varied: in Activity 1, the manipulation of the Geogebra interface, according to Figure 9, and the visualization of specific geometric properties.

In the second activity (Figure 10), in addition to viewing the properties, the pre-service teachers should build the geometric representation in the Cartesian plane using the Geogebra interface.

Um observador identifica três objetos que estão localizados no seu campo de visão, todos em uma mesma direção. Observe a figura.

Figura 87 – Ilustração da tarefa campo de visão



Fonte: Elaborado pela Autora

Utilizando o Geogebra reproduza uma representação geométrica no plano cartesiano, em seguida, análise essa construção com o seu grupo, indicando quais propriedades geométricas podem ser observadas, em seguida responda as questões.

A) Quais relações matemáticas podem ser estabelecidas a partir dos pontos que representam os objetos no plano cartesiano?
B) Qual a equação da reta determinada pelos pontos M_1, M_2, sabendo que $P(x, y)$ é um ponto que percorre a reta?

Figure 10.

Activity ***(****Freitas, 2019, p. 286)*

The idea is that they take as a starting point a mathematical model they already know or that is developed in this construction and through the free exploration of the software interface, reach the model of aligned points. In this model, the geometric construction allows the application of Thales' theorem or similarity of triangles, the notion of proportional segments, or even the linear dependence between oriented segments. This last concept brings the notion of vector, which was not part of the initial content under study.

Whichever path the subjects choose, they will arrive at the equation that represents the determinant, therefore, the condition for three-point alignment.

This same model must be the starting point, as a technique individuals use to find the equation of the line. We point out that this model for the deduction of the general equation of the line does not appear in the study of the reference model based on Briot and Bouquet (1860). In their 19th-century text, the authors immediately present the general equation of the line, and all subsequent constructions are based on this proposition to demonstrate that: $Ax + By + C = 0$.

By associating the geometric representation of the line (by three points) with the calculation of the determinant, this model emerged after the movement of modern mathematics, already with the influence of linear algebra.

To solve the question (a), the first step would be to build the geometric representation in Geogebra using the point option. In the interface, we can mark several points, select the option "line through two points" and choose two of the four points to draw the line. We outlined the segments that determine the location of each point by dashed lines, which will determine the coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) from each point M_1, M_2 and M_3, respectively (Figure 11).

The points are located on the same straight line or in the same direction, i.e., they are aligned. This means that the three points determine proportional segments on a straight line. The parallel lines that determine the coordinates of each point are a bundle of parallel lines cut by the line (r), and evoke Thales' theorem.

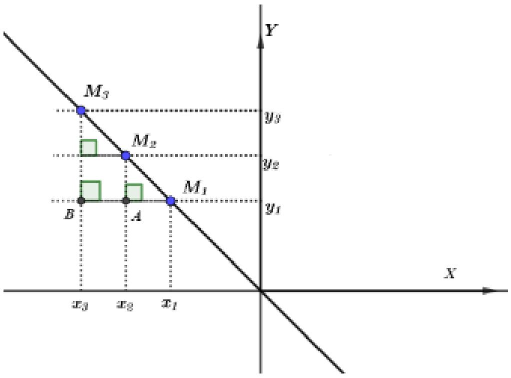


Figure 11.

Geometric representation of three aligned points, letter (A) (Freitas, 2019, p.286)

Similar triangles (rectangles) are also formed, M_1M_3B and M_1M_2A, and these geometric properties allow us to establish that: $\frac{M_1M_3}{M_1M_2} = \frac{x_3 - x_1}{x_2 - x_1}$ and $\frac{M_1M_3}{M_1M_2} = \frac{y_3 - y_1}{y_2 - y_1}$. The development of the equalities allows us to reach the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, which is precisely the condition for the three points to be aligned.

For question (b) (Figure 12), students could use the same technique, based on geometric similarity relations, on Thales' theorem, and find the equation of the line: $x(y_1 - y_2) + y\,(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0$ (I).

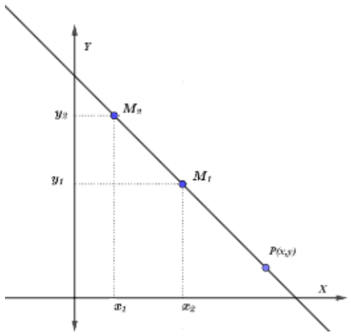


Figure 12.

Geometric representation, letter (B) (Adapted from Freitas, 2019, p.288)

The analysis of the equation, supported by Figure 12, allows us to state that we have constant values x_1, x_2, y_1 and y_2 and variable values$x \ e \ y$. In this way, we can substitute in equation (I), $y_1 - y_2 = a$, $x_2 - x_1 = b$ and $(x_1y_2 - x_2y_1) = c$, and we will have: $ax + by + c = 0$, the general equation of the line, principal relation sought. We can also directly develop the determinant $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ and come to the same conclusion.

After solving the task and determining the general equation of the line, the following question may be proposed by the researcher-professor: What happens to the equation of the line if the points M_1 and M_2 are points of intersection between the line and the x and y axes, respectively? Simulate and represent geometrically, using Geogebra, and show the resulting equation.

To answer this question, it is necessary to move points M_1 and M_2 so that they belong, respectively, to the x and y coordinate axes. This construction must be done in Geogebra to move the points, obtaining Figure 13.

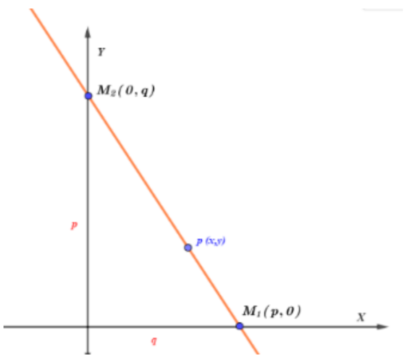


Figure 13.

The line that intersects the X and Y axes (Freitas, 2019, p.289)

Points M_1, M_2, and $p\ (x, y)$ are collinear and therefore belong to a line; so, we can conclude that: $\begin{vmatrix} x & y & 1 \\ p & 0 & 1 \\ 0 & q & 1 \end{vmatrix} = 0$. Developing this determinant, we have: $-qx - \ py + pq = 0$, and this equation is equivalent to: $qx + \ py - pq = 0 \Rightarrow qx + \ py = pq$.

We can divide the equation by the product pq to obtain an equation in which x and y are isolated: $\frac{qx}{pq} + \frac{py}{pq} = \frac{pq}{pq}$. So we have the segmental equation of line r: $\frac{x}{p} + \frac{y}{q} = 1$. Note that the denominators are exactly the measures of the segments that line r determines on the coordinate axes x and y

In summary, students must present mathematical relationships through line equations, which can be of three types:$y = ax$, i.e., a straight line passing through the origin of the Cartesian system, $y = ax + b$y; a line that cuts the y-axis at a point other than the point, origin of the Cartesian system; and a line that is parallel to the x-axis, defined $y = b, s$as $y = ax + b$y the generic algebraic model for the representation of any straight line. The reduced model of the equation of the line through the determinant also allows us to arrive at the segmented equation: $xp + yq = 1$, when the line intersects the coordinate points (x, 0) and (0, y). It is possible for students to develop this model as well.

The following stages of the study will be strongly influenced by this episode, as the students should develop a mathematical organization to study the line and its general equation. We did not predict in detail which geometric and algebraic constructions would be prioritized by students in the next session. The central question that guided the study is: How to teach the straight line equation to their potential students? The answer to this question will be the added synthesis of didactic objectives, which would consolidate the technological-theoretical moment.

Due to the limitations of this text, we do not present a thorough analysis of the resolutions of the tasks done by the pre-service teachers, nor the results of Episodes 7, 8, 9, and 10, which involved the study of the linear coefficient of the equation of the line, relations between points and lines, and positions of lines. What we can say globally is that they fulfilled the tasks based on the same model of techniques designed a priori. However, since the beginning of the study, they evolved into a process of debate around the technological-theoretical discourse of the techniques they used and the content thought in didactic terms for their potential students, something that was incorporated into the individuals' knowledge throughout the episodes at stake.

We highlight some results of Episodes 11 and 12, which were based on the construction of the task proposed as an activity that integrates the didactics (Figure 14).

Parte 1: Construa um esquema ou mapa conceitual, representando globalmente a evolução do estudo realizado, em termos dos objetos matemáticos estudados.

Parte 2: A partir das reflexões realizadas no processo de formação do ateliê de modelização didática, desenvolva uma organização didática de estudo para os seus alunos do 3º ano do ensino médio, com base na seguinte questão: ***como ensinar a geometria analítica do ponto e da reta?***

Figure 14 .

Integrative activity of the didactic (Freitas, 2019, p.295)

We illustrate, through one of the extracts from the dialogues of the pre-service teachers, the commitment to the development of the tasks from the mathematical and didactic points of view:

H1 - A gente elaborou a atividade do alinhamento de três pontos, cinco questões para eles identificarem se eles entenderam como esse... se é alinhado e essa questão aqui que é mais... que envolve também função exponencial do primeiro grau. Aí agora falta para terminar? A atividade dos outros... dos outros dias. Porque aqui, olha, no primeiro dia a gente só escolheu uma questão, que é essa daqui, além dos exemplos que a gente vai dando, que vai ser mais de construção. A segunda também é construção. E depois, na terceira, vem essa daqui, e depois, para entender quando a gente pega... como a gente pegou aqui, a equação geral da reta. Com... com a equação geral da reta tem aqueles casos, quando "B" é zero e quando é paralelo ao "X". Aí agora o que foi que a gente pensou? Em colocar isso em no Geogebra, para a gente desenhar uma reta. E aí, a partir do... quociente, eles veem a... acontecer no Geogebra. Não no quadro, como a gente está vendo.
H2 - Ah, sim, sim.
H1 - Porque até tem vezes que até o desenho do... do quadro fica um pouco complicado para eles visualizarem. Pode não ser o melhor... ((sobreposição de vozes))
H2 - Visualiza, não é?
H1 - Agora... agora aqui eu estou na dúvida. Se for para continuar... porque, se for para continuar o conteúdo, vê equação reduzida, vê ângulo, no caso na inclinação da reta, e o ângulo. Aí depois vem a reta reduzida e a reta paramétrica, só que como a gente não... no mapa conceitual a gente só foi até a equação geral, a gente vai também no plano só até equação geral.
H2- (Quem que escreve) [00:01:44] Geogebra?
H1 - O Geogebra, toda vez que a gente escreve fica assim.
(Transcrição de áudio, produção do Grupo A),

Figure 15 .

Audio transcription of the group's production (Freitas, 2019, p. 346)

We could identify in the pre-service teachers' audios, during the work, their task planning for the context of the didactic organization and the lesson plan, which they classified as exercises, construction tasks, and visualization tasks with the use of Geogebra. We questioned how far they would go in content, and they decided to comply with the proposal of the mental map built in one of the study sessions, according to Figure 15.

Conclusions

Given the extract presented in this text, regarding experimentation and the suppression of some sessions and some results, we highlight some conclusive aspects of the study episodes and general aspects identified as characteristics of the SRP carried out and consolidated as a theoretical and methodological device for research and professional education.

The planning moments identified in the activities executed by the pre-service teachers were quite significant. However, we could not identify in the audio recordings the possible praxeologies thought for the teaching of the PAG. Although the pre-service teachers found it challenging to express themselves in formal Portuguese, their work revealed the practice of being a teacher, a planner, and an organizer, from the didactic and mathematical points of view, taking into account, among other things, the pedagogical aspects.

Although the study activity was proposed for potential students, the pre-service teachers absorbed the activity for their own pre-service students, and at no time did they state that the proposal would not be real; on the contrary, some affirmed that they would put it into practice at the first opportunity they had, in a 3^{rd}-grade high school class, still in the supervised teaching practice.

Another noteworthy aspect identified in the constructions of the individuals is their (pre-service teachers) need to put themselves in the shoes of either teachers or their students to think about their teaching and learning. The oral explanation and the written construction of the conceptual map indicate that the pre-service teachers identified the mathematical elements developed throughout the study, and articulated them with those essential didactic and pedagogical elements to think about the teaching and learning of the chosen PAG objects.

This behavior was remarkable throughout all the teaching sessions mediated by the researcher. At each study session, in moments of sharing and institutionalization, the researcher encouraged and promoted the debate about learning and teaching, tasks, techniques, and strategies for the didactic approach to the themes addressed. Apparently, the pre-service teachers incorporate it along the way.

In the groups' work, the importance of the technologies that justify the techniques and the theoretical technological discourse became evident, as these technologies appear on the maps as an integral part of the study, in addition to the theory(ies) that justify the technologies, for example, Pythagorean theorem, Thales theorem, and similarity of triangles. The groups also identified the techniques mobilized for the tasks, the Sarrus rule, matrices, and determinants.

Another aspect evidenced in the construction of subjects was the incorporation of digital technologies as part of the process of thinking about teaching and learning mathematical objects, for example, the use of Geogebra, Winplot, and others for mathematics teaching.

We defined that part of the didactic knowledge mobilized by the pre-service teachers in the activities can be characterized as the teacher's ability to explain mathematical content, and make themselves understood by their students. About this skill, the survey work indicated that it can be enhanced by the praxeology of a task during the moments of sharing and socialization carried out in the different stages of the device.

With regard to the general aspects of the SRP carried out, we reinforce that we brought a part of it intending to characterize it as a constituted device, based on certain characteristics already described theoretically, which we synthesized as general characteristics that were present in the experimentation from a practical point of view to guide the constitution of this type of device:

- Process design, a priori provisional, replanning along the way;
- Inclusion of modelable situations of an interdisciplinary SRP;
- Connection between the mathematical and the didactic in modelable situations;
- Moments of sharing and socialization;
- Microgroups of work within the scope of the study collective;
- Establish connections between mathematics and didactics in moments of institutionalization;
- Didactic modeling of mathematical organizations;
- Defined guiding questions a priori as part of the researcher's control;
- The alternation between the didactic moments of an “open” SRP and a “closed” SRP, mixed SRP;
- Coexistence of a Regional SRP and other specific SRPs throughout the study.

We could only see this last item on the list at the end of the research, a good part of which was removed from this article. However, the notions of specific, regional, and global praxeology inspired us to define the mini-SRPs that took place within the scope of each activity and the groups' work in the scope of the study collective.

References

Artigue, M (1988). Ingénierie didactique. Recherches em Didactique des Mathématiques. **La Pensée Sauvage-Éditions,** 9(3), 281-308.

Ball, D. L.; D. L.; Thames, M. H. Phelps. G. (2008). Content Knowledge for Teaching What Makes It Special? **Journal of Teacher Education,** 59(5), 389-407.

Briot, C. ; Bouquet, J. C**.** (1860). **Leçons de Géométrie Analytique**. Dezobry, E. Magdeleine et C., Paris: Libraires-Éditeurs..

Chevallard, Y. (1999). Les analyses de pratiques enseignantes en théorie anthropologique de la didactique**.** Recherches en Didactique de Mathématiques. **La Pensée Sauvage-Éditions**, 19(2), 221-266.

Chevallard, Y. (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. Maury S. & Caillot, M. (Éd.). *Rapport au savoir et didactiques* (pp. 81-104)., Paris: Éditions Fabert. Acessado em de : http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Approche_anthropologique_rapport_au_savoir.pdf

Chevallard, Y. (2007a). Passé et présent de la théorie anthropologique du didactique. França, Acessado em 03 de julho de:

http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Passe_et_present_de_la_TAD-2.pdf.

Chevallard, Y. (2007b). **Um concept en émergence**: la dialectique es médias et des milieux. Université de Provence, França.

Chevallard, Y. (2009a). La notion d'ingénierie didactique, un concept à refonder. Questionnement et éléments de réponse à partir de la TAD. **15e École d'Été de Didactique des Mathématiques.** Clermont-Ferrand,. Acessado em 15 de abril de 2015 de: http://www.ardm.eu/book/export/html/676.

Chevallard, Y. (28 avril 2009b). La notion de PER: problèmes et avancées**.** Toulouse, . Acessado em 17 de abril de 2015 de:

http://yves.chevallard.free.fr/spip/spip/IMG/pdf/La_notion_de_PER___problemes_et_avancees.pdf.

Chevallard, Y. (2020). Some sensitive issues in the use and development of the anthropological theory of the didactic. **Educ. Matem. Pesq**., 22(4), 13-53.

Duval, R. (1995). **Sémiosis et pensée humaine**: Registres sémiotiques et apprentissages intellectuels. Bern: Peter Lang.

Freitas, Rita Lobo. (2019). **Dispositivo de pesquisa e formação profissional PEP-FP/TAD**: **constituição do conhecimento docente para o ensino de geometria analítica plana do ponto e da reta**. Tese de doutorado, Pontifícia Universidade Católica de São Paulo. São Paulo, Brasil.

Iezzi, G. *et al*. (2010). **Matemática Ciência e Aplicações**. São Paulo: Editora Saraiva,.

Maire, I. (1637). La Géométrie, in Discours de la Méthode pour bien conduire sa raison et trouver la vérité dans les sciences. Plus La Dioptrique, Les Météores et La Géométrie. Acessado em 28 de maio de 2018 de: http://gallica.bnf.fr/ark:/12148/btv1b86069594/f4.double.r.

Mishra, P.; Koehler, M. (2006). Technological Pedagogical Content Knowledge: A Framework for Teacher Teachers College. **Record,** 108(6), 1.017–1.054.

Sierra, T. Á. D. (2006). **Lo matemático en el diseño y analisis de Organizaciones didácticas:** los sistemas de Numeración y la medida de magnitudes.Tese de Doutorado, Universidad Complutense de Madrid. Madri, España.

Sierra, T. Á; GascON, J. (2018). Los recorridos de estudio e investigación para la formación del profesorado y la construcción de praxeologías matemáticas para la enseñanza. El caso de los sistemas de numeración. **Recherche en Didactique des Mathématiques**, 38(1), 79-117.

Shulman, L. S. (1986). Those Who understand: Knowledge growth in teaching. **Education Researcher**, 15(2), 4-14.

Shulman, L. (1987). Conhecimento e Ensino: Fundamentos da Nova Reforma. **Harvard Educational Review**, 57(1), 1-23.

Silva, M. J. F. da. Lima, G. L. de. (2015). Conhecimentos desenvolvidos em um curso de Licenciatura em Matemática na modalidade a distância. In *Anais* do XIV CIAEM-IACME, Chiapas, MX. México: CIAEM-IACME.

Tardif, M. (2012). **Saberes docentes e formação profissional**. Petrópolis, RJ: Vozes.

Appendix

Chart 1

Task Description

TASK DESCRIPTION CHART	
TASK TYPE	**DESCRIPTION**
$\boldsymbol{T_1}$	*Find the general equation of lines passing through a given point*
$\boldsymbol{T_2}$	*Through a given point, draw a line parallel to a given line*
$\boldsymbol{T_3}$	*Draw a line through two given points*
$\boldsymbol{T_4}$	*Find the point of intersection of two given points*
$\boldsymbol{T_5}$	*Find the general equation of lines passing through the point of intersection of two given lines*
$\boldsymbol{T_6}$	*Recognize whether three points are in a straight line. In current language we use the alignment of three given points*
$\boldsymbol{T_7}$	*Recognize whether three lines pass through the same given point*
$\boldsymbol{T_8}$	*Find the angle of two lines, what the authors seem to propose is to determine angles between two lines given in the plane*
$\boldsymbol{T_9}$	*From a given point, draw a perpendicular on a given line, and find the length of that perpendicular*
$\boldsymbol{T_{10}}$	*Through a point of intersection of two given lines, draw a line perpendicular to a given line*
$\boldsymbol{T_{11}}$	*Find the locus of points equally distant from two given points*
$\boldsymbol{T_{12}}$	*Find the tangent equation to any curve*
$\boldsymbol{T_{13}}$	*Find the equation of the tangent to the circle*
$\boldsymbol{T_{14}}$	*Take a tangent to the circle through an exterior point*
$\boldsymbol{T_{15}}$	*Take a tangent to the circle parallel to a given line*
$\boldsymbol{T_{16}}$	*Find the locus of points whose distances from two fixed points are to each other in a given relation*
$\boldsymbol{T_{17}}$	*Find the intersection points of two circles*