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The validation process in modelling activities: looking for a framework

Le processus de validation dans les activités de modélisation mathématique: à la recherche d'un cadre

El proceso de validación en las actividades de modelación matemática: en busca de una estructura

O processo de validação em atividades de modelagem matemática: em busca de um *framework*

Lourdes Maria Werle de Almeida
Londrina State University (UEL)
Pós-Doutorado em Educação Matemática
https://orcid.org/0000-0001-8952-1176

Rosangela Maria Kowalek
Londrina State University (UEL)
Doutoranda em Ensino de Ciências e Educação Matemática
https://orcid.org/0000-0002-2750-4829

Abstract

The article aims to investigate validation in mathematical modelling activities with the objective of organizing a framework related to validation in these activities through data triangulation to give coherence and cohesion to the results. For the organization of the framework, a literature review regarding the theme is realized. In addition, an empirical research is carried out with students of a Mathematics Degree course. The gathering of empirical data and the theoretical structure allow to characterize understandings, the importance and ways of carrying out validation in mathematical modelling. Although it can be recognized as the final step in the modelling cycles, validation cannot be thought of as a cumulative process that only begins at the end of the activity and from which only acceptance or refutation results. Instead, it can act as an iterative agent and guide students' decision-making processes, and validation mechanisms can be activated at different stages of the activity. What can be concluded, therefore, is that, even if specific actions are recognized, it is in the validation of the totality of the modelling that the efficiency of validation resides as a means of generating reliability in what can be said about a situation of reality by using mathematics.

Keywords: Mathematics education, Mathematics modelling, Mathematical model, Validation.

Resumem

El artículo aborda la investigación de la validación en actividades de modelación matemática. Por intereses de investigación en delinear una estructura de validación en estas actividades, se utiliza la triangulación de datos como una alternativa para dar coherencia y cohesión a lo que se pretende hacer. Para la organización de la estructura, se captura un marco teórico referente al tema. Además, se realiza una investigación empírica con estudiantes de un curso de Grado en Matemáticas. La recopilación de datos empíricos y el marco teórico permite caracterizar los entendimientos, la importancia y las formas de llevar a cabo la validación en la modelación matemática. Si bien puede reconocerse como el paso final en los ciclos de modelación, la validación no puede pensarse como un proceso acumulativo que solo comienza al final de la actividad y del cual solo resulta la aceptación o la refutación. En cambio, puede actuar como un agente iterativo y guiar los procesos de toma de decisiones de los estudiantes, y los mecanismos de validación pueden activarse en diferentes etapas de la actividad. Lo que se puede concluir, por tanto, es que, aunque se reconozcan acciones concretas, es en la validación de la totalidad de la modelización donde reside la eficacia de la validación como medio de generar fiabilidad en lo que se puede decir sobre una situación de la realidade a través de las matemáticas.

Palabras clave: Educación matemática, Modelación matemática, Modelo matemático, Validación.

Résumé

L'article porte sur l'investigation de la validation dans les activités de modélisation mathématique. Pour les intérêts de recherche dans la définition d'un cadre de validation dans ces activités, la triangulation des données est utilisée comme alternative pour assurer la cohérence et la cohésion de ce qui est censé être fait. Pour l'organisation du cadre, un cadre théorique concernant le thème est capturé. De plus, une recherche empirique est menée auprès d'étudiants d'un cours de Licence de Mathématiques. La collecte de données empiriques et la structure théorique permettent de caractériser les compréhensions, l'importance et les manières d'effectuer la validation en modélisation mathématique. Bien qu'elle puisse être reconnue comme l'étape finale des cycles de modélisation, la validation ne peut être pensée comme un processus cumulatif qui ne commence qu'à la fin de l'activité et dont il ne résulte que l'acceptation ou la réfutation. Au lieu de cela, il peut agir comme un agent itératif et guider les processus de prise de décision des étudiants, et des mécanismes de validation peuvent être activés à différentes étapes de l'activité. Ce que l'on peut donc conclure, c'est que, même si des actions spécifiques sont reconnues, c'est dans la validation de la totalité de la modélisation que réside l'efficacité de la validation comme moyen de générer de la fiabilité dans ce qui peut être dit d'une situation de réalité par les maths.

Mots-clés: Éducationdes mathématiques, Modélisation mathématique, Modèle mathématique, Validation.

Resumo

O artigo dirige-se à investigação da validação em atividades de modelagem matemática, tendo como objetivo a organização de um *framework* relativo à validação nessas atividades mediante uma triangulação de dados para dar coerência e coesão aos resultados. Para a organização do *framework* um quadro teórico relativamente à temática é capturado. Além disso, uma pesquisa empírica é realizada com alunos de um curso de Licenciatura em Matemática. A reunião dos dados empíricos e da estrutura teórica permite caracterizar entendimentos, a importância e modos de realizar a validação na modelagem matemática. Embora possa ser reconhecida como etapa final nos ciclos de modelagem, não se pode pensar a validação como processo cumulativo que só se inicia ao final da atividade e do qual apenas decorre aceitação ou refutação. Ao invés disso, ela pode atuar como agente de iteratividade e orientar os processos decisórios dos alunos, sendo que mecanismos de validação podem ser ativados em diferentes etapas da atividade. O que se pode concluir, portanto, é que, ainda que ações pontuais sejam reconhecidas, é na validação da totalidade da modelagem que reside a eficiência da validação como meio de gerar confiabilidade no que se pode dizer de uma situação da realidade por meio da matemática.

Palavras-chave: Educação matemática**,** Modelagem matemática, Modelo matemático, Validação.

The validation process in mathematical modeling activities: in search of a framework

Different movements to introduce mathematical modeling in mathematics classes have been noticed in recent decades. The structuring of a curriculum guided by the development of skills in which modeling[1] is one of them, as is the case in Germany (Ferri, 2018), the inclusion of a mathematical modeling component in mathematics teaching degree courses and the indication of the use of modeling in documents that guide mathematics teaching and in-service teacher education, as is the case in Brazil (Bassanezi, 2002; Mutti & Klüber, 2018; Brasil, 2018), are examples of these movements.

Mathematical modeling aims, in general, to understand, explain, or predict specificities of a real-world situation. This understanding means an explanation or prediction mediated by a mathematical model (Geiger et al., 2022; Blum, 2015). In other words, based on mathematics, the modelers (students and teachers in the classroom) propose a solution to a non-mathematical problem.

To characterize modeling as such, we must apply a set of procedures connected to high cognitive demand (Blum, 2015) and indicated by stages. The identification of these stages has been presented through analytical constructions called *modeling cycles*. On the one hand, these cycles are valuable as an epistemological reference model to specify these procedures. On the other hand, they have been used much when the aim is to look into students' performance when modeling (Barquero & Jessen, 2020).

The mathematical models produced during the stages of this cycle then function as coherently structured units of knowledge. From them, inferences, predictions, explanations, and even judgments about the situation are made explicit (Hestenes, 2010). These models have an epistemological status open to debate, and they are the verbalization of the relationship between reality, understood as the *world as experienced by the subjects* (Hesthenes, 2010). Therefore, they can be simplified descriptions of this reality depending on the modeler's vision and idealizations of a possible reality. However, they are constantly subjected to comparisons with the reality in question. These comparisons are, therefore, integrated into the stages of a modeling cycle and, in general, identified there as *validation*.

Nevertheless, what does it mean *to validate?* According to a Brazilian Portuguese language dictionary (Houaiss, 2021, p. 1), validate comes from the Latin *validare* and means "to make it valid based on the rules in force." The word validation is thus associated with the

[1] In the text, the word *modeling* always refers to mathematical modeling.

action or intention of making something valid and, as another dictionary suggests (Ferreira, 2009), making it legitimate or legitimize.

Validating in modeling activities seems to constitute a very challenging stage for students, which can be attributed precisely to the dynamic nature of the activity and the iterative nature provided by seeing and reviewing procedures (Czocher, 2018; Galbraith & Stillman, 2006).

Czocher (2013), Ikeda (2013), Czocher (2018), and Czocher, Stillman, and Brown (2018) consider validation to be a relevant stage of modeling. However, little has been investigated to point out how students can carry it out and how the teacher can encourage it, for example, since it is through validation that the weaknesses of a model or the lack of credibility of a response can be reviewed.

Given these considerations presented about validation in mathematical modeling, in this article, we aim to investigate the understanding, purposes, and how this validation occurs. Based on the theoretical backround relating to validation and the results of empirical research, we constructed a *framework*[2] pointing out possibilities, purposes, and ways to carry out validation in mathematical modeling activities.

Research methodology

For the research to outline a framework for validation in mathematical modeling, a data triangulation strategy[3] (Duarte, 2009) is used in order to building coherence and cohesion in what we intend to achieve. In this direction, we follow a constructivist paradigm, in which, according to Duarte (2009), reality is multiple and constructed, with inseparable subject and object of observation, and qualitative research prevailing as a situated activity in which, to understand the dynamics of what one intends to research, one must go into the field to capture this object from the perspective of the people involved (Godoy, 1995).

Under this methodological perspective, in this article the look at validation in modeling activities, on the one hand, addresses a theoretical backround regarding this object and, on the other, is based on empirical research that investigates how students understand validation, the importance they attribute to it, and the way they carry it out in modeling activities. In this sense,

[2] A *framework* is a set of possibilities, ideas, or beliefs used to deal with some subject, thus being identified as a possible conceptual structure relating to that subject (Eisenhart, 1991).

[3] According to Duarte (2009), data triangulation concerns considering different dimensions of time, space, context, and analytical level from which the researcher seeks information for his research. It is, therefore, the simultaneous exposure of multiple realities by the person who investigates.

the research subject (the authors of the article) and the object of observation (the validation in modeling activities) are interconnected through an interpretative process.

The theoretical structure is organized using publications in national and international mathematics education scientific journals on validation in modeling activities, and includes the Kowalek (2021) research about validation in mathematical modeling activities. Research results by Kowalek (2021), Czocher (2013), Ikeda (2013), Czocher (2018), and Czocher, Stillman, and Brown (2018) point to the still initial discussion regarding validation in mathematical modeling activities.

The empirical research, in turn, was carried out with students of the fourth year of a mathematics degree during the discipline Mathematical Modeling from the Perspective of Mathematics Education, taught by one of the authors of this article in the second semester of 2021. Together, the literature reviewed and the data from empirical research allowed us to organize the framework intended to signal the current state of research on validation in mathematical modeling activities.

Mathematical modeling and validation

The premise that mathematical modeling seeks to understand, explain, or propose predictions for a situation of reality and that a mathematical model mediates these actions (Geiger et al., 2022) permeates what this article proposes to discuss.

These actions, however, are always linked to particular interpretations of the modeler, and the results reflect their hypotheses about how *reality works*[4]. In other words, what is observed is not precisely the reality but the reality revealed by the modeler's method of interpretation. Thus, modeling is a human practice in which, to a certain extent, to model, the modeler modifies this reality and no longer considers it static and completely representable. However, despite admitting that there is a reality that is independent of our understanding, Bunge (1985, p. 167) states: "If we did not believe in the existence of the external world nor in the possibility of knowing it, even in part, we would not strive to make neither theories nor experiments nor at least we would not achieve any success in our exploration." Indeed, as Bunge (1969) also mentions, although well-referred to as *the father of mathematical modeling*, Galileo effectively did not mathematize nature, but rather the experimental techniques he used to understand it.

[4] The expression *functioning of reality* refers to the idiosyncratic possibility of referring to and pondering how reality is (Hestenes, 2010).

In the educational context, mathematical modeling, although not rigorous and univocally characterized, in general terms, follows a design of activities that include well-defined steps (generally associated with a modeling cycle): (1) understand the reality situation; (2) simplify what is necessary (or convenient) in this situation; (3) mathematize the situation; (4) build a mathematical model and perform mathematical resolutions; (5) interpret the solution and answer; (6) validate the model and response; (7) communicate the results in the classroom (Almeida, 2022).

Managing expectations that students include these steps in their *to-do* modeling and how they move when going through them is relevant for the teacher who aims to understand and identify the students' actions. Despite this expectation, the teacher needs to consider that there are some idiosyncrasies of students in the sequence of these steps (Ferri, 2018) and that previous experiences with modeling and the mathematics used in the activity influence students' actions (Almeida, 2018; Niss, 2010).

Mathematical models, as units of knowledge resulting from the *manipulation* of reality, can constitute what Bunge (1969) calls *suggestive metaphors*. However, as Almeida (2010) suggests, metaphorical thinking is valuable for mathematical modeling, and the resulting models are generated recursively and with recurrences in two domains: mathematics and reality. Mathematical models constructed in this way are, according to Lesh et al. (2000), systems that (a) include variables, (b) include relationships between variables, (c) require operations that indicate how these variables relate to each other, and (d) follow well-defined rules and standards.

These models, on the one hand, reflect modelers' decisions on simplifications, variables, and mathematization considered for the reality situation, and, on the other hand, they are the instruments on which mathematical and non-mathematical interpretations and responses to what you want to know about this situation are based. Considering the model, the process of its construction, and the response it produces, validation in modeling has its main object in these aspects, and it (validation) is a crucial procedure among the stages of a modeling cycle.

But what composes validation? How to accomplish it? What is your purpose in the modeling activity? Deliberating on answers to these questions is a way to guide the search for ways to carry out validation in mathematical modeling.

Beyond the meaning expressed in dictionaries "to make it valid based on the rules in force" (Houaiss, 2021, p.1) and to make something legitimate (Ferreira, 2009), validation has received meanings in areas such as mathematics, computing, philosophy and, the one that interests us most directly, mathematical modeling.

Thacker et al. (2006), mentioning validation in mathematics, suggest that it is a process that aims to quantify the efficiency of mathematical techniques, methods, and results concerning what is proposed to be evaluated or measured. These techniques and methods can be like tools that support the construction of (mathematical) models and thus, indirectly, constitute the totality to which some type of judgment of the model is directed.

But, as Parker (2020) suggests, models, including mathematical models, cannot be validated considering the premises of their construction as sure and invariably true, nor can they be validated using their good performance in previous situations to guarantee their efficiency in new situations. Although they can become reliable when the results they provide fit the observed data, what the models indicate are always *inherently partial* answers and, as Almeida, Sousa, and Tortola (2021) suggest, they are provisional answers so that, in what can be said about a situation in reality based on a mathematical model, there is some temporality.

The question from which Parker (2020, p. 4) presents an interpretation he calls *philosophical* for validation is: What should be evaluated –and therefore confirmed, or refuted in the process of validating a model if not the model itself? His considerations on the topic led him to reflect that for validation (from this philosophical perspective), what the modeler must have in mind is called *suitability for the purpose* so that the model is validated through a "decision process to define under what circumstances and when the model is suitable for a given purpose" (Parker, 2020, p. 5). This implies, then, that validation cannot focus only on the model or the answer but must take into account the question, that is, what the model should answer, in other words, the purpose of its construction.

Aligned with Parker's (2020) assertions, Alvarado (2017) does not separate the validation of the model from the purposes of its construction. For the author, as a stimulus to critical thinking, validation concerns "a domain of applicability and within a satisfactory range of precision" (Alvarado, 2017, p. 32), with the validity of the model determined under the purposes of the modeling itself from which this model is the result. This author also suggests that "absolute validation does not exist, but some particular techniques are established to validate aspects of the activity" (Alvorado, 2017, p. 33).

For example, this broader vision for validation seems to be highlighted in works that deal with it in mathematics and computer sciences. Indeed, Thacker et al. (2006), Hallerstede et al. (2018), Pace (2004), Marchi (2015), and Elaasar (2018) identify two distinct procedures for evaluating mathematical models and their construction process: validation and verification. The first concerns the quantification of the accuracy of the model when compared to experimental data or to the reality situation to which it is associated, considering what of that situation was included in the model; the second aims to evaluate and remove numerical or conceptual errors and, in general, internal to the model and the mathematics on which it is based. Thus, the model itself is not the only object to which validation is directed, but it also includes the modeling process in which it is constructed.

Within the scope of mathematical modeling understood as an activity whose execution follows some steps, validation is admittedly included as one of them, being identified, however, traditionally, as a final step and interpreted, in most cases, as a check directed both to the model and its response (Blum & Leisβ, 2007; Ferri, 2018; Almeida, Silva & Vertuan, 2012; Ferreira & Silva, 2019).

Hidiroğlu and Güzel (2013) indicate that the focus of validation in mathematical modeling should be the analysis of how well the model and the solution fit the particularities of the situation to which the model is addressed. These authors also identify verification aimed at evaluating the model and, as a result, evaluating the response. Therefore, in the authors' view, verification is a validation sub-process. From their empirical research, the authors conclude that teachers participating in the study, in general terms, expect their students to compare procedures between the model and the experimental data and the answer obtained to the problem concerning the expectations of the students themselves and the teacher.

Recent research, e.g., Geiger et al. (2022) and Castro and Almeida (2023), have shown that decision-making permeates the different modeling stages as an objective-oriented activity without a priori schemes and answers perceived. These decision-making processes are also highlighted in Czocher's research (2018). This author characterized validation as a continuous process that spreads across different modeling stages and encourages students' decision-making.

Considering this expansion of the validation process, Czocher (2018), based on empirical research with students from an engineering course, characterizes five distinct types of validation (V_1, V_2, V_3, V_4, and V_5), allocates them to student actions between different stages of a modeling cycle (Figure 1), and characterizes them as follows: in type V1, the actions focus on the mathematical results and the calculations and mathematical procedures are evaluated; in type V_2 actions include evaluating the mathematical model, its parameters and variables as well as its articulation with what in the modeling cycle is characterized as a model of the situation and corresponds to the idealized situation; in type V_3, these actions differ from those of V_2 since they evaluate the mathematical model in relation to the characteristics of the real problem; in type V_4, the actions aim to evaluate the response obtained regarding the reality situation considering empirical expectations and information about the situation; finally type V_5

comprises actions to evaluate results in relation to data and known information about the reality situation.

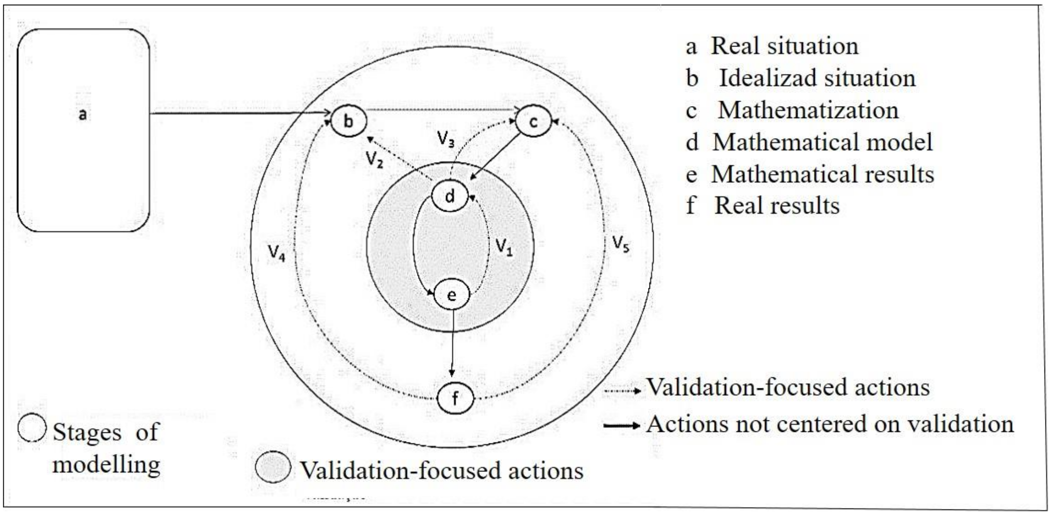


Figure 1.

Types of validation (adapted from Czocher, 2018, p. 151)

Validation appears as a process to analyze the procedures, the model, the results, and the answer to the problem related to the studied situation.

Ishibasshi and Uegatani (2019), based on mathematical modeling activities developed by basic education students in schools in Japan and seeking to identify the types of validation pointed out by Czocher (2018), concluded that students seem to create what the authors call a *fictional world* and validate the results obtained by modeling it so that an idealized situation becomes the parameter for analyzing the accuracy of the model. Continuing this research, Ishibasshi and Uegatani (2022) conclude that although we can talk about validation in a fictional world, validating can be a source of meaning for students, not just in the personal domain but also in the social domain, so that mathematical modeling goes beyond the classroom.

Students do not always perform the multifaceted actions indicated in the mentioned studies spontaneously. Thus, it is essential to recognize that validation requires students to follow complex procedures, which may fail without the teacher's intervention (Gurel & Bekdemir, 2022; Rehfeldt et al., 2019). In other words, validation cannot be neglected by the teacher. Still, they can offer means for the student to validate, verify, and make decisions in the modeling activity.

Empirical research

The empirical research was carried out with fourth-year students of a mathematics teaching degree course during the subject of "Mathematical Modeling from the Perspective of

Mathematics Education" taught by one of the authors of this article in the second semester of 2021, with the second author participating in all classes as a higher education practicum student[5]. The COVID-19 pandemic scenario at the time led classes to be held in a virtual environment through *Google Meet* and *Google Classroom.* The data on which the analytical process is based were obtained from transcriptions of class recordings generated by *Google Meet*, activity reports delivered by students in the *Google Classroom*, and students' responses to a questionnaire answered by the groups immediately after developing each activity and delivered in this virtual environment.

Twenty-four students enrolled in the course participated in the activities, identified by A_i, $1 \leq i \leq 24$. They formed seven groups identified as G_i, with $1 \leq i \leq 7$. The students organized in groups to develop one activity each besides a specific activity all groups developed. Considering the possible length of the text and the amount of data obtained with each group, we cite two activities, specified in Table 1.

[5] This is a mandatory activity for students in master's and doctoral programs and beneficiaries of a Capes scholarship.

Table 1.

The *mathematical modeling activities (produced by the authors)*

Activity theme	Students
Activity 1 The movement of the tide on Camboriú beach	A_1 and A_2 (G_1) A_3, A_4, A_5, A_6, A7, A_8 (G2 A_{10}, A_{11}, A_{12} (G3) A_{13}, A_{14} (G_4) A_{15}, A_{16}, A_{17}, A_{18} (G_5) A_{19}, A_{20}, A_{21} (G_6) A_{22}, A_{23}, A_{24} (G_7)
Activity 2 The giant zíper	A_5, A_{13}, A_{14}, A_{19}, A_{20}

Activity 1: The movement of the tide on camboriú beach[6]

Activity 1 was the first modeling that students developed in the subject. However, they had already read texts related to mathematical modeling and seen activities carried out by other students. In this way, they had information about possible stages of a modeling activity, for example. The teacher suggested the theme to the students. Given the news about the widening of the beach in the city of Camboriú, the students quickly became interested in the topic, even if, at first, they did not identify the situation of the tide movement as a problem that could be approached through mathematics.

Students perceived the possibility of understanding, explaining, or predicting specificities regarding this situation insofar as information and data about the enlargement work were shared. A fundamental aspect for the teacher and students to identify a problem in this situation was the information (Figure 2) about monitoring the work through cameras installed by the public authorities of the city along the length of the work. By interacting with this information and the dialogues between all the class participants, the problem was defined: Considering the height of the tide, at what time can the workers make more intense progress in the works to widen the beach's sand strip?

[6] Camboriú is a beach town in the state of Santa Catarina - Brazil

Tide movement at Camboriú Beach

High tide and low tide are a phenomenon associated with changes in sea water level and result in variations in the extent of sand not covered by water. This phenomenon is related to the lunar cycle. The explanation for this fact was given after Isaac Newton formulated the Law of Universal Gravitation, in 1687 (Any body exerts a gravitational force on another body that attracts it). The masses of the Sun and Moon exert forces that attract water from the oceans, thus causing the phenomenon of tides. The variation in the positions of the Moon and the Sun controls the tidal cycle, that is, the variation between low tide and high tide.

Since the works began at Camboriu beach, several communication channels have provided news about the progress of the works. But a difference is the installation of 13 cameras that show to population the progress of the work in real time. Through the camera images it is possible to observe how the work is carried out, which consists of enlarging the central beach by means of landfill in which sand is transported to the site through pipes connected to a dredger, for the entire length from the extreme point of Barra Sul to end of Barra Norte. The sand for the landfill will come from a deposit located at the bottom of the sea, 15km away. It is also possible to observe the factors that influence the work.

The students' problem: Considering the height of the tide, at what time can workers make the most intense progress in the works to widen the beach's sand strip?

Mathematical model: (f is the height of the wave and x is the time)

$$f(x)=\begin{cases}0,512+0,137sen(\pi x-\frac{3\pi}{2}) & \text{para } 1\le x<5 \text{ e } 9\le x<13;\\ 0,539+0,411sen(\pi x-\frac{3\pi}{2}) & \text{para } 5\le x<9 \text{ e } 13\le x<17;\end{cases}$$

Tide table for some days in August 2021 on Camboriú beach

Date	Time	Height (meters)
02-08-2021 (waning moon)	6h41min	0,3
	9h36min	0,7
	19h18min	0,4
	21h53min	0,6
09-08-2021 (new moon)	3h17min	1,0
	11h35min	0,01
	16h01min	1,0
	23h57min	0,2
16-08-2021 (waxing moon)	5h49min	0,4
	8h49min	0,7
	19h03min	0,4
	20h50min	0,6
23-08-2021 (full moon)	2h58min	0,9
	11h25min	0,1
	15h50min	0,9
	23h40mn	0,2

Figure 2.

Information from Activity 1 (produced by the authors)

After defining the problem, the students formed seven groups. These groups progressed in developing the activity through six classes (two per week). On each class day, part of the time was spent with all groups while another part was dedicated to specific work by each group in rooms created in the *Google Meet* classroom, whose recording was made available to the teacher.

The data-driven analytical process (transcriptions of class recordings, reports delivered by the groups, and responses to the questionnaire) in this activity aims to seek evidence in each group about (i) what students understand by validation, (ii) the importance they attribute to validation, and (iii) the way they carry out validation.

We read their report to infer what they understood by validation since they were asked to include what they had identified in each stage of the modeling. Table 2 contains information from five groups. Two groups did not include this information in the report.

Table 2.

What students understand by validation (reports delivered by groups)

Group	Excerpt from the report
G_1	We can say that validation is a moment to become aware of mistakes that have passed unnoticed both in the mathematical part and in formulating the answer to the problem.
G_2	Validation consists of a means of analyzing the response found and showing the relevance of hypotheses and simplifications.
G_4	Validation involves critically analyzing the answer to the problem to show whether it is consistent and appropriate.
G_6	Validation is the modeling stage in which it is necessary to analyze the mathematical representation, the mathematical model, found for the problem.
G_7	Validation consists of a review of the initial data along with hypotheses and mathematical processes used to develop the model. The model must be applicable and understandable.

Considering the data in Table 2, we can conclude that in this activity, students understand that validation (a) constitutes a moment to detect possible mistakes, whether mathematizing the situation or elaborating the answer to the problem (G_1), (b) consists of evaluating the answer obtained to the problem ($_G2$, G_4), (c) consists of analyzing the mathematical model (G_6), and (d) must include a broader review, including the data and mathematical processes used (G7).

To capture the importance that students attach to validation, we analyzed the answers to each questionnaire immediately after completing the activity. Students' answers are shown in Table 3.

Table 3.

Importance of validation in mathematical modeling activity (students' answers to the questionnaire)

Group	Answer to the questionnaire.
G1	Validation is essential to correct errors, especially when developing the model, as it must behave similarly to the given real data. Therefore, validation allows us to obtain a guarantee that the results found are consistent not only with mathematics but also with the problem solved by modeling.
G2	Validation is necessary because the evaluation can lead us to retake steps, reanalyze the data, change hypotheses, or modify the model. Therefore, it can be carried out several times during the activity.
G3	It is essential to validate the results found for the problem, but it is necessary to check whether the mathematical model is correct. Because if the model is not good, then answering the problem is useless.
G4	It is important to validate the results to demonstrate that the solution found is reasonable and consistent, therefore, reliable and appropriate for the problem.
G5	Validation results are crucial because they detect which model may not be the most appropriate for this situation; it can only adjust to a part of the data, which may generate different results from those expected.
G_6	Validating the results found for the problem is closely linked to validating the mathematical model; if the results are unsatisfactory according to the actual situation, the model must be reviewed, corrected, or reconstructed so the results are closer to the real data.

G7	Validation of the model is necessary to allow for the identification of errors and make necessary adjustments.

Students generally associate the importance of validation with the need to validate the mathematical model and its results. Some groups also refer to the analysis of the procedures used in building the model. One group also mentioned the need to validate procedures throughout the activity and not just at the end. Another group refers to the importance of validation as a means of offering a *guarantee* that the model and the response are adequate. In all answers, the relevance of validation is directed at the answer to the problem and the likelihood of the model results with the observed data, as if it were a comparison.

Finally, regarding how students carry out validation, our analytical movement considers the information from the delivered report and a mathematical modeling cycle that students were asked to include in the report, specifying what was done at each stage (Table 4).

Table 4.

Ways students carry out validation (prepared by the authors)

<table>
<tr><th>Group</th><th>Evaluation mode</th></tr>
<tr><td>G1</td><td>We validate the mathematical model to obtain the error between the observed data and the data estimated by the model. We also interpreted and analyzed the answer to the proposed problem and validated it by comparing it with internet data obtained from cameras installed throughout the area where the works were taking place.</td></tr>
<tr><td>G2</td><td>As results, we found that:
For days of a waning and waxing crescent moon, the most efficient and intense workers' working hours are between 6:41 am and 8:8 am and between 2:30 pm and 8:36 pm.
For full moon and new moon days, the most efficient and intense workers' working hours are between 7:26 am and 1:47 pm and 8:00 pm and 11:57 pm.
This is what we obtained from the model, but in the interpretation, we were not sure whether the work was actually at night at this time of the new moon. So, we only validated the model by comparison.</td></tr>
<tr><td>G3</td><td>We validated the model for the four phases of the moon (o quadro aqui inserido é da lua minguante e da lua crescente) and then looked at the workers' work reports to see if they worked during the shifts we found. The model is good, so we assume that the answer it will give is also good.
<table>
<tr><th>Data</th><th></th><th>Horário (horas/minutos)</th><th>Horári</th><th>Altura (metros)</th><th>Média</th><th>x</th><th>Erro</th></tr>
<tr><td>02/08/2021</td><td>1</td><td>6:41</td><td>6,68</td><td>0,3</td><td>0,375</td><td>0,375</td><td>0</td></tr>
<tr><td></td><td>2</td><td>9:36</td><td>9,6</td><td>0,7</td><td>0,65</td><td>0,649</td><td>0,15%</td></tr>
<tr><td></td><td>3</td><td>19:18</td><td>19,3</td><td>0,4</td><td>0,375</td><td>0,375</td><td>0</td></tr>
<tr><td></td><td>4</td><td>21:53</td><td>21,88</td><td>0,6</td><td>0,65</td><td>0,649</td><td>0,15%</td></tr>
<tr><td>16/08/2021</td><td>9</td><td>5:49</td><td>5,81</td><td>0,4</td><td>0,375</td><td>0,375</td><td>0</td></tr>
<tr><td></td><td>10</td><td>8:49</td><td>8,81</td><td>0,7</td><td>0,65</td><td>0,649</td><td>0,15%</td></tr>
<tr><td></td><td>11</td><td>19:03</td><td>19,05</td><td>0,4</td><td>0,375</td><td>0,375</td><td>0</td></tr>
<tr><td></td><td>12</td><td>20:50</td><td>20,83</td><td>0,6</td><td>0,65</td><td>0,649</td><td>0,15%</td></tr>
</table></td></tr>
<tr><td>G4</td><td>We have done:
Interpretation and validation of the answer to the problem –comparing the workers' working hours obtained by the model with those from the cameras installed by the city hall.
Model validation: analysis of the tidal height error (observed data) in relation to the results</td></tr>
</table>

	obtained by the model for this height.

G_5	The validation of the model was carried out by comparing the real-world data and those found by the model and occurred in the interpretation and validation phase of the results. We saw that the model efficiently responds to the proposed problem, as we reached an error of less than 1%, which is accepted. We made it clear that we disregarded days with climate changes, which are factors that influence the workers' working hours due to the tides. For the answer, we looked at the camera for a few days. Then, we saw that on September 14th, the machines carried out widening work in the morning until around 3 pm, which coincides with what happened in the model between the waxing and the full moon.
G_6	We validated the answer using data on how the workers worked and validated the model by comparing the found and the observed values. It is important to remember that other factors can interfere with the results, but we consider the model obtained to be adequate.
G_7	To validate the model and confirm the answer, we performed the following stages: - We reviewed the initial data together with the mathematical processes used in the development of the model; - We analyzed whether the model is applicable and understandable; - Comparing the data found at the beginning of the activity and the data obtained by the model, we observed little or no variation; - We concluded that the implementation of the model is reliable, so the answer to the problem is reliable.

Excerpts from the reports and the modeling cycle constructed for this activity indicate that in their actions towards validation, students include the evaluation of the mathematical model and the evaluation of the response. Regarding model validation, students compare the results with data relating to the height of the tide in the different phases of the moon obtained at the beginning of the activity.

To validate the response, however, the information that would enable them to validate (or not) the response was external to the activity. Indeed, the results obtained with the model were subjected to a comparison with information that was available on the city council site. Thus, this activity stage has a high degree of authenticity, as indicated by Almeida and Omodei (2022). Validation focuses on the purpose of building the model: determining the working hours of the construction workers. However, for the particular response, no attention was paid to the accuracy of this answer; but, given the available technique of evaluating the response in terms of video images available about workers' hours, the validation, although not absolute, was a moment of decision and reflection on the situation under study.

In general terms, the data obtained indicates that the actions related to validation occurred at the end of the activity. However, they used the expression *verification* to refer to the analysis of the mathematical model. Figure 3 provides a summary of the three aspects analyzed in this activity.

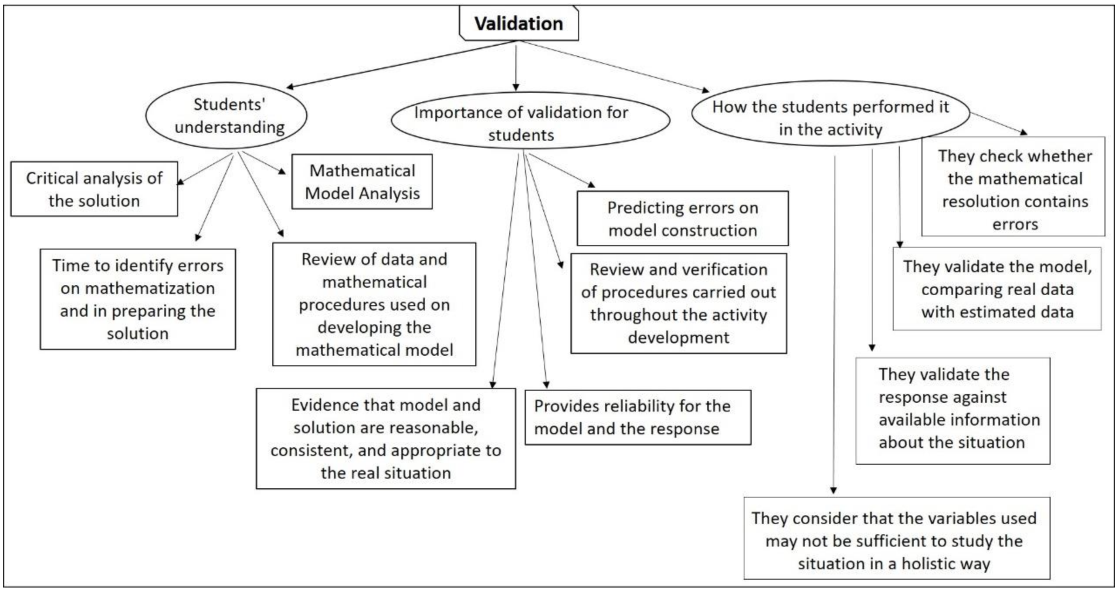


Figure 3.

Aspects of validation investigated in Activity 1 (produced by the authors)

Activity 2: the giant zipper

The activity with the theme *The Giant Zipper* was suggested by the teacher for the group with six students(A_5, A_{13}, A_{14}, A_{19}, A_{20} and A_{21}). The situation refers to a monument located at the entrance to the city of Cianorte[7], known as the *Capital do Vestuário* (The Clothing Capital). The monument is known as *Portal da Moda* (Fashion Portal) and contains an open zipper over the PR-323 highway supported, on one side, by a needle and a pin and, on the other side, by a cone.

The students had previous experience with modeling, and for the activity, the image (Figure 4) was made available to the group of students along with some questions, including: How long is this zipper? The students learned about the topic on the first day in the *Google Meet* classroom, which the teacher could record. Then, they defined steps such as the search for necessary auxiliary data, hypotheses, simplifications, and possible resolutions and deliberated

[7] Cianorte is a city in the north of Paraná, a Brazilian southern state. One of the city's main economic activities is clothing manufacturing. Probably for this reason, at the entrance to the city, there is a portal that includes various sewing utensils.

on how their answer could be validated. The students met two more times outside class hours, collected additional data about the highway where the portal is located, proposed a resolution and an answer to the problem, validated it, and, later, in a third class, presented it to the other students in the course.

The giant zipper

https://canaldacidade.com.br/monumento-ao-tropeiro-completou-55-anos/portal-cianorte-efc6e5c0-1/

Figure 4.

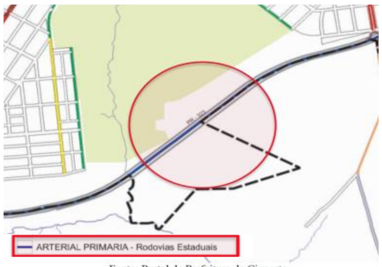
The situation to be studied (*survey data*)

For this activity, the gaze directed at the group's recordings of the classes and meetings, their report delivered, and the questionnaire students answered immediately after the last class aims to capture how the students carried out the validation as they had already answered two other items (what they understand by validation and what is its importance in modeling activities) in the activity about the movement of the tides.

Table 5.

Students' evaluation modes (*organized by the authors*)

Data source	**Purpose and action**	**Description**
Recording transcription	Define the reference measurement	A20: This value seems to be wrong! That can't be. A5: Yes, using the radar height won't work. (A5 refers to the radar shown in the image in Figure 4) A20: No! I think the radar is before the portal, so there is this difference in proportion. A5: True! They are different distances. A10: I think the radar being further ahead is not a good reference point. A20: Yes, we have to consider another point of reference. A10: It must be the width of the highway, see the photo (referring to the image in Figure 3).
Report	Define the reference measurement	To determine the actual length of the zipper, based on the image given, it is first necessary to establish a reference, which will be the basis for determining the measures required to solve the

		problem. At first, we thought about taking the radar as a reference, but we found that measuring the length of the radar would not be the most suitable since, in the image, the radar is in front of the portal, which possibly alters its size in the photo. That's why we researched and found data about the highway.  Fonte: Portal de Prefeitura de Cianorte
Report	Determine highway width	To check the measurement procedures used in the image to obtain the road width, we sought to find the actual width of the road. To do so, we initially researched the location/address of the portal. The research results showed that the road is PR-323 - Zone 3, Cianorte - PR, 87200-000 (Rodovia João Jorge Saad). Knowing that the portal is on PR-323, that is, on a state urban road, we tried to determine the type of this road. [...]. Thus, we determined that the width of each lane on the arterial road corresponds to 3.50 m.
Report	Validate the answer	Therefore, knowing that the actual length of the track is 3.5+3.5=7 *m*, we made the proportions with the measurements obtained in the image and obtained that the length of the zipper is approximately 28 m. To validate the zipper size, we used *Google Earth,* whereby we found the actual zipper length through the ruler of the system. We reached a value of approximately 28.05m, close to the length obtained through the proportion calculations. 
Recording transcript	Explain validation	Validation was the first of our procedures to obtain the 28m measurement, and we did this in two ways (the group referred to the two resolutions, one using a ruler to get the measurements and the other by inserting the image into the Geogebra software). After that, we used the measurement by *Google Earth* to check it. Because our interest was to know whether it was correct because 28 m is a lot for a portal over a 7 m highway. But then, there is something the teacher told us: Is our modeling no good for validating the measurement of *Google Earth?*

The information in Table 5 indicates that in this activity, signs of validation can be observed from the beginning, even though this may have occurred unconsciously by the students. At these moments, however, students would refer to verifications (of information and procedures). Validating the answer obtained (28 m) was important for the group, considering that the value alone did not have a meaning for them, but associating it with the size of the highway, for example, indicated the size of the portal. The group also used external information –measurement using Google Earth to validate the answer. Reflecting on a teacher's question was also relevant for the group to value their procedures and generate confidence in the methods used to obtain the answer.

Results – in search of a framework

The combination of arguments already shared in the literature brought to discussion in this article, with the results of empirical research aimed on validating in mathematical modeling activities, supports the organization of the framework relating to understandings, importance, and ways of carrying out validation in mathematical modeling activities.

In some aspects, the results regarding how students understand validation, the importance they attribute to it, and how they carry it out in the modeling activities they develop reflect elements already highlighted in previous research. However, in others, they bring specificities of validating that have not yet been highlighted.

The understanding and importance of validation for students, investigated in the activities they developed, reflect, on the one hand, verification signaling actions, such as identification of errors and their correction directed to mathematical procedures, as well as the analysis of the adequacy of data and hypotheses. On the other hand, they reveal students' emphasis on validation, giving it expressions such as guarantee, reliability, and relevance of the means used and the response obtained.

Indications that mathematical models produce *inherently partial* results, as Parker (2020) warns, that there is no absolute validation, as Alvarado (2017) suggests, and that modeling is relative to the world as *experienced by the modeler* (Hestenes, 2010), giving the model some temporality, as Almeida, Sousa and Tortola (2021) affirm, were not identified in the data collected from the students, even though they may have implicitly considered them.

However, students show an enthusiasm that makes them, through the ways in which they carry out validation, foster their decisions based on the means they use to validate. In both activities, the mathematical model is built with attention to the specificities of the situation, such as the phases of the moon in the activity of tidal movement and the search for reliable information about the width of the highway in the activity of the Cianorte city portal. The model and the answer are validated in both activities and include specific procedures. While validating the model in both activities, students consider mathematical and situational aspects. In validating the answer, they associate research beyond the modeling, seeking pertinent information outside the classroom.

Thus, while leading students to validate their internal procedures and the resulting final product (model and answer), validation also led them to reflect on external factors related to the activity, such as information on the city hall site and *Google Earth.* This aspect of validation was not identified in literature research, therefore being specific to the empirical research carried out here and opposite to what the work of Ishibasshi and Uegatani (2019) suggests, in which fictional worlds supported the validation process.

The empirical research indicates that the effectiveness of the model is associated with a purpose (determining workers' working hours and determining the size of the zipper), and validation must provide evidence that it meets this purpose. In other words, the acceptance or refutation of the model depends on the purposes for its construction. This question shifts the focus from validating the model itself to validating the modeling process as a whole since what could be considered the *purpose* is something defined, or even conjectured, during the different stages of the mathematical modeling activity. Therefore, modeling is the totality to which validation must be directed.

Although the analysis of the modeling activities referred to here was not aimed at identifying types of validation as specified in Czocher (2018), the information in Table 4 and Table 5 provides elements that indicate that students validated mathematical and non-mathematical procedures during the activity, verified calculations and data used, validating specific aspects of the modeling in addition to the answer obtained.

One framework relating to validation in modeling activities, including the research strategies undertaken in the article, is presented in Figure 4, revealing the understandings, importance, and ways of carrying out validation in activities of this type.

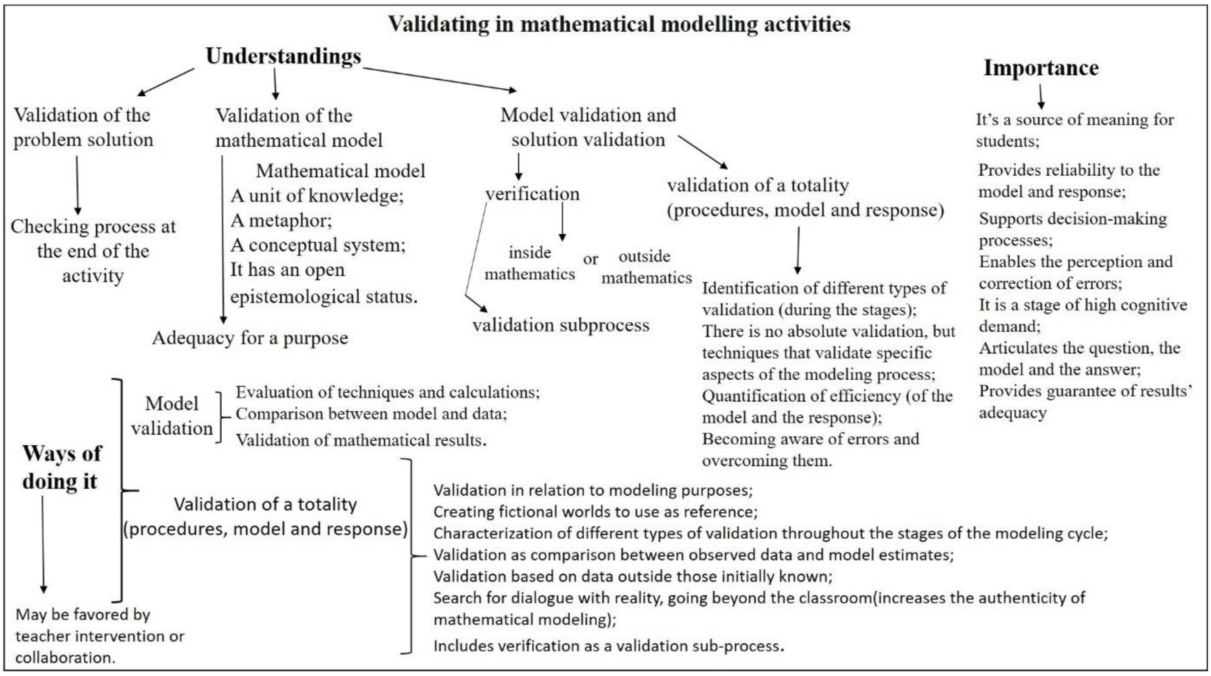


Figure 4.

Framework for validation in mathematical modeling (produced by the authors)

The framework we carried out points to what it means to validate in modeling activities as well as the ways to perform validation. On the one hand, the framework encompasses different points of view, including justification for some of them, but not others. Thus, it can serve as a guide for subsequent studies related to validating in mathematical modeling activities. On the other hand, what this framework point out can also be the subject of problematization in future research to maintain flexibility and debate within the scope of research in mathematical modeling.

Taking into account the various theorizations already noted in the literature and modeling practices, the framework proposed here indicates what is meant by validating and how validation occurs in the current historical and cultural moment and reflects the argumentative network that arises from the present research.

Final considerations

Although movements to promote the inclusion of mathematical modeling in the classroom are acknowledged, this inclusion, as a proposal that requires constructions and decisions for which there are no a priori schemes defined (Geiger et al., 2022; Almeida, 2018), still encounters resistance, including in indications, which can be observed in mathematics teaching materials (Doerr & Lesh, 2011; Mutti & Klüber, 2021).

Bring to discussion and knowledge the cyclical design, usually associated with modeling activities, can be a way to combat this resistance, helping to face the challenges that modeling poses. Even though each cycle stage has its specificities, validation, to some extent, functions as a regulatory action that demands decision-making and from which judgments arise. Thus, if mathematical modeling seeks to understand, explain, or propose predictions for a real-world situation, validation will indicate how well this explanation and predictions are carried out.

The purpose of the article, organizing and presenting a framework, is made possible through a data triangulation that adds to what is already recognized in the literature about validation, results of empirical research with data collected from mathematics teaching degree course students through different types of instruments. The result points to understandings, ways of doing things, and the significance of validation in mathematical modeling.

Although validation can be recognized as the final step in modeling cycles, it cannot be seen as a cumulative process that only begins at the end of the activity and from which only acceptance or refutation results. Instead, validation can act as an iterative agent, promote reflection, criticism, retakes, and revisions, and foster student success.

The results indicate that validation mechanisms can be activated at different stages of the modeling cycle. Sometimes, they can focus on mathematical procedures and evaluations, sometimes on the answer, or even on articulating, as intended, mathematics analysis and its impact on the answer. Thus, the verification process is distinguished from the validation process.

The constructed mathematical model, although supported by premises and hypotheses on the situation, only becomes the systemic structure it needs to be when rooted in mathematical operations and rules that, in a non-arbitrary way, consider the specificities of the situation based on the conjectures of the modeling students. Hence, they are approximations of reality and, therefore, have an epistemological status open to criticism and evaluation. Validation then acts as a quantification of the model's efficiency (Thacker et al., 2006).

We can conclude, however, that although specific actions are acknowledged as validation processes, on the validation of the totality (which includes the question to be answered, the model, its construction, and the answer it is capable of producing) lies the efficiency of validation as a means of generating reliability and, in the expression of the modeling students, *guarantee* that what can be said about a situation in reality through mathematical modeling constitutes an adequate answer, even if this adequacy may be incidental and provisional.

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