EMP Educação Matemática Pesquisa
ISSN 1983-3156
Qualis A1
http://dx.doi.org/10.23925/1983-3156.2025v27i1p217-246

Contribuições de uma organização de ensino de sistemas lineares via resolução de problemas no 2º ano do ensino médio

Contributions of teaching linear systems via problem solving in the 2nd grade of High School

Aportes de la enseñanza de sistemas lineales a través de la resolución de problemas en el 2do año de secundaria

Apports de l'enseignement des systèmes linéaires par la résolution de problèmes en 2ème année du lycée

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Abstract

The aim of this study was to analyze and describe the contributions of teaching via problem solving organization to learning the content of linear systems. The participants were 34 2nd grade high school students from a public school. The instruments and techniques used to record the data were: audio recording and transcription, photographs, student record sheets and field notes. The data analysis was based on descriptive and interpretive qualitative research and covered the students' performance and understanding in the four stages of the proposed teaching organization: (1) using the problem as a starting point; (2) concept formation; (3) content definition; and (4) application in new problems. The results showed that initially the groups of students used more trial and error, with little recourse to algebraic representations. In the following stages, the groups showed that they understood the conceptual aspects of linear equations and, above all, systems of equations, so that they were able to establish a relationship between conceptual language and formal language, based on algebraic representations. In stage 4, the groups performed well, but we found difficulties in the stages of executing and monitoring the problem-solving process. We conclude that the students surveyed demonstrated

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an understanding of the conceptual understanding and algorithmic processes of linear systems, revealing contributions from the proposed organization of teaching to encourage the construction of algebraic thinking.

Keywords: Mathematics teaching, Didactic sequence, Teaching approach, Linear systems.

Resumo

O objetivo deste estudo foi analisar e descrever as contribuições de uma organização de ensino via resolução de problemas para a aprendizagem do conteúdo de sistemas lineares. Os participantes foram 34 alunos do 2º ano do Ensino Médio de uma escola pública. Os instrumentos e técnicas de registro de dados foram: gravação e transcrição de áudios, fotografias, folhas de registros dos alunos e notas de campo. A análise de dados baseou-se na pesquisa qualitativa descritiva e interpretativa e abarcou o desempenho e compreensão dos alunos nas quatro etapas da organização de ensino proposta: (1) de uso do problema como ponto de partida; (2) de formação do conceito; (3) de definição do conteúdo; e (4) de aplicação em novos problemas. Os resultados mostraram que inicialmente os grupos de alunos utilizaram mais a estratégia de tentativa e erro, revelando pouca recorrência às representações algébricas. Nas etapas seguintes, os grupos revelaram compreender os aspectos conceituais de equações lineares e sobretudo de sistema de equações, de modo que puderam estabelecer relação entre a linguagem conceitual e a linguagem formal, baseados nas representações algébricas. Na etapa 4, os grupos tiveram bom desempenho, porém constatamos dificuldades nas etapas de execução e monitoramento do processo de resolução de problemas. Concluímos que os alunos demonstraram entendimento da compreensão conceitual e dos processos algorítmicos de sistemas lineares, revelando contribuições da proposta de organização do ensino para o favorecimento da construção do pensamento algébrico.

Palavras-chave: Ensino de matemática, Sequência didática, Abordagem de ensino, Sistemas lineares.

Resumen

El objetivo de este estudio fue analizar y describir las contribuciones de una organización docente a través de la resolución de problemas al aprendizaje del contenido de sistemas lineales. Los participantes fueron 34 alumnos de 2º año secundaria de un centro público. Los instrumentos y técnicas utilizados para registrar los datos fueron: grabación y transcripción de audio, fotografías, hojas de registro de los alumnos y notas de campo. El análisis de los datos se basó en la investigación cualitativa descriptiva e interpretativa y abarcó el desempeño y la comprensión de los alumnos en las cuatro etapas de la organización didáctica propuesta: (1) uso del problema como punto de partida; (2) formación del concepto; (3) definición del contenido; y (4) aplicación a nuevos problemas. Los resultados mostraron que, inicialmente, los grupos de alumnos utilizaron más bien una estrategia de ensayo y error, recurriendo poco a las representaciones algebraicas. En las etapas siguientes, los grupos mostraron una comprensión de los aspectos conceptuales de las ecuaciones lineales y, especialmente, de los sistemas de ecuaciones, de modo que fueron capaces de establecer una relación entre el lenguaje conceptual y el lenguaje formal, basado en representaciones algebraicas. En la etapa 4, los grupos tuvieron buen desempeño, pero encontramos dificultades en las etapas de ejecución y acompañamiento del proceso de resolución de problemas. Concluimos que los alumnos encuestados demostraron comprensión de los procesos de comprensión conceptual y algorítmica de sistemas lineales, revelando aportes de la organización de la enseñanza propuesta para incentivar la construcción del pensamiento algebraico.

Palabras clave: Enseñanza de las matemáticas, Secuencia didáctica, Enfoque de enseñanza, Sistemas lineales.

Résumé

L’objectif de cette étude était d’analyser et de décrire les contributions d’une organisation enseignante à travers la résolution de problèmes à l’apprentissage du contenu des systèmes linéaires. Les participants étaient 34 élèves de deuxième année de l'enseignement secondaire d'une école publique. Les instruments et les techniques utilisés pour enregistrer les données étaient les suivants : enregistrement et transcription audio, photographies, fiches d'élèves et notes de terrain. L'analyse des données était basée sur une recherche qualitative descriptive et interprétative et couvrait les performances et la compréhension des élèves dans les quatre étapes de l'organisation pédagogique proposée : (1) utiliser le problème comme point de départ; (2) former le concept ; (3) définir le contenu; et (4) l'appliquer à de nouveaux problèmes. Les résultats ont montré que, dans un premier temps, les groupes d'étudiants ont plutôt utilisé une stratégie d'essai et d'erreur, avec peu de recours aux représentations algébriques. Dans les étapes suivantes, les groupes ont montré qu'ils comprenaient les aspects conceptuels des équations linéaires et surtout des systèmes d'équations, de sorte qu'ils étaient capables d'établir une relation entre le langage conceptuel et le langage formel, basé sur des représentations algébriques. A l'étape 4, les groupes ont obtenu de bons résultats, mais nous avons constaté des difficultés dans les phases d'exécution et de suivi du processus de résolution de problèmes.

Nous concluons que les étudiants ont démontré une compréhension des processus conceptuels et algorithmiques des systèmes linéaires, révélant les contributions de la proposition d'organisation de l'enseignement pour encourager la construction de la pensée algébrique.

Mots-clés: Enseignement des mathématiques, Séquence didactique, Approche pédagogique, Systèmes linéaires.

Contributions of teaching linear systems via problem solving in the 2nd grade of high school

The BNCC – Base Nacional Comum Curricular (Brasil, 2018) leads that the teaching of Algebra has as one of its objectives the algebraic thinking, in a manner the student gets to understand, represent and evaluate problems involving quantities, in a way to represent them using the language of symbols and letters. In this sense, Coelho and Aguiar (2018) and Kuhn and Lima (2021) defended this algebraic thinking to make it possible to solve problems in different daily contexts.

However, the tests of Saeb 2019 (Brasil, 2019) and of Pisa 2018 (Brasil, 2021) revealed low rates of Mathematics proficiency by Brazilian students both in Elementary School and High School in the usage of Algebra. For Kuhn and Lima (2021), these results evidenced that possibly there is a fragmented, mechanical, out of context teaching of Algebra and which does not provide the student the development of competences concerning algebraic knowledge.

In this direction, studies such as that by (2018) and Proença et al. (2022), both of bibliographical nature referring to teaching involving algebra, showed teachers' postures still to be overcome. Borges (2018) indicated that the teachers are stuck to didactic books and algebraic techniques, and this is not presented in an interdisciplinary and contextualized manner. Proença et al. (2022) evidenced difficulties of the students for the understanding and problem solving, possibly caused by a teaching process which is based on revisitation and review of subjects, thus being necessary to approach the incorporation of a teaching approach which aims in larger degree the formation of concepts and not only procedures.

Specifically about the teaching and learning of the content linear systems, studied both in Elementary School (Battaglioli, 2008; Cataneo & Rauen, 2018; Negromonte et al., 2019; Valenzuela, 2007) and High School (Cunha Neto, 2020; Jordão, 2011; Maharani, 2020; Oktaç, 2018) pointed to a series of obstacles, such as: usage of didactic books stuck to algebraic calculations and students with difficulty to understand problems, which involves difficulties in the conceptual understanding and the usage of algebraic techniques.

Campos (2019) and Campos and Farias (2020) indicate that a path for the development of algebraic thinking (and consequently of linear systems), is in the activities proposal involving Arithmetics and Algebra with focus on problem solving, having in sight this allows establishing connections and relations in the path to reach the solution.

In this sense, the favoritism of algebraic thinking might be done through the usage of a teaching methodology which involves problem solving. Proença (2021) indicates a possibility for it, by presenting a proposal of teaching organization, which has as basis adopting the problem solving with focus on the conceptual development, making use of the for following stages: the usage of the problem as a starting point, concept formation, content definition, application in new problems. In terms of what the studies, previously cited, point out that following those stages appears as a possibility against the focus on teaching in techniques and mathematical procedures. As highlighted by Proença et al. (2022), this proposal brings the opportunity of the teacher approaching the conceptual development in tuning to the procedural knowledge of mathematical contents.

In this manner, our article has as objective an answer to the following guiding question: *which contributions are ranked from an organization of teaching via problem solving for the learning of the subject linear systems to the students of the 2nd grade of High School?* In order to do it, we structured the article in the following manner: the first section in which we present the theoretical references inherent to our research, in the following section, we describe the methodology we adopted, followed by the third section of results discussion, conclusions and references.

Algebraic thinking to understand linear systems

For Ponte, Branco and Matos (2009), the algebraic thinking is oriented in three demands of competences and skills: (i) representing - which involves algebraic reading, understanding, operation, translation and representation in different contexts; (ii) reasoning – understanding the capacity of realizing generalizations, understanding of rules and deductions; and (iii) solving problems and modeling situations – by mean of the usage of algebraic knowledge, functions, graphics, mathematical problem solving and of other domains. Ponte (2006) corroborates these ideas emphasizing that algebraic thinking must still concern establishing relations and representations between concrete and abstract objects. For Coelho and Aguiar (2018), algebraic thinking must have an object providing for the students to build a thought of search of analogies and patterns when facing problem situations of daily life.

In this context, the BNCC – Base Nacional Comum Curricular (Brasil, 2018) infers Algebra must have as objective the construction of the algebraic thinking in a path which allow them to understand, represent and evaluate problems involving quantities in a manner to represent them through a language of symbols and letters. Kuhn and Lima (2021), by reflecting on the construction of algebraic thinking in the BNCC, infer that for it to happen, students must be capable of formulate, employ and interpret Mathematics in different contexts. In this panorama, Pinheiro (2019) orients to the need that many schools must prepare in a way to implant teaching proposals which ensure the promotion of conditions that reach the skills and competences mentioned by the BNCC.

Concerning linear systems, the BNCC catches the attention that the student must be able to solve and elaborate daily problems, from Mathematics and of other areas of knowledge, which encompass linear equations making usage of algebraic and graphic techniques. In this sense, Pontes (2021) emphasizes that the teaching of linear systems has importance in measuring the students' comprehension and understanding to solve mathematical problems having knowledge of their applications in the daily context.

In this manner, the teaching of linear systems must allow a construction of algebraic thinking in the sense that the student is capable of reaching the demands pointed by Ponte, Branco and Matos (2009) as a form of: **representing –** through comprehension, understanding and acknowledgement of mathematical situations in different contexts which are equivalent to linear systems; **reasoning –** relative to the capacity of generalizing and planning strategies which allow solving linear problems; and, **solving problems** in a manner to use algebraic methods as an example, the Cramer's Theorem and the scaling for the resolution of linear systems as a manner to reach the solution of problems.

However, in Brazil, the teaching of linear systems presents a context of difficulties, starting by the didactic books. Battaglioli (2008) and Cataneo and Rauen (2018) when analyzing Mathematics didactic books for students of the 8th grade of Elementary School, pointed to the prioritization of algebraic calculations leaving aside contextualized problems. Furthermore, they perceived in the books the absence of activities of problem elaboration, graphic interpretation and inverse conversions.

In the work in the classroom, Delazari (2017), Negromonte et al. (2019) and Valenzuela (2007) and in the Elementary School and Jordão (2011) in the High School, present investigations in which the researched students presented difficulties in topics of linear systems, with obstacles in problem statement interpretation, in the resolution of equations with three unknown values and to classify in a correct manner a linear system.

This Brazilian panorama is also similar in pointings in international research both in Elementary School, High School and Higher Education. Maharani (2020) and Oktaç (2018) when researching High School students in Indonesia, observed students with obstacles to identify concepts and to solve problems which demanded more elaborated logical reasoning in the work with linear systems, these difficulties originated in algebraic contents from Elementary School.Reaching Higher Education, also in Indonesia, Dewi et al. (2021) investigated Mathematics teachers in formation pointing to individuals with difficulties to build relations between algebraic algorithms in the resolution of linear systems due to an immature conceptual knowledge. About this listed fact, the authors stress the need of research involving conceptual comprehension of algebraic contents, overall of linear systems.

In the aspect of teaching and learning of linear systems, the BNCC (Brasil, 2018) orients the teaching of linear systems must turn the student able to "solve and elaborate daily life problems, of Mathematics and of other areas of knowledge, which involve simultaneous linear equations, using algebraic and graphic techniques ". On the other hand, Proença, Campelo and Santos (2022) emphasize the BNCC does not bring the problem solving as a form of teaching, concerning only in taking students to an application and/or usage of the Mathematics learned daily in the classroom. In this sense, Proença, Campelo and Santos (2022, p. 17) argue that "if problem solving is integrated to the curriculum, then it must be understood as a strategy of teaching and learning, and not simply to apply contents".

Having in sight this panorama, we believe the approach of teaching by means of problem solving and with bias in the conceptual formation might be promising in the sense of changing this scene of difficulties pointed in the teaching of algebra and of linear systems. About this teaching approach is what we will discuss in the next subsection.

Problem solving in the teaching of Mathematics

About problem and exercise, Schoenfeld (1985) points that a problem generates an intellectual impasse in the person which tries to solve it, while in the exercise the solution scheme is easily accessed. Echeverría and Pozo (1998, p. 25) corroborate this idea stressing that for an exercise we already "dispose of and utilize mechanisms which lead us in an immediate manner, to the solution". In this manner, we perceive the function of an exercise in the teaching of Mathematics is linked to the application of formulas, algorithms, rules and models. About these ideas, Proença (2018, p.17-18) defends that:

> [...] a Mathematics situation becomes a problem when the person needs to mobilize mathematical concepts, procedures and principles previously learned to reach an answer. It is not about, however, the direct usage of a formula or known rules – when it occurs, the situation tends to be configured as an exercise (Proença, 2018, pp. 17-18).

In this panorama, Proença (2018) points to four stages for the resolution of a problem and which need the domain of cognitive knowledge, which we describe ahead:

Representation – which concerns the moment which the students understand or interpret the problem which they are trying to solve. In order to do it, it takes on their **linguistic**, **semantic and schematic knowledge**. In this sense, the **linguistic knowledge** encompasses the understanding of the mother language, of the recognition of the words and of the relation between subjects and objects. In their turn the **semantic knowledge** are those needed for the identification of mathematical terms in their relations. And, lastly, the **schematic knowledge** recognizes the essence of the problem based on previous mathematical knowledge, verifying then if the problem is of Geometry, Algebra, Arithmetics, among others (Proença, 2018).

Planning – it involves the usage of a strategy for problem solving. In order to do it, it is made the usage of **strategic knowledge** "[...] which suggests the act of generating and monitoring a plan of action" (Proença, 2018, p. 25). In this sense, the student can create strategies such as using trial and error, through drawing, Tables etc. *Execution* – it is the moment which the student executes the proposed strategy to solve the problem. In this manner, the **procedural knowledge** of the individual is necessary to use their logical thought in the search to establish spatial and quantitative relations (Proença, 2018). *Monitoring* – this last stage refers to the verification of the answer found in agreement to the question of the problem and the act of reviewing the followed solution (Proença, 2018).

About the teaching of Mathematics which utilizes the problem solving, we presented the proposal by Proença (2021) of teaching via problem solving, which corresponds to a proposal of organization of teaching, based on four stages which correspond to a sequence of classes to be approached by the teacher in the classroom to be known:

Stage 1 – Using the problem as a starting point: this first stage involves the work of teaching via problem solving, in which a problem is used as a starting point for a new mathematical content/concept/subject which are intended to be taught. For this stage, we take as reference the five actions of Teaching-Learning of Mathematics via Problem Solving (TLMvPS) by Proença (2018), to be known:

(a) *Choice of the problem:* in this action, the teacher must realize the choice of a mathematical situation (possible problem) with objective of guiding the students to use their previous mathematical knowledge, leading them to build the content/concept/subject which will be introduced and which provides that the students establish relations between this mathematical knowledge they use and the new knowledge learned. Still about this mathematical situation, Proença (2018) suggests it might present more than one answer, and which aim to predict different strategies to be adopted for their resolution.

(b) *Introduction of the problem:* this action involves the moment of contact of the teacher with the students in the environment of the classroom in which it is suggested the division of the students into smaller groups. Such facts aid the individuals to share their ideas, knowledge and experiences previously lived. The command by the teacher is that the students try to solve the manner they want or believe as being more convenient, being the moment the mathematical situation might be configured as a problem.

(c) *Aid to the students during the resolution:* In this third action , the teacher must aid the students about their frequent doubts, wrong interpretations and the rationality of the answer in the path of resolution. It is a moment in which the teacher acts as observer, encourager and director of the learning process. If necessary, the teacher could aid the students by means of strategies of resolution pre-established in the first action (of choice of the problem).

(d) *Discussion of the students' strategies:* this action approaches the socialization of the solutions adopted by the groups. Proença (2021) infers this moment is propitious so the students can perceive and build their relations between the knowledge they used. In this action, the teacher must point out difficulties and mistakes committed during the resolution and make the students synthesize what they understood and learned.

(e) *Articulation of the students' strategies to the content:* in this last action, the teachers must search to articulate the strategies used by the students to the mathematical content/concept/subject they want to introduce. About this moment, Proença (2018, p. 52) emphasized this looks for central points in the strategy(ies) to articulate a new content, discussing with the students the existing relation.

Stage 2 – Concept formation: This is the stage in which the students will develop their *mental constructs*, which are inherent to the construction of the learning of each person. Proença (2021, p. 04) stresses that for this moment "must start from some examples and non examples and create an activity to involve the students in the identification of characteristics which are part of the concept". In this sense, the author ranks some points to be followed in the activities to be developed by students as:

> a) exploring examples and non examples of concept to explore their characteristics; b) presenting a definition of the concept (mental construct), what will evidence their characteristics of the concept they mention; c) presenting other kinds or variations of concept, that is, presenting examples of the concept; d) presenting non examples of the concept, which allows amplifying the conceptual comprehension, because, as an example, to know what is a second degree equation, it is also important knowing what is not (Proença, 2021, p. 09).

Stage 3 – Content definition: This third stage involves the work of the teacher related to the definition and presentation of the algorithms which concern the content. Proença (2021) points out that in this moment, it must be approached the definition of the mathematical concept and of the algorithmic procedures of resolution. The author orients in this stage the teacher must have a concern in establishing the relation between formal mathematical language and the language adopted in the activities of conceptual formation of the second stage, focused on the collective debate about their understanding on mathematical structure until reaching a synthesis between the adopted representations.

Stage 4 – Application to new problems: Proença (2021) infers that this fourth and last stage is that which the teachers prepare their classes based on the usage of new and different situations which understand new problems or possible problems with the objective of transfer the students' learning, of the mathematical concept and the studied algorithmic processes. This stage is oriented towards an approach of teaching *for* problem solving, but the focus occurs due to the conceptual formation, being necessary to observe and evaluate the students in the process of problem solving. Lastly, Proença (2021) highlights that for this stage, the worked mathematical situations must be contextualized involving daily life, social life, politics, economics and different scientific areas subjects.

Methodology

The present research involves the classroom environment, in which the researcher (first author) was the teacher of the class involved. Thus, the study had as participants 34 students (22 of the female gender and 12 of the male gender) of the 2^{nd} grade of High School in a public school of the State of Paraná. These participants and their legal guardians signed the TCLE – Termo de Consentimento Livre e Esclarecido (Informed Consent Form) and the TALE – Termo de Assentimento Livre e Esclarecido (Informed Assent Form). Such process was previously followed by the Comitê Permanente de Ética em Pesquisa com Seres Humanos da Universidade Estadual de Maringá – Paraná (Permanent Ethics Committee for Research with Human Beings of the State University of Maringá – Paraná),under the substantiated opinion number 5,716,066 granted by the institution. In this regard, the identity of all investigated students was preserved, with the designation A1, A2, A3, and so on.

The adopted instruments of data collection and registry were: recording and transcription of the classes, pictures of the realization of the activities, students' activity sheets and field notes. In this panorama, Duranti (1997) emphasizes the recording and transcription of audio open ways to the understanding of how the individuals use the speech and other quotidian instruments. According to Martins (2008), the usage of photos in social research helps in the understanding of comprehension of human and social phenomena in a process which opens ways to analyze questions and experiments. Lastly, Godoy (1995, p. 29) justifies that the usage of field notes allows the researcher to identify "dimensions, categories, trends, patterns and relations, unveiling their meanings".

In the classroom, we elaborated and developed a proposal of teaching about the content of linear systems, taking as North the four stages of teaching organization via problem solving, proposed by Proença (2021). Table 1 shows the chronogram with the activities worked in the classroom in each stage:

Table 1.

Chronogram of activities realized in the classroom (Organized by the authors 2025)

Stages by Proença (2021)	Classes	Mathematical Content	Description of the activities
Usage of the problem as a starting point	1, 2, 3 and 4	● Linear systems with 3 equations and 3 unknowns	***Situation 1:*** *Solving the problem equivalent to a linear system of the2x2 kind.* ***Situation 2:*** Solving the problem equivalent to a phased linear system 3x3.
Concept Formation	5 and 6	● Linear equation and its characteristics. ● Solution of a linear equation.	***Activities 1:*** Given Table 1 with linear equations and Table 2 with nonlinear equations, pointing to the characteristics observed from a linear equation and a nonlinear equation. ***Activities 2:*** Given the items a, b and c, signal which equations are linear and justify which are not. ***Activities 3:*** Discussing the meaning of solving a linear equation.
	7 and 8	● Linear system and its characteristics. ● Solution of a linear system.	***Activities 4:*** Review the linear systems in the situations 1 and 2 and point out the characteristics observed. ***Activities 5:*** Giving examples of linear systems of the 2x2 and 3x3 kinds. ***Activities 6:*** Presenting the definition of the solution of a linear system and analyzing an example. ***Activities 7:*** Realizing the synthesis of the studied concepts of linear equation, linear system and its solutions.
Content Definition	9 and 10	● Linear system mxn. ● Solution of a linear system. ● Linear system 3x3. ● Classification of a linear system. ● Cramer 's Theorem.	***Activity 8:*** Reading the formal definition of: a linear system mxn; the solution of a linear system; and of a 3x3 of a linear system 3x3. Presenting the classifications of a linear system. Solving linear systems using Cramer's Theorem.
	13 and 14	● Phasing of linear systems.	***Atividade 9:*** Solving linear systems using Phasing.
Application in new problems	15 and 16	Problems involving linear systems	***Atividade 10:*** Solving diverse problems equivalent to linear systems.

Table 2 ahead shows overall the proposed activities by the students during the classes of our teaching organization:

Table 2.

Activities developed in the 4 stages of teaching organization (organized by the authors (2025)

STAGE 1 – USAGE OF THE PROBLEM AS STARTING POINT

Situation 1: At the Bookstore A, Paulo made two budgets:
Budget 1: a rubber from the brand Pitágoras and a pencil of the brand Newton. **Total:** R$8.00.
Budget 2: two rubbers from the brand Pitágoras and four pencils from the brand Newton. **Total:** R$27.50.
What are the prices of the rubber Pitágoras and of the pencil Newton at Bookstore A?

Situation 2: On the other hand, at |Bookstore B, Paulo made three more budgets:
Budget 1: a rubber from the brand Pitágoras, two pencils from the brand Newton and a pen from the brand Leibniz. **Total:** R$9.00.
Budget 2: two pencils from the brand Newton and two pens from the brand Leibniz. **Total:** R$11.00.
Budget 3: three pens from the brand Leibniz. **Total:** R$10.50.
What are the prices of the Pitágoras rubber, the Newton pencil and the Leibniz pen at Bookstore B?

STAGE 2 – CONCEPT FORMATION

Activity 1:
a) In the Quardo1 below, we observe examples of equations we can call linear:

QUADRO 1
$I)\ x + 4y = 10$
$II)\ a + 2b + c = \frac{-1}{5}$
$III)\ \frac{m}{2} - 4n = -2{,}5$
$IV)\ \sqrt{3}x - 1{,}15y + \pi z = 7$

What characteristics do you observe a linear equation **MAY** have?

b) Now, in Quadro 2 we have equations which are **NONLINEAR**:

QUADRO 2
$I)\ x.y.z = 8$
$II)\ a^2 + 2b^3 + c = -9$
$III)\ \frac{5}{t} + 10u - v^3 = 0$
$IV)\ \sqrt{3}d^2 - 9{,}5w + \pi z = 7$

What characteristics do you observe a linear equation **CAN NOT** have?

Activity 2: Check which equations below are non linear, then justify why the other equations are non linear.

$A)\ (\ \ )\ x + y + z = 10$
$B)\ (\ \ )\ a^2 + b^3 + \sqrt{c} = 25$
$C)\ (\ \ )\ m + np = 10$
$D)\ (\ \ )\ a^3 + b^2 = 10$
$E)\ (\ \ )\ x + y - 9z = \frac{3}{5}$
$F)\ (\ \ )\ \frac{10}{a} + b = 6$
$D)\ (\ \ )\ a^3 + b^2 = 10$
$E)\ (\ \ )\ x + y - 9z = \frac{3}{5}$
$F)\ (\ \ )\ \frac{10}{a} + b = 6$

Activity 3
a) In the linear equation $x + y = 10$, what values for x and for y turn the linear equation into a true sentence?
b) And in the linear equation $a + b - c = 0$, what values for a, b and c turn the linear equation into a true sentence?
c) What do the numeric values which turn the linear equations into true mathematical sentences mean to them?
d) Observing the items a and b what can you observe in relation to the amount of solutions in a linear equation?

Activity 4: Let us review the systems of linear equations which we have seen in previous classes:
Linear system from Situation 1: **Linear system from Situation 2:**

$$\begin{cases} b + l = 8 \\ 2b + 4l = 27{,}50 \end{cases} \qquad \begin{cases} b + 2l + c = 9 \\ 2l + 2c = 11 \\ 3c = 10{,}5 \end{cases}$$

Answer: Which characteristics do you observe in the mathematical form of a linear system?

Activity 5: Give an example:
a) Of a linear system with two equations and two unknowns.
b) Of a linear system with three equations and three unknowns.

Activity 6:
In the previous class you understood the concept of solving a linear equation, how would you explain what is a solution of a linear system now?

Activity 7:
Having as basis what you studied in the last two classes. Write in your own words, what do you understand?
a) Linear equation.
b) Solution of a linear equation.
c) Linear system.
d) Solution of a linear equation.

STAGE 3 – CONTENT DEFINITION
Linear System m x n
A group of **m** linear equations and **n** unknowns $x_1, x_2, \ldots, x_n$ is called a **linear system m** x **n** (IEZZI et al., p. 103, 2016).
Solution of a linear system
We say that a sequence of real numbers $(\alpha_1\ \alpha_2, \ldots, \alpha_n)$ is **solution of a linear system** of **n** unknowns if the solution is of each one of the equations of the system (IEZZI et al., p. 103, 2016).
Classification of a Linear System
The board below presents the classifications, possible abbreviations of a linear system in relation to its solution:

Classification	**Abbreviation**	**Solution**
Possible and Determined System	PDS	Allows a single solution.
Possible and Undetermined System	PUS	Allows infinite solutions.
Impossible System	IS	Does not allow any solution.

Activity 8: Solve the linear systems below using Cramer's Theorem:

$$A)\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ 4x + y + 2z = 12 \end{cases} \quad B)\begin{cases} x + 2y + z = 9 \\ 3x + y = -5 \\ 2y + z = 11 \end{cases} \quad C)\begin{cases} 2x - y + z = 3 \\ x + y + 2z = 9 \\ -x + 3y + z = 7 \end{cases}$$

Activity 9: Phase, classify and solve the linear systems below:

$$A)\begin{cases} x + 2y + z = 8 \\ 2x + 5y - z = 9 \\ 3x + y + z = 8 \end{cases} \quad B)\begin{cases} 2x + y - z = 8 \\ x - 3y = 0 \\ 3x - 2y - z = 2 \end{cases} \quad C)\begin{cases} x + y + z = 6 \\ 3x - y + z = 2 \\ 4x + 4y + 4z = 2 \end{cases}$$

$$D)\begin{cases} x + 2y + z = 9 \\ 2x + y - z = 3 \\ 3x - y - 2z = -4 \end{cases} \quad E)\begin{cases} x + y + z = 1 \\ 2x + y - 3z = 4 \\ 3x + 2y - 2z = 5 \end{cases}$$

STAGE 4 – APPLICATION IN NEW PROBLEMS
Activity 10: Transform the enunciations of the questions ahead in a linear system (those which still do not have one). Then, solve the systems using Cramer's Theorem or the algebraic technique of phasing you learned in the previous classes.
A) (FUVEST/1992 - Adapted)
Carlos and his sister Andreia went with their dog Bidu to their grandfather 's pharmacy. There they found an old scale with malfunction, which only correctly indicated weights superior to 60 kg. Then, they weighed themselves two at a time and obtained the following marks:
● Carlos and the dog together weight 87 kg;

● Carlos and Andreia weight 123 kg;
● Andreia and Bidu weight 66 kg.
Determine the weight of each one of them.
B) (UFMS/2018 - Adapted) The system ahead was built based on the monthly sales of three salesmen (A, B and C), in which the values of x, y and z are the quantities sold by each salesman.

$$\begin{cases} A: x + 2y + 3z = 14 \\ B: 2x - 3y + 2z = 2 \\ C: -2x + y - 5z = -15 \end{cases}$$

Determine the product of the sales of the three salesmen.
C) (ENEM/2020-Adapted) In a country, the traffic violations are classified accordingly to their gravity. Violations of the *light* and *medium* kinds add, respectively, 3 and 4 points to the licence of the perpetrator, besides the fines to be paid. A driver committed 5 traffic violations. Consequently he had 17 points added to his licence.What is the reason between the number of violations of the *light* and the number of *medium* kinds committed by this driver?
D) (UFRN/2001- Adapted) Three friends, denominated X, Y and Z, utilized the computer every night. In relation to time in hours, in which each one uses the computer, each night, it is known: • X's time plus Z's time exceeds Y's time by 2; • X's time plus four times Z's time is equal plus 3 plus double Y's time; • X's time plus 9 times Z's time exceeds 10 Y's time.. Determine the sum of the number of hours using the computer by the three friends each night.

As it was discussed in Table 1, we highlight that 16 classes of 50 minutes happened, divided into two daily classes on Tuesdays and Thursdays between the months of July and September of 2022. However, these days encompassed a rainy period in which it was not possible to establish the same number of participant teams in the four stages researched, due to the lack of students in the days of our research. Because of this, the analysis boards of the next section presented different numbers of teams participating for each analyzed stage.

In face of the classes realized in our proposal of teaching organization, the present research is of qualitative approach, having in sight the collected data aimed the students' comprehensions of the concept of linear systems, It is of inductive and descriptive characters, in which the teacher of the class is the main actor, due to his interpretations (Bogdan; Biklen, 1994). With this, the procedures of data analysis of the data generated by our study were based on the descriptive and interpretative analysis. For Gil (2007), the descriptive research aims overall at realizing the description of the characteristics of a phenomenon or of a population or of the relations which are established between them. On the other hand, the interpretative character, according to Rosenthal (2014), brings new possibilities to the investigator, allowing him other views for the phenomena investigated, rebuilding correlations and the senses of concrete particular cases.

Results and discussions

In this section we are going to present the results and discussion found in our research. In order to do it, we separated the analysis by stage in a manner to contribute with the best understanding by the reader.

Stage analysis 1 – using the problem as a starting point for the teaching of linear systems

In relation to the results of stage 1 of *using the problem as a starting point* of our teaching organization, Table 3 ahead shows how the performance of the 15 participant teams was relative to the situations 1 and 2, in which the number in front of each item represents the quantity of teams referring to this item:

Table 3.

Performance of the 15 teams in the resolution of the situations 1 and 2 from stage 1 (Created by the authors 2025)

	Stages of Problem Solving					
Activity	*Representation*	*Planning*	*Execution*	*Monitoring*	Strategy and procedure of execution	FS
Situation 1	☑ 14 x 00 NF 01	☑ 14 x 00 NF 01	☑ 14 x 00 NF 01	☑ 14 x 00 NF 01	TE: 10 EQ+TE+TAB: 01 SL+MS: 02 TE+SL+MS: 01	☑ 14 NF 01
Situation 2	☑ 14 x 00 NF 01	☑ 14 x 00 NF 01	☑ 14 x 02 NF 01	☑ 14 x 02 NF 01	TE: 10 EQ+MS: 04 NF: 01	☑ 12 x 02 NF 01

Caption:
EQ: Equation; MS: Method of Substitution; TE: Trial and error; TAB: Table;
FS: Final Solution; ☑: Correct; x: Incorrect; NF: Did not do.

Table demonstrates that from the 15 participant teams, 14 were correct in situation 1, obtaining success in the four stages of the problem solving (Proença, 2018). Between the strategy and the procedure of execution adopted: 10 teams used the strategy of trial and error: two teams wrote a linear system and used the method of substitution to solve it; and two teams used more than one strategy as a path to find the solution. Lastly, a team did not do the resolution of situation 1.

In relation to situation 2, we verified that 12 of the 15 participant teams did the resolution correctly, reaching the correct final solution. However, as happened in situation 1, in situation 2, 10 teams also favored the strategy of solving through trial and error in a manner to reach the correct resolution of the proposed problem. There was also the occurrence of four teams which chose to write equations for the proposed problem and used the method of substitution to solve it. However, we still observed that two teams had difficulties in **procedural knowledge** in the stage of *execution* and, still in the stage of *monitoring the* problem solving by not reviewing the rationality of the answer found, in a way they did not manage to reach the correct final solution. Lastly, only one team did not manage to realize the activity of solving situation 2.

About the results in this stage, we see the usage of the trial and error strategy prevailed , followed by 10 of the 15 participant teams for the resolution of situations 1 and 2. In this sense, Maharani (2020), in the research with 40 students of High School from Indonesia, verified students with limitations to choose strategies to solve linear system problems. Falcon (2009), when investigating nine students from High School in the State of Texas in the United States of America, observed they would rather use the strategy of trial and error for problem solving equivalent to linear systems before having contact with such content. In this panorama, Töman and Gökburun (2022) when investigated 82 students from High School in Turkey about algebra knowledge and the establishment of algebraic thinking, perceived students with low level os algebraic thinking associated to obstacles in the transition of Arithmetics into Algebra.

Our research in its turn, the difficult of the algebraic thinking was given in three more demands advocated by Ponte, Branco and Matos (2009): in the demand of **representing,** because the students were not capable of translating the informations from the mother language to the algebraic language using their previous knowledge; in the demand of **reasoning,** because they had difficulties in relating the situation proposed to an algebraic content; and, lastly, in the demand of **solving problems,** due to not adopting their algebraic knowledge for the resolution of the proposed problems. About this fact, Töman and Gökburun (2022) suggest with emphasis the need of a teaching and learning process with connection between the algebraic contents based on studies made in this direction.

Analysis of stage 2 – forming concepts of linear systems

Stage 2, of *concept formation*, had a moment in which the students were able to point out the characteristics, the examples and non examples, the definition of the concept and its variations for the content of linear equations and its solutions and of linear systems and its solutions. Chat 4 below clarifies how the performance and the comprehension of the 14 participant teams happened when they realized the activities 1 to 7 from stage 2:

Table 4.

Performance and understanding of the 14 teams in the activities in stage 2 (Created by the Authors 2025)

Activities	Aspects of the Concept	Answers/ pointed ideas	NT
Activity 1			
Pointing Characteristics of EL	Pertinent Characteristics	*The variable must have exponent 1.*	11
		The numerical coefficients are real numbers	10
		The EL may have addition and subtraction	8

		operations between the terms.	
	Irrelevant characteristics	*We can use different letters in an EL.*	8
		The variables may be in the numerator.	5
		We have positive and negative signals in the EL.	4
Pointing characteristics of nonlinear equations	Pertinent characteristics	*The variables may have exponent different from 1.*	12
		The variables may be in the denominator.	10
		We can multiply the variables.	10
Activity 2			
2.1 Signal the EL and justify the nonlinear equations	Pertinent comprehension	Correctly signaled item a.	14
		Correctly justified items b and c.	10
2.2 Signal the EL and justify the nonlinear equations	Pertinent comprehension	Correctly signaled item e.	12
		Correctly justified items d and f.	12
2.3 Signal the EL and justify the nonlinear equations	Pertinent comprehension	Correctly signaled the items h and i.	11
		Correctly justified item g.	11
Activity 3			
a) Pointing the solution of an EL	Pertinent comprehension	*Presented correct answers.*	14
b) Pointing the solution of an EL	Pertinent comprehension	*Presented correct answers.*	13
c) Pointing the meanings of values which turn an EL into a true mathematical sentence	Pertinent characteristics	*Turn the equation into a true one.*	2
		The solutions of the equations.	10
		Found the result.	1
	No answer	*Did not answer.*	1
d) Pointing the amount of solutions an EL has	Pertinent characteristics	*Has infinite solutions.*	8
		Has various solutions.	3
		Has more than one solution.	1
		There are many and diverse solutions.	1
	No answer	Did not answer.	1
Activity 4			
Pointing the characteristics observed in the SL	Pertinent characteristics	*SL are formed by EL.*	10
		The EL of the SL are grouped in keys.	9
Activity 5			
a) Presenting an example of a 2x2 SL	Pertinent comprehension	Presented the example correctly.	11
		The example presented is partially correct.	4
b) Presented an example of a 3x3 SL	Pertinent comprehension	Presented the example correctly.	14
	Partial comprehension	The example presented is partially correct	1
Activity 6			
Pointing the concept of solution of a SL	Pertinent characteristics	*The solution makes every equation a true one.*	4
		It is turning all equations true.	4
		The solution of an SL is the solution of every equation in this system.	2
		Turns a sentence true.	2
		Makes you discover the unknowns.	1
		The solution is turning all the sentences as true ones.	2
Activity 7			
a) Presenting with your own words what is an EL	Pertinent characteristics	*They are 1st degree equations.*	13
		Contains operations of addition and subtraction.	5
		The variable stays in the numerator of a fraction.	5
		There can not exist multiplication between the unknowns.	4
	Irrelevant	*We can use different letters for the unknown.*	3

	characteristics		
b) Presenting with your own words what is the solution of an EL	Pertinent characteristics	*It makes the equation true.*	15
		Substitution of numbers for unknowns which made the result true.	3
c) Presenting with your own words what is a SL	Pertinent characteristics	*It is a group of EL.*	14
		Grouped by a key.	3
d) Presenting with your own words what is the solution of a SL	Pertinent characteristics	*Makes all the EL from the SL true.*	13

Caption:
NT: Number of teams who answered; EL: Linear equation; SL: Linear system.

Table 4 allows us verifying that by mean of the seven proposed activities, the researched teams could realize the pointing of different characteristics, examples and non-examples both of linear equations and their solutions and linear systems and their solutions, in a way that in the end they presented the definition of the concept (mind construct) of this studied content. On the other hand, in a look at the literature, Dewi et al. (2021) and Oktaç (2018) when researching students of Elementary School, High School and Higher Education pointed to individuals with obstacles of concepts and linear systems and obstacles by establishing relations between algorithms inherent to this content. About this fact, the authors input as the main cause of the immature conceptual knowledge of the students inherent to the content of linear systems. In this sense, we believe that stage 2 of *concept formation* allowed us to ease these difficulties by having in sight the comprehension and understanding of the different concepts ranked by Table 4.

In this context, such a stage also allowed advancing in the development of the algebraic thinking of our investigated teams. This can be observed in the demand of **representing** pointed by Ponte, Branco and Matos (2009), having in sight the students were able to acknowledge and interpret the concepts of linear systems in different contexts, since the characteristics, presentation of examples and non-examples and the definition for the concept (mind construct). In this panorama, we still observed the advance in the demand for reasoning, because the research teams were able to generalize the concept of linear systems by pointing examples and non-examples of what was and what was not a linear system.

Analysis of stage 3 – realizing the content definition of linear systems

This stage of our organization aimed to bring the formal definition and the algorithmic processes concerning the content of linear systems. In this sense, we search to, according to what is oriented by Proença (2021), establishing relations between mathematical language of this stage with the mathematical language of the stage of conceptual formation (stage 2) in a manner to promote collective debates on the understanding of the mathematical structure and the different representations.

About the classes of stage 3, in a manner of presenting the formal definitions and the algorithmic processes inherent to linear systems, the teacher explained on the board the item *a* from activity 8, referring to Cramer's Theorem and, in the same manner, the items *a, b* and *c* of activity 9 encompassing the resolutive process of phasing. In this manner, the students stayed with the items *b* and *c* of activity 8 and with the items *d* and *e* of activity 9 as the proposed and analyzed activities for this stage. Table 5 below demonstrates the performance of the 10 teams of students in the realization of the items proposed in activities 9 and 10:

Table 5.

Performance of the 10 teams in the resolution of the activities from stage 3 (Created by the authors 2025)

Activities	**Performance**	**NT**	**Difficulties**
Activity 8 Solving and classifying the 3x3 SL using CT			
b) $\{ x + 2y + z = 9\ 3x + y = -5\ 2y + z = 11$	Correct resolution	7	-
	Did not do	3	-
c) $\{ 2x - y + z = 3\ x + y + 2z = 9\ -x + 3y + z = 7$	Correct resolution	5	-
	Incorrect resolution	3	Mistakes in algebraic calculation.
	Did not do	2	-
Activity 9 Solving and classifying the 3x3 SL using the PM			
d) $\{x + 2y + z = 9\ 2x + y - z = 3\ 3x - y - 2z = -4$	Correct resolution	5	-
	Partially correct resolution	1	Did not write the solution and did not classify the SL.
	Incorrect resolution	4	Errors in algebraic calculation.
e) $\{x + y + z = 1\ 2x + y - 3z = 4\ 3x + 2y - 2z = 5$	Correct resolution	5	-
	Partially correct resolution	1	Did not write the solution and did not classify the SL.
	Incorrect resolution	4	Errors in algebraic calculation.

Caption:
NT: Number of teams; SL: Linear Sistema Linear; CT: Cramer 's Theorem; Phasing Method.

By means of Table 5, we see that in activity 8, 7 of the 10 participant teams did the correct resolution of item b, being that 3 teams did not do their resolution. For item c in its turn, 5 teams realized the correct resolution, 2 did not do it and 3 teams did the resolutive process in a wrong way pointing to obstacles in **procedural knowledge**. Figure 1 below, shows the difficulty in team 3's resolution in the item c of activity 8:

:

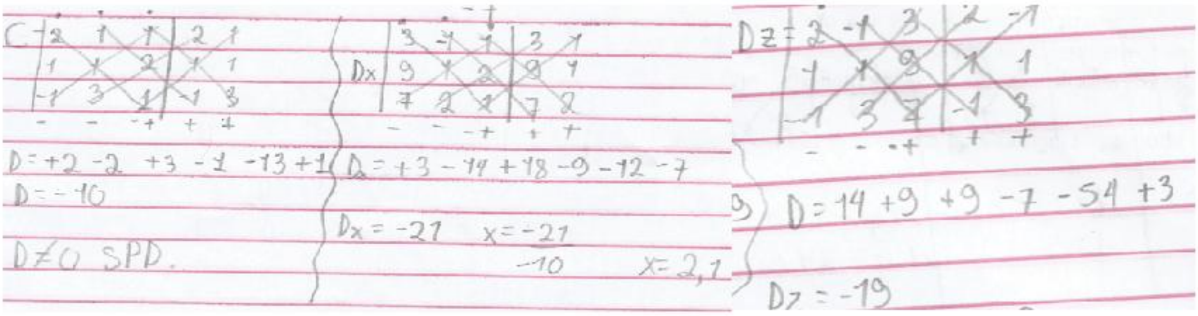

Figure 1.

Team 3's difficulties in activity 8-c (Created by the authors 2025)

By means of Figure 1 we see team 3 had mistaken the calculation of the determiners D, Dx and Dz. In determiner D, the mistake occurred by putting the coefficient 1 for the monomial -y in the first line. In Dx, the mistake happened in the third line in which the correct coefficient of the monomial 3y was 3 and not 2 as noted by the team. The mistake in the determiner Dz in its turn occurred due to the signal of the term -7 which would be +7.

Advancing to activity 9 which involved the resolution of linear systems by means of the algebraic method of phasing, we see in Table 5 that 5 of the 10 researched teams realized in a correct manner the resolution of the items d and e. On the other hand, a team did not present a solution, and the classification of the linear systems of the items d and e occasioned a partially correct resolution for the activity. We still had 4 teams which realized the incorrect resolution of these items also indicated obstacles in **procedural knowledge.** Figure 2 ahead demonstrates the errors in algebraic manipulation of Team 2 in the resolution of the item and of activity 9:

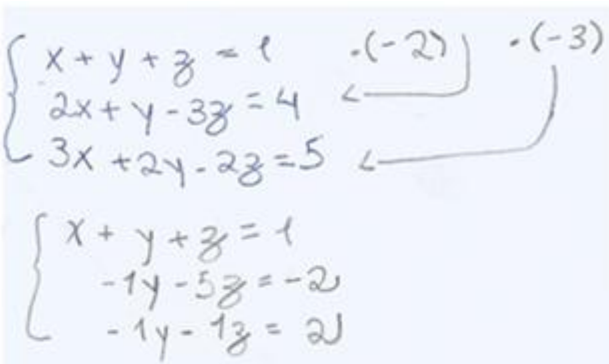

Figure 2.

Difficulty of team 2 in activity 9-e (Created by the authors 2025)

In Figure 2 above we verified that team 2 committed mistakes in the elimination of the unknown x in the second and third lines of the linear system which led them to find the equation

-1y-5z=2 in equation II which the correct independent term is -2. In the same manner, the team pointed to -1y-1z=2 for equation III which the correct equation would be -1y-5z=2. All those mistakes of algebraic manipulation in activities 8 and 9 point to the researched students' difficulties converging in **procedural knowledge.**

Likewise, inside this context of difficulties found by our study, Cury (2013) and Freitas and Guadagnini (2013) when investigating algebraic contents with students from the 9th grade of Elementary School pointed to students with algebraic mistakes coming from algebraic manipulation. For Cury (2013, p. 17) these results "demonstrate these students still did not develop algebraic thinking, because they do not know how to express abstractions and generalizations". Advancing to the High School, Fonseca (2016) in his research with the content of arithmetic progression with students of the 2nd grade of High School, found mistakes in algebraic calculations in the proposed activities. About this fact, the author points to the need of future works which come to contour such obstacles pointed.

For our research, we believe that the obstacles found in the resolution of the activities done by the researched teams in activities 8 and 9, can be avoided by means of improving the construction of the algebraic thinking of the students in the demand of **reasoning** advocated by Ponte, Branco and Matos (2009). Having in sight the need of the comprehension of the rules and algorithms needed for the algebraic manipulation demanded by the method of Cramer's Theorem and of phasing in the resolution of linear systems.

Analysis of stage 4 – application in new problems of linear systems

About that moment, the student did the resolution of activity 10 which involved problem solving which demanded usage of linear systems. Table 6 ahead presents the development of the 15 teams which participated in this stage in activity 10. In Table 6, the number in front of each item represents the number of teams which refer to this item:

Table 6.

Performance of the 15 teams in the resolution of activity 10 of stage 4 (Created by the Authors 2025)

	Stages of Problem Solving					
Activity 10	*REPRES*	*PLAN*	*EXEC*	*MONIT*	Strategy and procedure of execution	FS
Item a expected SL	☑ 12 x 00 NF 03	☑ 12 x 00 NF 03	☑ 08 x 04 NF 03	☑ 08 x 04 NF 03	SL + SM: 09 SL + PM: 01 SL + TE: 01 SL + CT: 01	☑ 08 x 04 NF 03

$\{x + y = 87\ x + z = 123\ y + z = 66$					NF: 03	
Item b Expected answer: Product: x.y.z = 6	☑ 15 x 00 NF 00	☑ 15 x 00 NF 00	☑ 13 x 02 NF 00	☑ 01 x 14 NF 00	SL + PM: 15	☑ 01 x 14 NF 00
Item c SL and expected answer: $\{L + M = 5\ 3L + 4M = 17\ \frac{L}{M} = \frac{3}{2}$	☑ 13 x 00 NF 02	☑ 13 x 00 NF 02	☑ 11 x 02 NF 02	☑ 04 x 09 NF 02	SL + AM: 08 SL + SM: 01 SL + CT: 02 TE: 02 NF: 02	☑ 04 x 09 NF 02
Item d SL and expected answer: $\{x + z = y + 2\ x + 4z = 3 + 2y\ x + 9z = y + 10$ $x + y + z = 6$	☑ 14 x 00 NF 01	☑ 14 x 00 NF 01	☑ 14 x 00 NF 01	☑ 04 x 10 NF 01	SL + PM: 14 NF: 01	☑ 04 x 10 NF 01

Caption:
SL: Linear System; AM: Addition Method; SM: Substitution Method;
PM: Phasing Method; TE: Trial and error; TC: Cramer's Theorem.
FS: Final solution; ☑: Correct; x: Incorrect; NF: Did not do.

Through Table 6, we verified that in item a, 12 of the 15 participant teams had success in the stages of *representation* and of *planning* of solving the proposed problem. However, only eight teams succeeded in the stages of *execution* and *monitoring* considering that four teams had difficulties in algebraic calculation, indicating an obstacle in **procedural knowledge**. On the other hand, nine teams adopted the strategy of writing a linear system and utilizing the substitution method for the resolution of the problem, indicating maturing in the demands of **representing, reasoning** and **solving problems** in the construction of the algebraic thinking, advocated by Ponte, Branco and Matos (2009).

This maturing got more evident in item b, in which we perceived that the 15 teams adopted the strategy of writing a linear system equivalent to the problem proposed for, then, use the resolutive method of phasing for its resolution, being that all the 15 teams had success in the stages of *representation* and of *planning* of the resolution of the proposed problem. However, when they reach the stage of *execution*, two teams made mistakes in algebraic calculation, revealing, more than once, difficulties in the usage of **procedural knowledge.** In the state of *monitoring*, 14 of the 15 investigated teams were not aware about the fact of the enunciation demanding the proposed value of the product x.y.z and ended missing the final solution of what was proposed when they did not review if the answer was in accord to the command demanded by the enunciation.

Similarly in item c, 13 teams succeeded in the stages of *representation* and of *planning* of the resolution of the proposed problem and, once again, 2 teams had obstacles in the stage of *execution* and 9 teams in the stage of *monitoring* by not presenting the reason between the number of light and medium violations as asked in the enunciation.

Lastly, in item d we observed that 14 of the 15 teams adopted the strategy of writing a linear system and solving it by means of phasing doing the correct stages of *representation, planning* and *execution* of the problem resolution. However, once more obstacles occurred in the stage of *monitoring*, in which 10 teams mistaken the final answer by not observing the problem asked for the value of the sum of the variables x+y+z.

About the difficulties in the stage of *execution* (usage of **procedural knowledge**) in activity 10, we observed in a similar way that Ramos and Curi (2015) when they worked the contents of Algebra with students from the 1st grade of High School also pointed to investigated students with obstacles with the execution of algebraic calculation. In the same manner, Amka (2020) researched High School students from Indonesia about the problem solving which encompassed linear systems with three equations and three unknowns, indicating some students with obstacles in the process of execution and phasing and which were not able to substitute variables, not fulfilling then, the strategy of planned solving for the complete and correct resolution of the problem.

In this context, Rakhmawati et al. (2019) also performed a similar investigation with 30 High School students from Indonesia and pointed to students with difficulties in the resolution of mathematical calculation (*execution* stage of problem solving) and in order to verify the veracity of the answer found (*monitoring* stage*)*. About this fact we believe such obstacles need to be investigated and approached in future research to be understood and treated. As a form to control this difficulties, Lima (2019) points out to the need to use methodologies different from the traditional with rescue of mathematical contents from Elementary School and of High School, and Ramos and Curi (2015) emphasize the demand for a work focused in the analysis of mistakes in a manner the teacher is able to aid their students in overcoming difficulties.

About the difficulties in the stage of *monitoring* in the items b, c and d, Roberts and Le Roux (2019, p. 13, our translation) when they investigated students from 8th and 9th grades of Elementary School in South Africa, pointed that “for all the students, the ending of the calculation is the appearance of the solution”. In this sense, we see the intervention and aid by the teacher is needed in the teaching approach in a manner the students are able to overcome these obstacles.

In a general view of this stage 4, we observed a path into the construction of the researched students' algebraic thinking caused by the transfer of the learning of contents coming from our approach of teaching in stage 2 (*concept formation)* and in stage 3 (*content definition).* Such a fact is observed by the choice of most of the teams in the strategy of writing a linear system and using an algebraic method for the resolution of the proposed problems, greatly advancing the demands of **representing, reasoning** and **solving problems** pointed by Ponte, Branco and Matos (2009). However, the difficulties found in the stages of *execution* and *monitoring* in problem solving with obstacles in **procedural knowledge** need to be researched in a way to be circumvented and rebuilt.

Conclusion

This work had as objective answering the following guiding question: *which contributions are ranked from an organization of teaching via problem solving for the learning of the subject linear systems to the students of the 2nd grade of High School?*
Thus, our research had as North the four stages of organization of teaching via problem solving, proposed by Proença (2021).

The first stage, of *usage of the problem as starting point*, pointed to teams which, in their majority, opted for strategies of problem solving by means of trial and error, being absent the demand of **representing** the algebraic thinking oriented by Ponte, Branco and Matos (2009). In this sense, we had two teams with difficulty in the areas of *execution* and of *monitoring* in the problem solving proposed with obstacles in **procedural knowledge** However, we saw that 14 and 12 teams got it right, respectively, the resolution of situations 1 and 2 demonstrating success in every stage of the problem solving.

Advancing to the second stage, of *concept formation,* we could observe the students could present a definition (mental construct) of the concept of linear system, revealing the development of algebraic thinking in the scope of **representing**. This was evidenced in the realization of the activities, in which the studies indicated their comprehension and understanding by pointing characteristics in a correct manner, examples and non-examples of the learned concept.

Moving forward to the third stage, the *content definition,* we verified the students were able to establish relations between formal mathematical language and the mathematical language used in the *concept formation* stage, what we can point as what amplified the comprehension of the students on **representing algebraic** thinking. However, the obstacles in algebraic manipulations with origin in the usage of **procedural knowledge** (proper to the algorithmic calculations) were found in the realized activities. These difficulties occurred mainly in the work of the resolution process of linear systems with three linear equations and three unknowns by using algorithms of escalation and Cramer's Rule, revealing difficulties concerning the **reasoning** of algebraic thinking.

Finally, the fourth and last stages of *application in new problems*, showed that most of the students adopted the strategy of writing a linear system orienting to the build the demand of **representing algebraic** thinking. Then, for the resolution of the proposed problems, we see the teams adopt in their majority an algebraic strategy (Cramer's Theorem, escalation, addition or replacement method) demonstrating the construction of the demand **reasoning** of the algebraic thinking. Lastly, most of the research teams realized a resolution passing through the stages of *representation, planning, execution and monitoring* of the problem solving in a manner to get to the solution of the mathematical orienting, thus, to the demands of **reasoning** and **solving problems** into the construction of algebraic thinking.

However, in answer to our research question, we can rank contributions of the proposed teaching from the subject of linear systems which concern to contributions for the field of research, to be known: a) in the teaching of algebra and the construction of the algebraic thinking, the study made it possible the comprehension and understanding by the students of mathematical concepts and of LE and LS and the capacity of relating them to algorithmic processes of algebraic manipulations allowing, then, the resolution of problems in different contexts; b) in the teaching of linear systems, the contribution points to a non-fragmented teaching approach, which privileged the realization of contextualized activities and not only stuck to algebraic calculations, in a manner to ease the previously cited obstacles in our theoretical references, culminating to the development of the algebraic thinking by the students; c) in relation to the problem solving in the teaching, one of the contributions we observed is the conceptual formation proposed by the work by (2021) and that was not approached by other studies encompassing the teaching and learning of linear systems, which contributed strongly to the development of the algebraic though in the scope of **representing**; c) lastly, one of the important contributions of this study were pointing the difficulties related to the stages of *execution* inherent to **procedural knowledge** and of *monitoring* in the problem solving of linear systems, which needed to be studied, with the objective of being rebuild for an even larger advance in the demands of **reasoning** and **problem solving** concerning algebraic thinking.

About the limitations found in the research, a limitation happened due to the fact that we investigated students of the 2nd grade of High School which had the 9th grade of Elementary School and a good part of the 1st grade of High School by mean of emergency remote education due to the Covid-19 pandemic, the students revealed difficulties in mobilizing previous knowledge, overall with the usage of algebraic strategies. In this manner, the obstacles cited by the authors of our bibliographic review about the teaching of Algebra and linear systems were, in our study, reviewed and the conceptual and procedural knowledge were empowered.

Therefore, we expect our study works as a base to new research which sustains the investigation in the elaboration and realization of proposals of teaching via problem solving in the four stages of teaching organization by Proença (2021). This is because this teaching organization works as a base to Lay the foundations of the development of algebraic thinking since the resolution of a problem as the starting point, passing, over all, through the conceptual formation (still not much explored in the classes and in works present in scientific literature) and favoring the usage of mathematical concept and procedures in the resolution of new problems

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