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Rational number interpretations as one of the central elements in the development of proportional reasoning: an approach with Frac-Soma

Iinterpretaciones de números racionales como uno de los elementos centrales del desarrollo del razonamiento proporcional: una aproximación con Frac-Soma

Interprétation des nombres rationnels comme l'un des éléments centraux du développement du raisonnement proportionnel : une approche avec Frac-Soma

Interpretações do número racional como um dos elementos centrais do desenvolvimento do raciocínio proporcional: uma abordagem com o Frac-Soma

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Abstract

This article aims to analyze understandings of students of the 7th grade of elementary school when solving activities that emphasize the rational number interpretations, quotient and operator. For this purpose, a qualitative approach was employed to assess 11 activities that utilized Frac-Soma and were conducted with a class of ten students from a public school in Sobradinho, RS. Data production considered protocols, recordings (audio and video), photographs, and the teacher/" 'researcher's logbook. Among the results was that notions related to fair sharing were understood, as the students established connections between the requested quantities and the partitioning process, necessary for understanding the quotient interpretation and the development of proportional reasoning. On the other hand, obstacles were identified in relation to the comparison notion, as conclusions were not systematized regarding

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the notions of “the more”/“the less” one quantity is greater/smaller than another. To minimize these difficulties, during the teacher/” ’researcher’s interventions, it was necessary to emphasize the unitization process, which is fundamental to understanding the equivalence notion and this to proportional reasoning. In activities involving operator interpretation, it was found that in the action of partitioning the integer, as well as “exchanging” Frac-Soma parts for others, there was a “loss” of the unit reference. After discussions in the groups and teacher/researcher interventions, signs of understanding that the operator is a function capable of transforming the unit into another similar one were noticed. Thus, it could be concluded that the students presented an understanding of these rational number interpretations, although the notions of operator and comparison are still a challenge for some.

Keywords: Mathematics education, Partitioning, Sharing, Comparison, Unitization.

Resumen

Este artículo tiene como objetivo analizar las comprensiones de estudiantes de 7º grado al resolver actividades que enfatizan las interpretaciones del cociente y del operador de números racionales. Para ello, se optó por un abordaje cualitativo para apreciar 11 actividades que hicieron uso del Frac-Soma y fueron dinamizadas en una clase de diez estudiantes de una escuela pública de Sobradinho/RS. La producción de datos consideró protocolos, grabaciones (audio y vídeo), fotografías y el cuaderno de bitácora del profesor/investigador. Entre los resultados, se constató la comprensión de nociones relacionadas con el reparto equitativo, ya que los estudiantes establecieron conexiones entre las cantidades solicitadas y el proceso de partición, necesarias para la comprensión de la interpretación del cociente y el desarrollo del razonamiento proporcional. Por otro lado, se identificaron obstáculos en relación con la noción de comparación, ya que no se sistematizaron las conclusiones sobre las nociones de "cuánto más"/"cuánto menos" una cantidad es mayor/menor que la otra. Para minimizar estas dificultades, durante las intervenciones del profesor/investigador fue necesario hacer hincapié en el proceso de unitización, fundamental para comprender la noción de equivalencia y ésta para el razonamiento proporcional. En las actividades que involucraron la interpretación del operador, se constató que, en la acción de partición del entero, así como en el "cambio" de pedazos de Frac-Soma por otros, hubo "pérdida" de la unidad de referencia. Después de las discusiones en los grupos y de las intervenciones del profesor/investigador, se evidenció la comprensión del operador como una función capaz de transformar la unidad en otra semejante. Así, se puede concluir que los estudiantes presentaron comprensiones sobre estas interpretaciones del número racional, aunque las nociones de operador y comparación aún sea un desafío para algunos.

Palabras clave: Educación matemática, Particionamiento, Compartir, Comparación, Unitización.

Résumé

Cet article vise à analyser les connaissances des élèves de 7e année lors de la résolution d'activités qui mettent l'accent sur les interprétations du quotient et de l'opérateur des nombres rationnels. À cette fin, une approche qualitative a été choisie pour évaluer 11 activités utilisant Frac-Soma et dynamisées dans une classe de 10 élèves d'une école publique de Sobradinho/RS. La production de données a pris en compte les protocoles, les enregistrements (audio et vidéo), les photographies et le journal de bord de l'enseignant/chercheur. Parmi les résultats, il a été constaté que les notions liées au partage équitable étaient comprises, car les élèves établissaient des liens entre les quantités demandées et le processus de partition, nécessaires à la compréhension de l'interprétation du quotient et au développement du raisonnement proportionnel. En revanche, des obstacles ont été identifiés en ce qui concerne la notion de comparaison, car les conclusions n'ont pas été systématisées sur les notions de "combien de plus"/"combien de moins" une quantité est plus grande/moins grande que l'autre. Pour minimiser ces difficultés, lors des interventions de l'enseignant/chercheur, il a été nécessaire de mettre l'accent sur le processus da nonutilisation, qui est fondamental pour comprendre la notion d'équivalence et donc le raisonnement proportionnel. Dans les activités impliquant l'interprétation de l'opérateur, il a été constaté que l'action de partitionner l'entier, ainsi que "l'échange" de morceaux de Frac-Soma contre d'autres, entraînait une "perte" de la référence à l'unité. Après les discussions au sein des groupes et les interventions de l'enseignant-chercheur, il est apparu que l'opérateur comprenait qu'il s'agissait d'une fonction capable de transformer l'unité en une autre unité similaire. On peut donc conclure que les étudiants ont compris ces interprétations des nombres rationnels, même si les notions d'opérateur et de comparaison restent un défi pour certains d'entre eux.

Mots-clés: Enseignement des mathématiques, Cloisonnement, Partage, Comparaison, Nom utilisation.

Resumo

Este artigo objetiva analisar os conhecimentos produzidos por alunos do 7º ano do ensino fundamental ao resolverem atividades que enfatizam as interpretações do número racional quociente e operador. A pesquisa é norteada por uma abordagem qualitativa e a produção de dados considerou protocolos de 11 atividades que foram desenvolvidas em uma turma de dez alunos de uma escola pública de Sobradinho/RS, além de gravações (áudio e vídeo), fotografias e diário de bordo da professora/pesquisadora. Dentre os resultados, constatou-se que noções relativas à partilha justa foram compreendidas, pois os alunos estabeleceram conexões entre as quantidades solicitadas e o processo de particionamento, necessárias à compreensão da interpretação quociente e ao desenvolvimento do raciocínio proporcional. Em contrapartida, identificaram-se obstáculos em relação à noção de comparação, pois não foram sistematizadas conclusões acerca das noções de "quanto mais/menos" uma quantidade é maior/menor que a outra. Para minimizar essas dificuldades, durante as intervenções da professora/pesquisadora, foi necessário enfatizar o processo de unitização, fundamental à compreensão da noção de equivalência e ao raciocínio proporcional. Nas atividades envolvendo a interpretação operador, verificou-se que na ação de particionar o inteiro, bem como "trocar" peças do Frac-Soma por outras, ocorreu "perda" da referência da unidade. Após discussões nos grupos e intervenções da professora/pesquisadora, perceberam-se indícios de entendimentos de operador como uma função capaz de transformar a unidade em outra semelhante. Em conclusão, observou-se que os alunos apresentaram entendimentos sobre essas interpretações do número racional, embora, as noções de operador e comparação ainda sejam um desafio para alguns.

Palavras-chave: Educação matemática, Particionamento, Partilha, Comparação, Unitização.

Rational number interpretations as one of the central elements in the development of proportional reasoning: an approach with Frac-Soma

The development of proportional reasoning has been the subject of studies by international and national researchers (Lesh, Post & Behr, 1988; Lamon, 2007, 2008; Oliveira, 2009; Maranhão & Machado, 2011; Oliveira, 2014; Soares, 2016) due to its importance for mathematics, social practices, and other areas of knowledge. It is known that numerous situations in these fields require a qualitative and quantitative analysis of the investigated phenomenon, and generally, these analyses reveal proportional principles. For Lesh, Post, and Behr (1988, p. 1), proportional reasoning is:

> a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. [...] Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought. [...] the essential characteristics of proportional reasoning involve reasoning about the holistic relationship between two rational expressions such as rates, ratios, quotients, and fractions. This invariably encompasses the mental assimilation and synthesis of various components of these expressions and an aptitude to infer about the equality or inequality of pairs or series of these expressions, based on analysis and synthesis. It also involves the ability to successfully produce the missing components, regardless of the numerical aspects of the problem.

From this perspective, proportional reasoning encompasses concepts of covariance (recognizing that the variation of variables occurs together), enabling multiple comparisons (both numerical and non-numerical) between rational expressions to assess their equality or inequality. Furthermore, it implies the skillful resolution of situations with missing values without relying on numerical values. In this conceptualization, researchers highlight a connection between proportional reasoning and the various meanings/interpretations of rational numbers in fractional representation (rate, ratio, quotient, fraction), with the term “fraction” approaching the concept of part-whole.

Lamon (2007, 2008) also mentions this relationship. According to the researcher, understanding rational numbers and the associated multiplicative concepts is directly linked to the development of proportional reasoning. Likewise, the ability to reason proportionally serves as an indicator of the comprehension of these numbers. Thus, it is necessary to understand that in a single symbol $\left(\frac{a}{b}\right)$ lies a world of meanings, multiple interpretations, representations, and ways of thinking and operating (Lamon, 2008). The existence of this diversity of meanings/interpretations and representations, as noted by Graça, Ponte, and Guerreiro (2021), is the source of most of the difficulties students face in the study of rational numbers.

Another factor that makes the understanding of rational numbers more complex, and consequently the development of proportional reasoning, as various researchers indicate (Lamon, 2007, 2008; Oliveira, 2014; Soares, 2016; Graça, Ponte, & Guerreiro, 2021), relates to the emphasis placed on only one of the meanings/interpretations of rational numbers in fractional representations, namely, the part-whole relationship. According to Lamon (2008), students whose teaching and learning process centered on the part-whole interpretation[4] have a limited understanding of rational numbers and, consequently, encounter difficulties in mobilizing aspects related to proportional reasoning.

In this context, the teaching and learning process needs to consider working with the different interpretations of rational numbers, namely: part-whole[5], measurement[6], ratio[7], quotient[8], and operator[9]. This is because it enhances the understanding of which arithmetic operations are valid when dealing with some of these interpretations in mathematical problems, thereby expanding knowledge of this numerical set and stimulating the development/mobilization of aspects of proportional reasoning (Oliveira, 2014). Associated with the various interpretations of rational numbers, Lamon (2007, 2008) describes central structures of mathematical knowledge that interrelate to form a network of concepts, contexts, representations, and ways of reasoning, essential for the development of proportional reasoning, with interpretations of the rational number being one of the "nodes" in this network.

The resources that teachers choose to explore the various interpretations of rational numbers and other "nodes" in the network proposed by Lamon (2007, 2008) are of fundamental importance. Frac-Soma[10] is one of the resources that can contribute to the understanding of rational numbers and, consequently, the development of proportional reasoning. This manipulative material consists of 235 pieces arranged in 18 strips, one of which is whole, and the others are partitioned equitably according to multiples of 2, 3, and/or 5, ranging from 1 to 30. The partitioning activity[11], as per Lamon (2008), is the central point in understanding

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[4] This is the term that Lamon (2007, 2008) used to address the various meanings assumed by the fractional representation of rational numbers in different contexts. Henceforth, we will use the term proposed by Susan Lamon throughout the text.

[5] Signifies the rational number based on the relationship between the quantity of equal parts of a unit in relation to the total number of parts into which it is divided.

[6] Associates the rational number with points on the number line, based on the idea that a rational number $\frac{a}{b}$ is the value assigned *to* length intervals $\frac{1}{b}$.

[7] Defines the rational number as an ordered pair of numbers expressing the relative sizes of two quantities.

[8] Relates the rational number as the result of a division, in other words, when a certain number of objects needs to be distributed equally among a specific number of groups.

[9] Directs the rational number to the measure of some change in a quantity from a previous state.

[10] The original version of this material was proposed by Baldino (1983).

[11] We will discuss this concept in the next section.

rational numbers because these numbers are a quotient field. Furthermore, this activity generates other interpretations of rational numbers and contributes to assigning meaning when studying operations with these numbers. Hence, the creation of Frac-Soma through partitioning actions is essential for students to comprehend rational numbers. To achieve this, it is necessary to consider it as a partial representation of the mathematical object, and concept acquisition occurs only when students establish relationships between the manipulative material and other representations (numeric, pictorial, algebraic) through a process of abstraction and generalization. Given this context, this study aims to analyze the understanding of 7th-grade students when solving activities that emphasize the interpretation of rational numbers as quotients and operators.

Interpretations of the rational number from the perspective of Susan Lamon: interconnection with the development of proportional reasoning

As mentioned in the introduction, the development of proportional reasoning, according to Lamon (2008), involves central structures of mathematical knowledge (Figure 1), which are interconnected, forming a network of concepts, procedures, representations, and ways of thinking, expressed by seven "nodes" (interpretations of the rational number, measurement, quantity and covariation, unitization, relative reasoning, sharing and comparison, progressive and regressive reasoning).

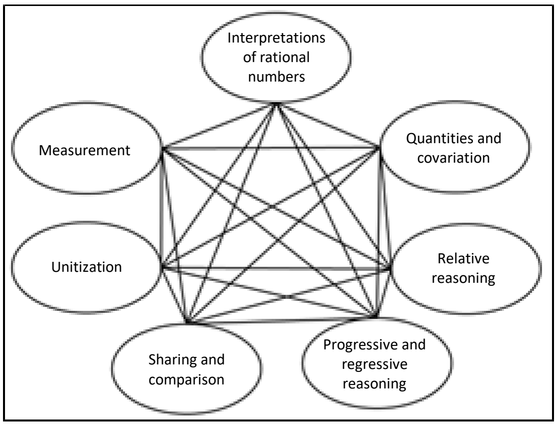


Figure 1.

Central knowledge structures (Adapted from Lamon, 2008)

According to Lamon (2008), there is a relationship between the understanding of rational numbers and the development of proportional reasoning. In this study, we will present some considerations related to the interpretations of rational numbers, especially quotients and operators, as previously mentioned, as well as the "nodes" of the network, namely: *measurement*, due to its close connection with the different interpretations of rational numbers in fractional representation; *sharing and comparison*, for the connections with the quotient interpretation and the resource used in the development of activities, namely the Frac-Soma; and *unitization*, for its relationship with the idea of equivalence and operations with rational numbers. To do so, we draw on the ideas of other authors to complement Lamon's (2007, 2008) perspective.

Cyrino et al. (2014), based on Lamon (2008), assert that the concept of *measure* or *measurement* is present in the formation of knowledge of rational numbers in fractional representation and, consequently, at the core of the development of proportional reasoning. The various interpretations of rational numbers, one of the "nodes" in Lamon's (2007, 2008) network, can be considered as various ways of making measurements because:

> a *part-whole* fraction [comparison] measures the multiplicative relationship of a part to the whole to which it belongs; a *ratio* measures relative magnitude; a *rate* such as speed is a quantification of motion; a *quotient* is a measure of how much 1 person receives when m people share n objects; an *operator* is a measure of some change in a quantity from a prior state; as a *measure*, a rational number directly quantifies a quality such as length or area (Lamon, 2008, p. 40).

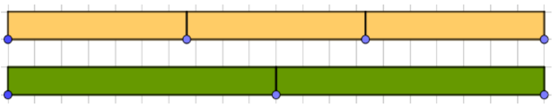
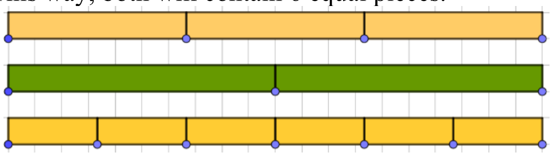
Lamon (2008), by analyzing the teaching and learning process, particularly in the United States of America, found that the act of measuring is underestimated, especially at the beginning of the study of rational numbers in fractional representation, as the proposed activities do not facilitate an understanding of the principles of measurement (conservation of distance and area; displacement and partitioning; relationship between units). On the other hand, the proposed activities with Frac-Soma, presented in the two sections discussing the results of this study, prioritize the principles of measurement, especially area conservation and partitioning.

The process of partitioning, as described by Lamon (2008), is a central concept in understanding rational numbers in fractional representation, since these numbers originate from the concept of *sharing*, which involves partitioning a unit (continuous or discrete) into disjoint and equal parts. If partitioning is important, so is comparing, because, for example, in *partitioning* activities, in addition to shading a figure to visually judge that $\frac{2}{3}$ of a unit is more than $\frac{1}{2}$ of the same unit, one also needs to determine "how much more" it is.

In this regard, it is desirable that the analysis of the activity goes beyond qualitative judgments (more/less) and involves quantitative assessments (how much more/how much less). It is worth noting that the concept of making divisions (or partitions) within a unit and then establishing *comparisons* between these divisions is also related to *measurement* (Oliveira, 2014). Based on these understandings, Lamon (2008) proposes *sharing* and *comparison* as one of the “nodes” in the network of central aspects in the development of proportional reasoning (Figure 1). Table 1 presents an activity that involves sharing and comparison, using Frac-Soma as a resource.

Table 1.

Activity involving sharing and comparison (Adapted from Lamon, 2008)

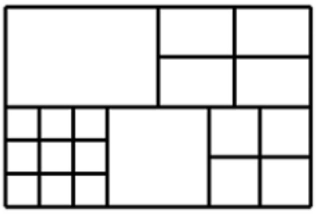
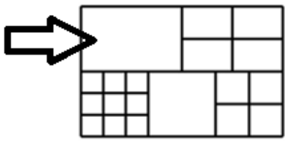
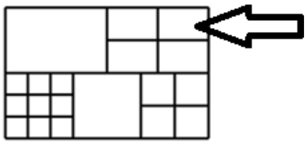
Activity	*A possible solution*
a) Determine how to cut the strips below so that the pieces are the same size. b) Determine how much more is taken if two pieces from the yellow strip and one from the green strip are considered. 	a) To cut the strips so that the pieces are the same size, simply cut each piece of the first strip in half and cut each piece of the second strip into three equal parts. This way, both will contain 6 equal pieces.  b) $\frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$

The activity presented in Table 1 demonstrates one of the key concepts related to rational numbers and the development of proportional reasoning, which is equivalence, as it is essential to recognize that the partitions made in the strips (yellow and green) create pieces that correspond to $\frac{1}{6}$ of the whole (of the strip), allowing us to express $\frac{2}{3}$ as $\frac{4}{6}$ and $\frac{1}{2}$ as $\frac{3}{6}$ and find that $\frac{2}{3}$ is $\frac{1}{6}$ greater than $\frac{1}{2}$.

The “node” referring to *unitization* involves the process of reorganizing quantities, regrouping them in a way that represents the exact total quantities, with the reference units remaining the same but expressed in different fractional forms (Lamon, 2007, 2008). Thus, *unitization* plays a fundamental role in various processes necessary to understand rational numbers in fractional representation, especially in *sharing* (partitioning), equivalence, and operations with rational numbers. The activity in Table 2 involves *unitization*.

Table 2.

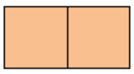
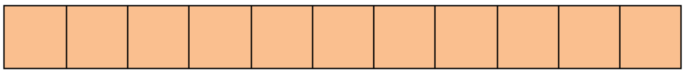
Activity involving unitization (Adapted from Lamon, 2008)

Activity	*A possible solution*
Describe how you can "see" $\frac{1}{4}$ and $\frac{1}{16}$ in the figure below. 	Look at the top left rectangle; it corresponds to $\frac{1}{4}$ of the unit.  Look at the "small" rectangle on the top right; it corresponds to $\frac{1}{16}$ of the unit. 

In this case, it is essential to propose activities that encourage the unitization process, as it enables thinking about any given quantity, choosing or anticipating the best way to solve it. However, at the same time, it is essential to highlight the importance of the *unit* in the study of rational numbers in fractional representation and development of proportional reasoning, as, according to Lamon (2008), its teaching has typically focused on it as a single object. However, when it comes to rational numbers in fractional representation, the unit can be a set of objects representing it. The activity in Table 3 exemplifies this statement.

Table 3.

Activity involving a set of objects as a unit (Adapted from Lamon, 2008)

Activity	*A possible solution*
If the unit of measurement is:  and the strip is:  What is the length of this strip?	The unit fits 5 and a half times in the strip, that is, $5\frac{1}{2}$.

As mentioned, one of the "nodes" in the network proposed by Lamon (2008) refers to the different *interpretations of the rational number*, which can be considered a *measurement*. Among these interpretations, this study considers *quotient* and *operator*, seeking connections with the "nodes" of the network and with Frac-Soma.

The *quotient interpretation* of the rational number is based on *partitioning* or dividing an integer into equal parts (equal division). The understanding of this process accompanies students throughout their school years, as it involves situations related to fair sharing, which is the act of dividing an object in a way that all involved parties receive an equal amount (Lamon, 2008). From this perspective, rational numbers, in the *quotient interpretation*, can be understood as *partitioning*, where a is a quantity and b is a parameter; thus, in the representation $\frac{a}{b}$, the quantity a is operated as indicated by the parameter b (Oliveira, 2014).

It is worth noting that, unlike situations involving the part-whole concept, in this concept (quotient), the quantity a can be smaller, greater, or the same as b, and it can represent different objects (Silva; Almouloud, 2018). In other words, while in the *part-whole* and *measurement interpretations* the same unit is involved — that is, only one variable —in the quotient interpretation, there are two different variables. An example involving this interpretation is in Table 4.

Table 4.

Example of an activity involving quotient interpretation (Research Data)

Look at the comic strip and answer:

I had...	*I divided a strip between...*	*I divided the other two strips among...*
	Guilherme and Natiéli	Daniel, Vitória, Anna Jullya and Bryan

a) Which piece of the Frac-Soma did Guilherme receive? What fraction does it represent?
b) What fraction represents the quantity that Daniel, Vitória, Ana Julia, and Bryan received?
c) Who received more: Vitória or Natiéli? Why?

In the activity (Table 4), a number of objects (three strips from Frac-Soma) needs to be evenly distributed among a certain number of groups (one strip for two people and two strips for four people). When dividing the strip between Guilherme and Natiéli, we can observe that the piece from the Frac-Soma that Guilherme will receive corresponds to $\frac{1}{2}$ of the whole ($1 \div 2$). On the other hand, the piece that Daniel receives corresponds to $\frac{2}{4}$ of the whole ($2 \div 4$), which is equivalent to the piece received by Guilherme. Since the piece Vitória receives is the same as the one Daniel receives (resulting from equal partitioning), and the piece Natiéli receives is the same as the one Daniel receives (resulting from equal partitioning), these pieces are equivalent. Thus, the girls receive pieces of the same size (equivalent).

Thus, the *quotient interpretation* is subject to the quotient function rule, and here, the *fraction bar* is a symbol for this function $\frac{x}{y} \equiv quociente(x, y)$, commonly written as $x \div y$, where the dividend x and the divisor y symbolize its arguments (Onuchic & Allevato, 2008, p. 88, our emphasis). In the activity (Table 4), $\frac{2}{4} \frac{tiras\ (t)}{pessoas\ (p)}$ means $\begin{array}{l|l} 2t & 4p \\ \cline{2-2} & \frac{2}{4}\frac{t}{p} \end{array}$, because $\frac{2}{4}\frac{t}{p} \cdot 4p = 2t$. Indicating the quotient $\frac{2}{4}$ with "fraction bar" notation reinforces the fact that in $\mathbb{Q}$, all divisions have a remainder of zero. It is worth noting that the *quotient interpretation* of rational numbers extends beyond partitioning elements; it enables comparative analysis (Lamon, 2008; Soares, 2016), as seen in part c) of the activity (Table 4).

According to Lamon (2008), the *operator interpretation* is a set of instructions for executing a process, capable of shrinking or enlarging, contracting or expanding, multiplying or dividing, transforming quantities into others. For example, "$\frac{3}{5}$ *of*" can be seen as a single operation on a quantity x. It can also be considered as multiplying x by 3 and then dividing the result by 5, or as dividing x by 5 and then multiplying the result by 3. In other words, "$\frac{3}{5}$ *of* x", with x equal to 20, should be interpreted as a composite function. Therefore, "$\frac{3}{5}$ *de* 20" corresponds to: $\frac{3}{5} \times 20 = 3 \times (20 \div 5) = (3 \times 20) \div 5 = 3 \times 4 = 60 \div 5 = 12$. From this perspective:

> The fraction bar notation $\frac{a}{b}$, [...], is used to symbolize a particular class of composite functions defined by $\frac{a}{b} \times x = a \times (x \div b) = (a \times x) \div b$, where a and b are constants, and x is a numerical expression for some quantity. The fraction bar is neither a functional symbol nor a delimiter but a symbol for the operation of function composition (Onuchic & Allevato, 2008, p. 93, our translation).

The *operator interpretation* differs from the others because, in this case, "[...] the significant relationship is the comparison between the quantity resulting from an operation and the quantity that is acted upon" (Lamon, 2008, p. 153). Understanding this comparison facilitates comprehension of the multiplication operation involving rational numbers, which can be explored through the manipulation of Frac-Soma pieces.

To exemplify this statement, consider "$\frac{1}{3}$ *of*" a quantity already established, in this case, $\frac{1}{2}$. To do this, we should take "$\frac{1}{3}$ *of* $\frac{1}{2}$". Figure 2 illustrates this situation using Frac-Soma. First, we take a part of a whole, divided into two equal parts (the green strip), applying the operator

$\frac{1}{2}$. Then, by using $\frac{1}{3}$ as an operator on the established quantity, we look for the piece that fits 3 times in $\frac{1}{2}$ and, consequently, 6 times in the whole (yellow strip). The piece we are looking for corresponds to $\frac{1}{6}$, so $\frac{1}{3}$ *of* $\frac{1}{2}$ is $\frac{1}{6}$.

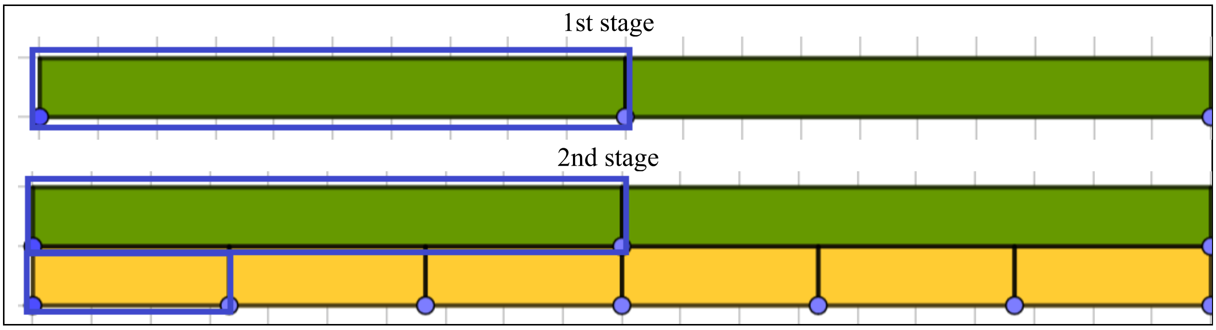


Figure 2.

Multiplication of rational numbers (Research data)

The action carried out in the Frac-Soma illustrates that multiplying rational numbers means taking pieces of pieces of a unit. In light of the aspects presented, based on Lamon (2007, 2008), we can say that teaching rational numbers and developing proportional reasoning pose a challenge from both cognitive and mathematical perspectives. This demands that teachers and/or researchers develop activities that, while addressing mathematical and cognitive complexity, also mitigate learning difficulties (Soares, 2016; Graça, Ponte, & Guerreiro, 2021).

Methodological Aspects

This research is characterized as qualitative because it is situated in an educational context that accommodates a diverse range of procedures, supported by different conceptions of reality and knowledge (Bicudo, 2012). Hence, conducting qualitative research involves seeking to understand aspects of reality and social relationships, which provide us with more descriptive information that emphasizes the meaning attributed to actions (Borba & Araújo, 2019).

For this purpose, we designed and implemented a sequence of activities to promote understanding of rational numbers in fractional representation through *measurement, part-whole, quotient, operator*, and *ratio interpretations*, highlighting the use of various representational systems triggered by Frac-Soma. The initial activities of the sequence involved creating, recognizing the properties, and naming the pieces of an adapted version of Frac-Soma, which the students made out of white cardboard with individual strips containing 1 to 15 pieces, as suggested for partitioning. The remaining activities, due to the need for precision in the size of each piece, were carried out using a complete Frac-Soma (containing 235 pieces) in blue (Figure 3), which a professor/researcher, the first author of this study, produced in EVA.

Figure 3.

Composition of the Frac-Soma manipulative material (Personal collection of the professor/researcher)

The choice not to differentiate the strips by colors, as in the original Frac-Soma (Baldino, 1983), is based on two aspects: firstly, because it does not allow the association between the number of pieces in each strip and its correspondence to a specific color and its respective multiple; secondly, the fact that, when positioning the pieces, it is necessary to identify the congruent lengths and check that the height of all the pieces is the same.

To achieve the goal of the sequence, each interpretation was initially emphasized using the Frac-Soma, followed by other activities that did not involve manipulative materials but focused on conversions between pictorial, numerical, and algebraic representations. In total, 56 activities with 257 items were proposed, with 26 activities (127 items) utilizing Frac-Soma, as shown in Table 5.

The implementation of the sequence took place during regular Mathematics class hours in a 7th-grade class at a public school in Sobradinho, Rio Grande do Sul, Brazil. The class consisted of ten students, referred to in this study as Student A, Student B, ..., Student J, respectively. Since the students had experienced a remote learning scenario during the 5th grade and part of the 6th grade due to the global pandemic of the SARS-CoV-2 coronavirus, which causes COVID-19, they studied aspects related to rational numbers and their interpretations for the first time through the sequence of activities in this research.

Table 5.

Summary of activities and their implementation (Prepared by the authors)

Topic	Activities	Items	Class hours	Date
Creation of the material	A1 to A4	28	6	June 20 and 21, 2022
Measurement Interpretation	A5 to A15	66	6	June 22, 28, and 29, 2022
Part-whole Interpretation	A16 to A29	69	4	July 5 and 6, 2022
Quotient Interpretation	A30 to A38	30	4	July 13 and 20, 2022
Operator Interpretation	A39 to A50	48	4	August 2 and 3, 2022
Ratio Interpretation	A51 to A56	16	2	August 10, 2022

Regarding data production, to encourage discussions during the development of the activities, the students were arranged into four groups, each consisting of two or three members, named G1, G2, G3, and G4, in order to uphold ethical principles and maintain participant anonymity[12]. During the development of the sequence, the activities were projected on the classroom board, and the professor/researcher read them aloud. Among the sources of data production, we considered the following: students' protocols, systematized during the meetings on auxiliary sheets; audio and video recordings that capture dialogues and gestures during the resolution of activities; photographs showing moments of Frac-Soma manipulation; and a logbook kept by the professor/researcher, containing reflections on the development of the sequence.

Through these sources, we identified and described episodes that included transcriptions of some dialogues between the students and the professor/researcher (P) during the implementation of the activities. At times, there are markings like [...], indicating omissions in the transcription due to language vices or because they were not understandable. Furthermore, to reference speech excerpts, we used the student's designation, the numerical code of the activity, and its respective item, followed by the month and year of activity implementation. For example, Student A_A30_c_August, 2022.

To compose this study, we considered 11 activities related to the *interpretations of quotient* and *operator*, all of which use Frac-Soma. Through the analyses, we composed two

[12] The research was registered and approved by the Human Research Ethics Committee and has a Certificate of Presentation for Ethical Consideration (CAAE).

sections. The first one focuses on *notions of sharing and comparison*. The second section emphasizes the *operator interpretation and its relationship with the multiplication of rational numbers*.

Notions of sharing and comparison

Activity 30 (A30 - Figure 4) involved exploring rational numbers in fractional representation seen as a *quotient*, that is, it proposed the equal sharing/division of the piece representing the whole in Frac-Soma among a certain number of people. To do this, the students needed to use the pieces of the resource to find the rational number corresponding to the sharing/division they had made.

30) Distributing Frac-Soma pieces from the strip that represents the whole: a) Divide the whole equally between Bryan and Vitória. How can we make this distribution? What fraction of the whole did each person receive? b) Now the strip needs to be divided among Daniel, Guilherme, Anna Jullya, and Andressa, so that each one receives equal parts. Which piece corresponds to each of the parts received? What fraction does this piece represent? c) The classmates Ruana, Yasmin, and Kauana joined the group! Therefore, the whole must be distributed equally among Daniel, Guilherme, Ana Jullya, Andressa, Ruana, Yasmin, and Kauana. In this division, what fraction represents the part that each one received? d) Is there a piece in Frac-Soma that corresponds to the part each of the seven classmates received? Why? e) Now, another classmate, Natiéli, is going to receive the pieces. So, the whole needs to be divided again, now into eight parts. What fraction represents the part Natiéli received?

Figure 4.

Activity involving the distribution of Frac-Soma pieces[13] *(Personal collection of the professor/researcher)*

The students immediately mentioned that, to divide the whole between two people, *"[...] you need to cut the piece of the whole right in the middle, to give half to each person"* (Student D_A30_a, July, 2022). When relating it to the resource, they pointed out that *"[...] in Frac-Soma, the division of the whole piece in half is done with these two pieces"* (referring to the pieces corresponding to $\frac{1}{2}$) (Student G_A30_a, July, 2022). Thus, all groups correctly determined the requested pieces and the related rational numbers (items A30_and A30_b).

Furthermore, in A30, to determine the requested answers, all groups counted the number of students mentioned in the instructions for each item. Then, they identified the corresponding pieces in Frac-Soma, thereby determining the rational numbers involved in the situations. However, when faced with the fact that seven people had to receive the divided pieces, the students realized that *"[...] this cannot be done with the pieces we already have in Frac-Soma*

[13] In the instructions of the activities, we used the term "fraction" to refer to rational numbers in fractional representation, considering that Frac-Soma allows working with non-negative rational numbers, and the students are familiar with this term.

because there is no strip with seven pieces. To make the division, it would be necessary to cut this whole into seven parts, where each person will receive $\frac{1}{7}$." (Student A_A30_c_July, 2022).

To consider the requested distribution and stimulate the discussion initiated in A30, activity 31 (A31 - Figure 5) was proposed. To do this, the students received a strip of paper with the exact dimensions as the piece that represents the whole in Frac-Soma, and they had to use it to create a new strip. Initially, G1 determined that *"[...] the new strip of material needs to be blue and have seven equal pieces"* (Student F_A31_a_July, 2022), associating these characteristics with the pieces already included in Frac-Soma.

31) Creating new Pieces for Frac-Soma: When we divided the whole among seven classmates, it was not possible to identify pieces in Frac-Soma that represented the part each one received:
a) Name at least two characteristics that this Frac-Soma strip must have!
b) Now you will receive a strip of paper the same size as a whole, and you will create the new strip of Frac-Soma that you mentioned earlier. How are you going to partition it?
c) What fraction does each piece represent?
d) Which Frac-Soma strip could be created from this strip?

Figure 5.

Activity to make a new strip for Frac-Soma (Personal collection of the professor/researcher)

With the strip representing the whole, the students proceeded to partition it, without using any measuring instruments. A discussion about the developed strategies is described in the following episode:

> P: How are you going to divide the piece into seven parts?
> Student E_G4: *I think we can fold the strip like this* (indicating that the folds should always be in the middle) *and then cut it. But we will not have seven pieces.*
> Student G_G2: *You can't do it by folding it in half. If you fold it in half, you get an even number of pieces, and seven is an odd number.*
> P: Does anyone have any other ideas about how we can divide it into seven parts?
> Student D_G2: *We can use the pieces we already have in Frac-Soma and make them a bit smaller than this one* (pointing to the piece that represents $\frac{1}{5}$).
> Student B_G2: *It's easier to use this other piece, which is the strip with six, because it'll be closer to the piece with seven. Or we can use the pieces from the strip with eight.*
> Student B_G2: *The piece with seven will need to be larger than the one with six and smaller than the one with eight.*
> Student G_G2: *It's the opposite, because if it's going to be larger than the one with six, it will be the same as the $\frac{1}{5}$.*
> Student A_G1: *And also because the more pieces the strip has, the smaller the piece becomes, so the pieces from the strip with seven need to be smaller than the one with six.*
> Student G_G2: *That's true, and the new pieces need to be smaller than the ones with $\frac{1}{6}$ and a bit larger than the ones with $\frac{1}{8}$.*
> Student A_G1: *And it will work because seven will be between six and eight.*

It is possible to observe that students partially consolidated the notions of sharing/dividing and comparison during this activity, as they emphasized aspects related to comparing two rational numbers in fractional representation, but did not consolidate the idea of "how much more" one number is greater than the other. According to Lamon (2007), partitioning requires comparison, and for that, it is essential that the student can identify that $\frac{1}{5}$ is greater than $\frac{1}{6}$, for example, and understand "how much more" one rational number is greater than the other. In the case of Frac-Soma, the identification that one rational number is greater than the other is visual (qualitative), while the understanding of "how much more" refers to quantitative aspects and requires notions related to equivalence, involving the process of *unitization*, which we did not identify in the dialogue.

Furthermore, according to Lamon (2007), even if students have good intuitive strategies for the partitioning/division process, it is essential to consider some basic characteristics of partitioning:

> The unit must be divided into equal shares; - If a unit consists of more than one item, the items must be the same size; - Equal means equal in amount, but shares do not always have the same number of pieces; - Equal shares do not have to have the same shape. (Lamon, 2008, p. 82).

In the described episode, through the partitioning of the unit, the students concluded that the more sections within a unit, the smaller the size of each of them will be. Similarly, the students concluded that the smaller the size of each share, the greater the quantity of pieces required to form the whole. These aspects, through partitioning, align with the principles of measurement (Lamon, 2008). It is noteworthy that the students justified their actions during partitioning by considering that a measurement is always an approximation. Therefore, it is possible to manipulate units of measurement to make them as precise as needed.

Thus, by considering the elements they had available at the time, all the groups established approximate measurements for the piece representing $\frac{1}{7}$, carrying out the partitioning reasonably. When asked about the next strip to be made from this one (A31_d), G2 mentioned that *"[...] the next one is the strip with 14 pieces because we just need to cut all these in half"* (referring to each of the pieces of $\frac{1}{7}$) (Student G_A31d_July, 2022). On the other hand, G4 and the other groups did not consider the order, that the next strip should be made from the one already constructed, and pointed out that the next strip *"[...] should have 11 pieces in total because there is the strip with ten and the one with twelve. We don't have one with 11 pieces yet and we can use those two to make it."* (Student A_A31d_July, 2022).

Activity 32 (A32 - Figure 6) was carried out with the aim of considering the partitioning of the pieces, taking the piece corresponding to $\frac{1}{2}$ as the new unit.

32) Let's continue dividing pieces of Frac-Soma, now using as a basis the piece that represents the fraction $\frac{1}{2}$ a) How can we divide it equally between Guilherme and Vitória? b) Is there a piece in Frac-Soma that corresponds to each part of this division? Which piece is it? How much does it represent? c) What if we want to partition $\frac{1}{2}$ among Ana Julia, Kauana and Bryan? Is it possible? d) What fraction represents the piece each one received? Why? e) Who received the biggest piece: Guilherme or Bryan? Why?

Figure 6.

Activity of sharing/dividing the piece corresponding to $\frac{1}{2}$ (Personal collection of the professor/researcher)

Similarly to the previous activities, the groups did not encounter obstacles in determining the requested rational numbers with the help of the Frac-Soma pieces, using the process of overlapping pieces associated with equal sharing. Therefore, they relied on the idea that *"[...] if we divide the piece representing $\frac{1}{2}$ between two classmates, we just need to cut it in half. With the pieces of Frac-Soma, we can exchange the $\frac{1}{2}$ for two pieces of $\frac{1}{4}$ and give one to each of them"* (Student A_A32_b_July, 2022), and that *"[...] to divide the piece representing $\frac{1}{2}$ among three classmates without cutting it, we found the piece that fits three times in it. So, we exchanged it for these three from the 1/6 $\frac{1}{6}$ strip, and each one will receive one"* (Student D_A32_c_July, 2022). When asked which of the students had received the largest piece (A32_e), the groups determined the answer by overlaying and comparing their sizes, thus visually identifying it without using the notion of equivalence.

However, when asked about "how much more" one will receive compared to the other, the groups hesitated to answer. In a way, "although children may quickly see that one value is larger or smaller than another, it takes some time before they are able to quantify differences" (Lamon, 2008, p. 84). This fact highlights the difficulties students encountered in the process of *unitization*, as they did not use notions of equivalence in the comparison between $\frac{1}{4}$ and $\frac{1}{6}$, for example, which are fundamental ideas for the development of proportional reasoning.

Finally, activity 33 (A33 - Table 4) was explored to establish divisions in the strips and compare quantities resulting from the partitioning process. Thus, initially, one strip should be shared/divided between two children, and subsequently, two strips among four children, while establishing equivalence relationships between the results.

By analyzing the answers, we concluded that only G2 considered the sharing/division of two strips among four children, splitting each strip in half and distributing half to each child. The other groups divided each of the strips into four parts and then distributed two pieces to each child, one from each strip. Thus, the groups differed in their answers to A33_b. G2 stated that the received part was $\frac{1}{2}$, while G1, G3, and G4 answered that it was $\frac{2}{4}$. However, when prompted to identify the pieces in Frac-Soma resulting from the partitioning and asked which child would receive the largest piece of strip, they all agreed that *"[...] they will receive the same size, even if it is two pieces, when they are put together, they also make up half. The pieces they receive will have the same size"* (Student A_A33_c_July, 2022).

Efficient partitioning is a form of "mental comparison of the amount of stuff to be divided to the number of shares. It involves knowing that you have enough stuff that you will not run out if you give each person a little more or make each share larger" (Lamon, 2008, p. 84). The results indicate that, after interventions by the professor/researcher, students perform partitioning effectively, similar to the participants in the research by Graça, Ponte, and Guerreiro (2021). We understand that activities that value the process of partitioning are essential, as they allow students to develop different strategies to carry out equal sharing, as well as to relate the situation to the division operation. In other words, partitioning activities enable students to understand that an equal sharing situation can be represented by a fraction, in which the numerator (a) represents the quantity to be distributed, the denominator (b) represents the number of people to distribute to, and the expression $\frac{a}{b}$ represents the result of the sharing (Graça, Ponte, & Guerreiro, 2021).

Operator interpretation and its relationship with the multiplication of rational numbers

Activity 39 (A39 - Figure 7) addressed, using Frac-Soma pieces, concepts related to determining rational numbers in fractional representation based on previously established quantities. It developed ideas related to the *operator interpretation* and, consequently, explored the operation of multiplying rational numbers in fractional representation.

During the resolution, the groups determined the quantities requested in A39_and A39_b, obtaining half and double of $\frac{1}{2}$, using the Frac-Soma pieces. In this regard, they emphasized that the half would be defined by *"[...] cutting the piece right in the middle. This will be the same as taking the piece of* $\frac{1}{4}$*"*, and for the double *"[...] you need to take two pieces of ½"* (Student A_A39_August, 2022). To identify the triple of $\frac{1}{2}$ (A39_c), the groups argued that *“[...] you need three pieces of 1/2. And since we don’t have it in Frac-Soma, we need to take another strip of the whole and cut it in half. This will make the strip bigger than the whole”* (Student G_A39_August, 2022). When asked about the rational number associated with these pieces, Student G emphasized that *“[...] since there will be three pieces of $\frac{1}{2}$, the fraction for the triple will be $\frac{3}{2}$”* (Student G_A39_c_August, 2022).

39) Obtaining other Frac-Soma pieces through transformations with the piece $\frac{1}{2}$ a) I want half of the piece $\frac{1}{2}$. How would you represent this using Frac-Soma? What fraction indicates this half? b) I want to double the amount of $\frac{1}{2}$. Explain how to obtain this quantity with a piece of Frac-Soma. Represent the resulting piece numerically. c) Give me triple the amount of $\frac{1}{2}$. How to determine this quantity? What fraction does it represent? d) If you didn't have Frac-Soma, how could you represent the previous transformation mathematically? Which numerical operation should be used in this process? e) I need one-third of $\frac{1}{2}$. How can you calculate the value of this piece? Let's write this relationship on the board, including the final result! f) Give me one-fourth of $\frac{1}{2}$. How can you represent this relationship using the Frac-Soma pieces? What fraction corresponds to $\frac{1}{4}$ of $\frac{1}{2}$? g) Now, give me $\frac{2}{5}$ of $\frac{1}{2}$. What fraction is associated with this quantity? How can you determine $\frac{2}{5}$ of $\frac{1}{2}$ without Frac-Soma?

Figure 7.

Activity about operator interpretation through transformations (Personal collection of the professor/researcher)

In the presented arguments, we can observe the process of partitioning in the student’s statement when he mentioned “cutting” to obtain half of $\frac{1}{2}$. However, it is not clear from the students’ statements that they fully understood that they were taking parts of parts of a whole. Furthermore, in this activity, there are indications of an understanding of aspects related to the *operator interpretation* as they identified the transformations (enlargement) that occurred when obtaining double and triple the quantity of $\frac{1}{2}$. This understanding is explicit during the resolution of item d), when the students were prompted to establish a mathematical relationship that justified the rational number obtained, as well as the operation related to the transformation, when student G mentions that assuming triple $\frac{1}{2}$ is *“[...] the same as doing three times a half, which will result in three halves”* (Student G_A39_August, 2022). This procedure was corroborated by the other students in the class and confirmed during the discussion for resolving item e):

> Student D_G2: *It was easier to make the triple, we just had to take three pieces of the $\frac{1}{2}$. Now, this one I found more difficult.*

P: What if we think about what you have already done? [...] how did you determine half of $\frac{1}{2}$?
Student D_G2: *We just cut it in half.*
P: And did you cut the piece in half?
Student D_G2: *No. We exchanged the $\frac{1}{2}$ piece for two pieces of $\frac{1}{4}$ because one one-fourth piece equals half.*
P: So, if we now want the third part of $\frac{1}{2}$, how could we do this exchange?
Student G_G2: *We need to find three pieces that complete $\frac{1}{2}$.*
⋮
Student A_G1: *[...] three pieces of the $\frac{1}{6}$ strip fit into $\frac{1}{2}$.*
Student G_G2: *[...] since we only want one-third, the answer will be only one piece. The answer will be $\frac{1}{6}$.*

As with item a), we can see that the students carried out the partitioning process to obtain the desired result. However, there is still no evidence that they understand ideas related to the *operator interpretation*; in other words, they do not realize that if we take a piece that fits three times into $\frac{1}{2}$, this piece will fit 6 times into the whole. When partitioning/exchanging pieces, there is a "loss" of the unit as a reference.

Items e), f), and g) of activity A39 were revisited with the whole group in order to explore regularities in the answers based on the numerical representation. When prompted to observe the relationships between rational numbers in fractional representation, the students realized that *"[...] the result is the answer of the numbers on top when we do the multiplication, and also the ones at the bottom"* (Student D_A39_g_August, 2022). Even though the operation of multiplication of two rational numbers in fractional representation was not explicitly explored, the students realized that there was a relationship of multiplicity between the numerators and denominators.

Activity 40 (A40 - Figure 8), like A39, explored relative enlargement and reduction transformations related to the piece corresponding to $\frac{1}{3}$.

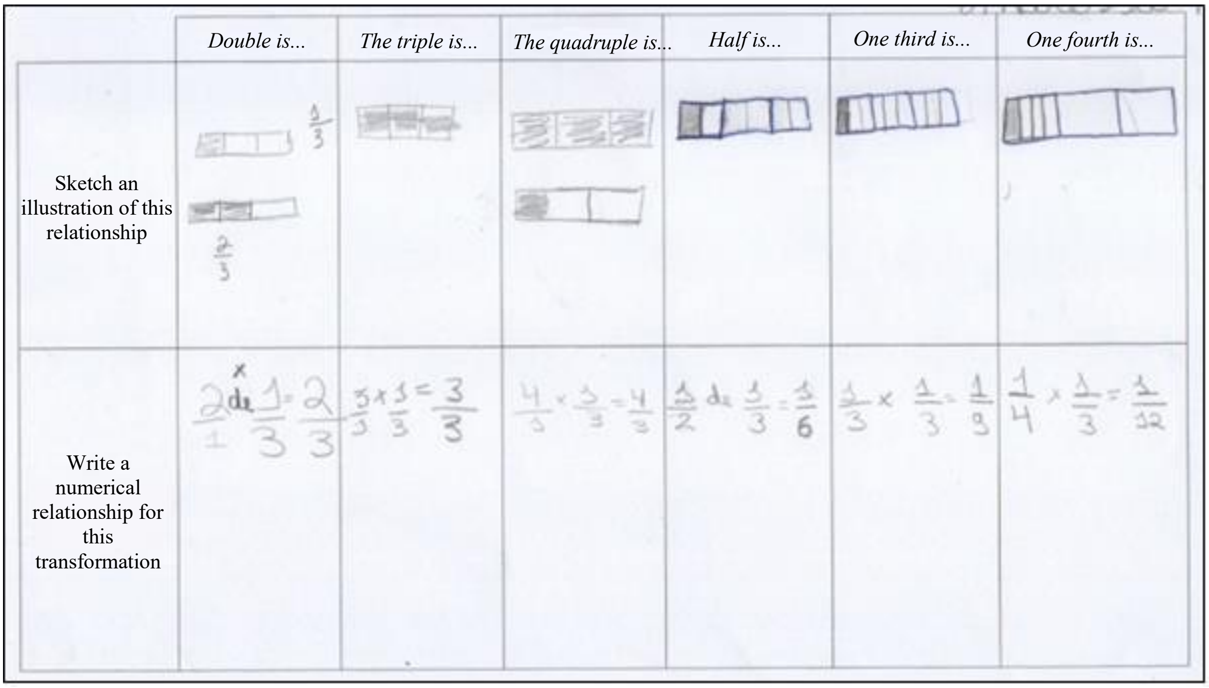


Figure 8.

Protocol of G1 in A40 (Personal collection of the professor/researcher)

It is worth noting in the protocol (Figure 8) that different students responded to each of the alternatives in A40. Therefore, there are different notations in the numerical representation, as well as visual representations that reveal different understandings. In the numerical relationships for double and half, the students who answered indicated the term "*of*" by using it to "replace" the multiplication symbol. Still in the numerical relationships, the other students in the group associated the process performed with the Frac-Soma pieces with the multiplication operation. In the pictorial representation, there are some indications of an understanding of the *operator interpretation*. However, there are some difficulties previously mentioned (in A39), such as the "loss" of the unit as a reference, as evident in the pictorial representation of one-fourth, where the unit is divided into three parts, but only one of the three parts is divided into four parts.

Activity 41 (A41 - Figure 9) in addition to what was previously covered in A39 and A40, also explores the concept of comparison. To do this, the students received two strips of paper with the same dimensions as the piece representing the whole in Frac-Soma.

41) You will now receive two strips of paper of the same size as the piece that represents the whole: a) On the first strip, you have to fold and cut out $\frac{1}{2}$ of $\frac{1}{6}$. What fraction does this piece represent? b) On the second strip, represent the quantity $\frac{1}{3}$ of $\frac{1}{4}$ with a piece. What fraction does it represent? c) Which fraction is larger?

Figure 9.

Relationships of sharing and comparison based on the operator interpretation (Personal collection of the professor/researcher)

By analyzing the students' dialogues and protocols, we can see that there were two ways of solving items a) and b). One of them used the strategy of dividing, for example, the unit into six pieces and then dividing one of these pieces in half, thus verifying that the piece $\frac{1}{12}$solves the problem, without focusing on the unit. Another one used the regularities observed in the numerical representation, multiplying $\frac{1}{2}$ by $\frac{1}{6}$ to obtain $\frac{1}{12}$ and then finding the corresponding piece in Frac-Soma. Therefore, it was necessary to revisit, with the whole group, the set of instructions to carry out the process, using the manipulative material to understand how quantities are transformed into others. When prompted to make a comparison between the rational numbers corresponding to the created pieces (A41_c), all groups immediately concluded that they were equal since they had the same size.

Finally, it is noteworthy that the sequence included four more activities similar to those previously presented, related to the *operator interpretation*. In these activities, there was evidence of progress in the understanding of the *operator interpretation*. In particular, in transformations that resulted in reductions of the given initial quantity.

Final Considerations

This study aimed to analyze the understanding of 7th-grade students when solving activities that emphasize the interpretation of rational numbers as quotients and operators. Through the analysis of the data obtained in this research, we observed that the groups consolidated the notions related to fair sharing/equal division, as the students established connections between the requested quantities and the partitioning process. This process is essential to understanding the *quotient interpretation* and the development of proportional reasoning. By establishing these connections, the groups understood that a situation of fair sharing/equal division can be represented by a rational number in fractional representation, $\frac{a}{b}$, where a represents the quantity to be distributed and b represents the number of people who will receive these parts.

On the other hand, more than just partitioning units, it is extremely important to understand the concept of comparison. This, in turn, was one of the difficulties that the students presented. We observed that the comparisons students made were visual, meaning they compared the sizes of the Frac-Soma pieces without considering the quantitative aspects of the numbers associated with them. To overcome these difficulties and promote quantitative comparison, during the interventions of the professor/researcher, it was necessary to emphasize the process of unitization, which contributed to understanding the concept of equivalence, essential for the development of proportional reasoning.

Another relevant aspect identified in the resolutions is that, when the groups performed unit partitioning, they established ideas related to measurement, emphasizing that the greater the number of sections in the unit, the smaller the size of each of them, and vice versa. This is consistent with what Lamon (2008) emphasizes about the *measurement interpretation*, which plays an important role in understanding rational numbers and the structures associated with multiplicative reasoning.

When considering the *operator interpretation*, the groups engaged in partitioning processes to find the requested results and transformations. It is worth noting that, in the act of partitioning the whole and "exchanging" pieces of Frac-Soma for others, there was a "loss" of the unit as a reference. However, after the interventions by the professor/researcher, there were signs in the analysis of the protocols that pointed towards an understanding of the *operator interpretation*, especially regarding the understanding of the operator as a function capable of transforming the unit into another similar one. Therefore, we can conclude that the students demonstrated an understanding of these interpretations of rational numbers, although the concepts of operator and comparison still pose a challenge for some.

Finally, it is essential to emphasize that creating situations where only one interpretation is used by students is challenging, especially when using Frac-Soma, as these interpretations are interconnected. Thus, it is of utmost importance that working with rational numbers includes situations that are characteristic of each interpretation, considering the knowledge and possibilities associated with them.

References

Baldino, R. R. (1983). *Material Concreto:* Frac – Soma 235. Casquinha – Material de Apoio Pedagógico.

Bicudo, M. A. V. (2012). A pesquisa em Educação Matemática: a prevalência da abordagem qualitativa. *Revista Brasileira de Ensino de Ciência e Tecnologia*, *5*(2), (mai-ago), 15-26.

Borba, M.C & Araújo, J. L. (2019). *Pesquisa qualitativa em Educação Matemática*. Editora Autêntica.

Cyrino, M. C. C. T.; Garcia, T. M. R.; Oliveira, L.; Rocha, M. R. (2014). Formação de Professores em Comunidades de Prática: frações e raciocínio proporcional. Londrina: UEL, 37-63, 2014.

Graça, S. I., Ponte, J. P. da & Guerreiro, A. (2021). Quando As Frações Não São Apenas Partes de Um Todo…! *Revista Educação Matemática Pesquisa. 23*(1), 683-712.

Lamon, S. J. (2007). Rational Numbers and Proportional Reasoning: Toward a Theoretical Framework for Research. In F. K. Lester (org.), *Second Handbook of Research on Mathematics Teaching and Learning:* a Project of the National Council of Teachers of Mathematics. Charlotte: IAP/NCTM.

Lamon, S. J. (2008). *Teaching Fractions and Ratios for Understanding:* Essential content knowledge and instructional strategies for teachers. New York: Routledge.

Lesh, R., Post, T. & Behr, M. (1988). Raciocínio Proporcional = Proportional Reasoning. In J. Hiebert & M. Behr (orgs.), *Number Concepts and Operations in the Middle Grades.* Reston: Lawrence Erlbaum/National Council of Teachers of Mathematics.

Maranhão, C. & Machado, S. (2011). Uma Meta-Análise de Pesquisas sobre o Pensamento Proporcional. *Educar em revista* (n. especial), 141-156.

Oliveira, I. (2009). Proporcionalidade: estratégias utilizadas na Resolução de Problemas por alunos do Ensino Fundamental no Quebec. *Boletim de Educação Matemática (Bolema),* 22(34), 57-80.

Oliveira, L. M. C. P. de. (2014). *Aprendizagens no Empreendimento Estudo do Raciocínio Proporcional* [Dissertação de Mestrado em Ensino de Ciências e Educação Matemática, Universidade Estadual de Londrina]. https://pos.uel.br/pecem/wp-content/uploads/2021/08/OLIVEIRA-Lais-Maria-Costa-Pires-de.pdf

Onuchic, L. R. & Allevato, N. S. G. (2008). As diferentes "Personalidades" do número racional trabalhadas através da resolução de problemas. *Boletim de Educação Matemática (Bolema), 21*(31), 79– 102.

Silva, M. J. F. da & Almouloud, S. A. (2018). Números racionais: concepções, representações e situações. In G. P. Oliveira. *Educação Matemática epistemologia, didática e tecnologia*. Editora da Livraria da Física.

Sores, M. A. da S. (2016). *Proporcionalidade um conceito formador e unificador da Matemática*: uma análise de materiais que expressam fases do currículo da Educação Básica [Tese de doutorado em Educação nas Ciências, Universidade Regional do Noroeste do Estado do Rio Grande do Sul]. http://bibliodigital.unijui.edu.br:8080/xmlui/handle/123456789/4963