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Aspects of the differential thinking of high school students with GeoGebra

Aspectos del pensamiento diferencial de estudiantes de secundaria con GeoGebra

Aspects de la pensée différentielle des lycéens avec GeoGebra

Aspectos do pensamento diferencial de estudantes do ensino médio com o GeoGebra

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Abstract

In the research reported in this article, we seek to explore aspects of differential thinking and thinking-with-GeoGebra that emerge when high school students investigate activities on the calculation of areas and volumes. Initially, we discuss mathematical and differential thinking issues, proposing a perspective on differential thinking formed by four aspects: The notion of limit and continuity, the notion of infinitesimal, the defined integral concept, and the visual-geometric design. From the methodological point of view, based on qualitative research, we developed teaching experiments with six pairs of students from the 10, 11 and 12 grades of high school, considering the elaboration of a task composed of five activities. In the article, based on the notion of sampling in qualitative research, we discuss the investigation developed by one pair of 12th graders about the "The Problem of Volume" activity. In the data analysis, we highlighted the role of visualization and experimentation-with-technologies in developing students' thinking. We emphasize how the resources or potentialities of the software offered

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ways for students to articulate the four aspects that make up differential thinking in the proposed perspective. This study contributes to the production of knowledge about the use of digital technologies in mathematics education, especially in relation to the teaching of calculus in high school.

Keywords: Mathematics education, Digital technology, GeoGebra, Calculus, Mathematical thinking.

Resumen

En la investigación reportada en este artículo, buscamos explorar aspectos del pensamiento diferencial y del pensamiento con GeoGebra que surgen cuando los estudiantes de secundaria investigan actividades sobre el cálculo de áreas y volúmenes. Inicialmente, discutimos cuestiones sobre el pensamiento matemático y el pensamiento diferencial, proponiendo una perspectiva sobre el pensamiento diferencial formada por cuatro aspectos: noción de límite y continuidad, noción de infinitesimal, concepto de integral definido y concepción visual-geométrica. Desde un punto de vista metodológico, con base en una investigación cualitativa, desarrollamos experiencias didácticas con seis parejas de estudiantes de 1°, 2° y 3° de secundaria considerando la elaboración de una tarea compuesta por cinco actividades. En este artículo, a partir de la noción de muestreo en la investigación cualitativa, discutimos la investigación desarrollada por una de las parejas de 3° grado en relación con la actividad titulada: "El Problema del Volumen". En los resultados destacamos el papel de la visualización y la experimentación con tecnología en el desarrollo del pensamiento diferencial de los estudiantes. En particular, destacamos cómo los recursos o capacidades que ofrece el software significan para los estudiantes articular esos cuatro aspectos que componen el pensamiento diferencial en la perspectiva propuesta. Finalmente, este estudio contribuye a la producción de conocimiento sobre el uso de las tecnologías en la Educación Matemática, particularmente en relación con la enseñanza del Cálculo Integral en la Educación Secundaria.

Palabras clave: Educación matemática, Tecnologías digitales, GeoGebra, Cálculo, Pensamiento matemático.

Résumé

Dans la recherche rapportée dans ce article, nous avons cherché à explorer les aspects de la pensée différentielle et de la pensée avec GeoGebra qui émergent lorsque les élèves du secondaire étudient des activités sur le calcul des aires et des volumes. Dans un premier temps, nous discutons des questions sur la pensée mathématique et la pensée différentielle, en proposant une perspective sur la pensée différentielle formée par quatre aspects : notion de limite et de continuité, notion d'infinitésimal, concept d'intégrale définie et conception visuelle-géométrique. D'un point de vue méthodologique, basé sur une recherche qualitative, nous avons développé des expérimentations pédagogiques auprès de 6 binômes d'élèves de 1ère, 2ème et 3ème année du Lycée en considérant l'élaboration d'une tâche composée de 5 activités. Dans cet article, basé sur la notion d'échantillonnage en recherche qualitative, nous discutons de l'investigation développée par l'un des binômes de 3e année en relation avec l'activité intitulée : « Le problème du volume ». Dans les résultats, nous soulignons le rôle de la visualisation et de l'expérimentation de la technologie dans le développement de la pensée différentielle des élèves. En particulier, nous soulignons comment les ressources ou les capacités du logiciel offert signifient pour les étudiants d'articuler ces quatre aspects qui composent la pensée différentielle dans la perspective proposée. Enfin, cette étude contribue à la production de connaissances sur l'utilisation des technologies dans l'enseignement des mathématiques, notamment en relation avec l'enseignement du calcul intégral au lycée.

Mots-clés : Enseignement des mathématiques, Technologies numériques, GeoGebra, Calcul, Pensée mathématique.

Resumo

Na pesquisa relatada neste artigo buscamos explorar aspectos do pensamento diferencial e do pensar-com-GeoGebra emergentes quando estudantes do ensino médio investigam atividades sobre o cálculo de áreas e volumes. Incialmente, discutimos questões sobre pensamento matemático e pensamento diferencial, propondo uma perspectiva sobre pensamento diferencial formada por quatro aspectos: noção de limite e continuidade, noção de infinitésimo, conceito de integral definida e concepção visual-geométrica. Do ponto de vista metodológico, com fundamentação na pesquisa qualitativa, desenvolvemos experimentos de ensino com seis duplas de alunos de 1ª, 2ª, e 3ª séries do ensino médio considerando a elaboração de uma tarefa composta por cinco atividades. No presente artigo, com base na noção de amostragem na pesquisa qualitativa, discutimos a investigação desenvolvida por uma das duplas de 3ª série em relação à atividade intitulada: “O Problema do Volume”. Nos resultados destacamos o papel da visualização e da experimentação-com-tecnologias no desenvolvimento do pensamento diferencial dos estudantes. Em especial, enfatizamos como os recursos ou potencialidades do software ofereceram meios para que os estudantes articulassem os quatro aspectos que compõem o pensamento diferencial na perspectiva proposta. Por fim, este estudo contribui com a produção de conhecimentos acerca do uso de tecnologias digitais em Educação Matemática, em particular com relação ao ensino de cálculo no ensino médio.

Palavras-chave: Educação matemática, Tecnologias digitais, GeoGebra, Cálculo, Pensamento matemático.

Aspects of the differential thinking of high school students with GeoGebra

Research in mathematics education has explored different aspects of mathematical thinking at varying levels of education, such as elementary education and higher education (Henriques, 2010; Wielewski, 2005; Krutetskii, 1976). Many concepts and terms related to mathematical thinking, such as advanced mathematical thinking, appear in the literature to broaden understandings of students' thinking and their complexities. According to Henriques (2010),

> The nature of mathematical thinking is necessarily intertwined with the cognitive processes that give rise to mathematical knowledge. Understanding, as it happens, is a process taking place in the individual's mind. It can be quick, a click in the mind, but it is often based on a long sequence of learning activities during which various mental processes occur and interact. Therefore, mathematics education researchers have become aware of the importance of cognitive processes and their interactions in understanding advanced mathematics. (Henriques, 2010, p. 19, our translation)

We understand that Henriques (2010) conceptualizes advanced mathematical thinking to understand mathematics conceived as "advanced" from a theoretical framework that relates mathematical thinking to advanced mathematical thinking. In fact, expressions such as advanced mathematical thinking and elementary mathematical thinking began to be intensely discussed in mathematics education by one of the working groups of the "International Group for the Psychology of Mathematics Education" (Hardel, Selden, & Selden, 2006), considering that authors such as Dreyfus (1991), Tall (1991), and Gray et al. (1999) had constructed a view that mathematics explored by students at younger ages presents a certain continuity focused on advanced mathematics and this fact is evidenced by several characteristics present in it. The authors justify this perspective through the complexity of the processes inherent to mathematical thinking and the cognitive changes that can be identified in the individual in mathematical activity. In this way, the thinking involved in learning a given mathematical concept or object goes through articulations between elementary and advanced mathematical thinking. According to Sad (2000),

> […] many of the processes (transition and mental reconstruction, generalization and abstraction, intuition, rigor, analysis, and synthesis) of advanced mathematical thinking can be found at a more elementary level, and what makes the shift from elementary mathematical thinking to the advanced one is the transition from describing to defining and from convincing to proving, in a logical way based on the definitions taken. (Sad, 2000, p. 3, our translation)

According to Reis (2001), advanced mathematical thinking discusses the intuition and rigor present in this advanced thinking. This discussion, of an epistemological nature, addresses two essential elements in the construction of mathematical knowledge, particularly involving Differential and Integral Calculus and Mathematical Analysis curriculum components. About Calculus, we can say that the approach to the themes is based on an applied perspective, with the intuitive interpretation of concepts, appeal to geometric visualization, with formal reservations and rigor. On the other hand, in mathematical analysis, topics are usually treated according to a logical-formal perspective, with the rigorous definition of concepts and several propositions demonstrated, with very little geometric visualization and little appeal to intuition and rigor.

To understand this discussion between rigor and intuition, Reis (2001) analyzed some teaching materials and semi-structured interviews with four teacher-researchers, namely, Roberto Ribeiro Baldino and Geraldo Severo de Souza Ávila (interviewed in 1998), and Djairo Guedes de Figueiredo and Elon Lages Lima (interviewed in 1999). Respecting part of the interviews, when asked about the perception of this relationship between rigor and intuition in the teaching of mathematics, the interviewees presented a view that considers cognitive, moral, and cultural issues and breaks conceptions of rigidity in the face of this relationship, highlighting the existence of different levels of rigor that need to be mediated by teachers. Emphasizing aspects related to the dichotomy between calculus-differential thinking and analysis-analytical thinking, Reis (2001, p. 157, our translation) questions “the possibility of the existence of a balance point between intuitively constructed differential thinking and rigorously constructed analytical thinking”, assuming that “due to a mistaken preconception on our part, we tried, in any case, to associate intuition with differential thinking and rigor with analytical thinking. As all respondents demonstrated, this association cannot be made in a dichotomous or reductionist way” (Reis, 2001, p.157, our translation).

Thus, it is possible to consider that intuition and rigor assume both a dimension of antagonism and complementarity. Another element to be highlighted in the interviews by Reis (2001) is the unfolding presented by the interviewees in relation to differential thinking. Professor Roberto Ribeiro Baldino admitted that he still did not have a complete characterization of differential thinking, but despite that, he assumed that he considers differential thinking as something that, by introducing the decomposition of magnitudes, goes beyond algebraic thinking. Baldino also commented in the interview that the genesis of differential thinking is more difficult to follow from the 19th century onwards.

Professors Djairo Guedes de Figueiredo and Elon Lages Lima did not comment on a characterization, even historical, of differential thinking and analytical thinking. They chose to identify several elements of comparison between calculus and analysis from a didactic-pedagogical point of view. Thus,

> As we can see, due to the very difficulty of discussing and characterizing the differential and analytical thoughts demonstrated by our interviewees, the attempt at analysis and the difficulty of carrying it out, on our part, show that the conceptual mastery of these two forms of thought is still little known / explored in the production of mathematics education. (Reis, 2001, p.159, our translation)

In search of an explanation about which elements are associated with differential thinking, we can discuss some reflections proposed by Sad (2000). The researcher, who defended her doctoral thesis in 1998 presenting an epistemological approach to aspects of differential and integral calculus, discusses in the article *Uma abordagem epistemológica do cálculo* [An Epistemological Approach to Calculus] the production of meanings and knowledge of calculus (Sad, 2000). Thus, reinforcing the point made by Wielewski (2005) that thinking is influenced by various elements (internal and external) and that it manifests itself in various ways, Sad (2000) draws attention to the fact that "diversification in modes of production of meaning is made in apparent relation to the 'same' produced object [...] and symbolically represented [...], but which, however, can reproduce different meanings" (Sad, 2000, p. 1, our translation).

In this sense, many aspects have been considered in the conceptualization of various forms of mathematical thinking. Such aspects concern the very nature of mathematics, the elements that constitute mathematical concepts or content and even events in the classroom, task designs, problem solving, roles, and knowledge of teachers and students, etc. Within the scope of mathematical thinking involving calculus, Sad (2000) defined special names (local stipulations) for cores of semantic fields related to the topic. Such denominations refer to the notions of limit, infinitesimals, geometric visualizations, and algorithms. Specifically, we present the categorization proposed by Sad (2000) in Table 1.

Table 1.

Nucleus associated with calculus (Sad, 2000, p. 7-8)

Nucleus in a semantic field	**Characterization**
Local stipulations regarding limit	When we have the Weierstrassian definition[4] of the limit of a function of a real variable in the core.

[4] $\lim_{x \to c} f(x) = L \coloneqq (\forall \varepsilon)(\varepsilon > 0 \wedge \varepsilon \in \mathbb{R}) \Rightarrow (\exists \delta)(\delta > 0 \wedge \delta \in \mathbb{R} \wedge (\forall x)(x \in \mathbb{R} \wedge x \neq c \wedge 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$

Local stipulations regarding infinitesimals	When we have in the core elements based on the notion of infinitesimal[5].
Visual-geometric local stipulations	When we have geometric principles or results at the core, graphs, and drawings of flat or spatial figures.
Algorithm-like local stipulations	When algorithms are at the core (rules, formulas, sequences memorized "by heart"), without relating to mathematical understanding and justification.

Considering these categories presented by Sad (2000), but mainly the nature of the data produced and analyzed in the research reported in this article, we propose a new categorization referring to aspects that we consider relevant and emerging from differential thinking. Such categorization, presented in Table 2 below, starts at a high school scenario but can be conceptualized at other levels of education.

Table 2.

Aspects of differential thinking (elaborated by the authors)

Aspect	**Characterization**
Notion of limit and continuity	When there is a manifestation of reasoning and conclusions involving a process of approximation to a maximum or minimum value, showing uniformity in the constructed investigation elements.
Notion of infinitesimal	When there is the manifestation of ideas related to the reduction in the size of a measure until it is as small as possible, but without canceling itself out. This reduction implies an increase in the distribution of an interval or in the number of sides of a polygon (inscribed and/or circumscribed), for example.
Definite integral concept	When there is the manifestation of proposals involving calculating area and/or volume through well-defined intervals.
Visual-geometric design	When there is the manifestation of construction of flat and/or spatial figures, as well as the analysis of figures for the manifestation of reasoning and conclusions.

Therefore, the research reported in this article comprises aspects of differential thinking, such as the manifestation of differential and integral calculus concepts that emerge formally and/or intuitively in high school students when investigating activities involving the calculation of area and volume using the GeoGebra software. This process emphasizes the thinking-with-GeoGebra notion proposed by Borba, Scucuglia, and Gadanidis (2018), highlighting the role of media (such as GeoGebra) in knowledge production. This perspective is based on the human-

[5] Sad (2000) explains the notion of infinitesimal as conceived since Newton (of infinitesimal monads, of infinitely small increments), or as for Leibniz (a class of numbers smaller than any other designated, sometimes also expressed as differentials or as distances infinitely small).

beings-with-media concept, proposed by Borba and Villarreal (2005). Although the theme “teaching calculus” has been explored for decades in the mathematics education field, several current studies raise questions or concerns considered very pertinent (Trevisan & Araman, 2021; Santos et al., 2022), including specificities in the high school context (Siple, Figueiredo, & Herbst, 2023).

Methodological Aspects

This research was approved by the Research Ethics Committee (CAAE: 20196619.8.0000.5466). To investigate aspects of differential thinking emerging in a group of high school students when doing thinking-with-GeoGebra activities that involve investigating the area of regions limited by axes and curves and calculating the volume of regions limited by flat surfaces and curves, we sought answers to the following research question: What aspects of differential thinking and thinking-with-GeoGebra emerge when high school students investigate activities about the calculation of areas and volumes?

Bicudo (1993) states that researching mathematics education is not researching mathematics or education. Even if it addresses something pertinent to both, it expresses concern “with understanding mathematics, with doing mathematics, with elaborate interpretations of the social, cultural, and historical meanings of mathematics” (Bicudo, 1993, p. 19-20, our translation). For this task, this research sought to follow a path that explores the proposed activities beyond a single resolution, with a focus on providing moments of discoveries and revelations, foreseen or not, that value the participants' understanding of the mathematical contents explored on specific tasks. This route presents evidence of the research methodology proposed by Steffe and Thompson (2000), called teaching experiment.

The methodology of teaching experiments is directed at exploring and explaining the activity according to the mathematics of the students who, immersed in a collaborative scenario, have the freedom to share and explore conjectures and negotiate common goals and objectives. In this methodology, Steffe and Thompson (2000) understand a teaching session as a sequence of meetings that follow some specific criteria: each meeting has the presence of the teaching agent (researcher) and one or more students, a witness, a method of recording the environment and the activities carried out, in addition to the authors suggesting a duration of one semester or a year. These teaching sessions comprise the teaching experiment research methodology (Steffe & Thompson, 2000).

The scenario for data production was organized into two teaching sessions with each of the six participating pairs (12 students). Each teaching session contemplated one pair at a time.

Observations of all teaching sessions were recorded through footage of the environment and the OBS Studio software[6]. In addition, all constructions made in the GeoGebra software by the students were saved separately, serving as a data production record. Table 3 below presents the instruments used to produce data for this research, followed by the justification.

Table 3.

Data production instruments (authors' elaboration)

Instruments	Justification
Video recording	The choice to record the data through filming, aware that this instrument is not immune to problems, agrees with the idea that the video can capture valuable behaviors of the participants, verbal and non-verbal manifestations, and allows the continuous revisitation of the researcher.
File generated in GeoGebra	It presents all the construction, organization, and advancement of students' thinking. It can be cross-revised with audiovisual files and/or paper notes.
Field Notes	Even though it is not recurrent in all teaching sessions and/or pairs, this instrument complements the process of revisiting and clarifying possible thoughts and strategies addressed by the participants.

The teaching sessions occurred in 2019 at a State public school in an up-country city in the state of São Paulo, where the first author of the article works as a teacher. That year, she taught Mathematics classes for Grade 11 and Grade 12 at the school. She occasionally had contact with students from the other classes (other 11 and 10 graders) through dialogues. Despite the above, the invitation to participate in the production of data for this research was extended to all students at the school. Among those who expressed interest within the stipulated period, 12 were selected through a draw, four from each high school grade. During the development and closure of this study, 11 students effectively participated in the production of data, who were divided into five pairs and one individual (one of the participants in one of the Grade 11 pairs had an unforeseen event on the day of the first teaching session, but another member proposed to develop the activities individually), according to the grouping: two pairs of Grade 10 students; one pair and one individual of Grade 11 students; two pairs of Grade 12 students. We opted for this type of grouping and to develop two teaching sessions with each pair in the afternoon, on different days and separately, so that the data produced could be as detailed as possible, since there was a limited amount of electronic equipment. Besides, the school held only morning shifts. Also, we considered that placing the participants in an unfamiliar scenario would bring greater comfort and dialogue between those involved.

[6] Free software that provides real-time source and device capture, scene compositing, encoding, recording and live streaming. Available at: <https://obsproject.com/pt-br/download>. Last accessed on: 28 Apr. 2021.

Five activities or tasks based on the use of GeoGebra were elaborated in this research: two exploring limited regions and three exploring solids of revolution. All the activities were designed so the students turned to dialogue, manipulation, and mathematical investigation, aiming to approximate and generalize the presented problems. The activities were proposed in the sequence[7]:

> Grade 10 - High school
> - Activity 1: Squaring the circle (investigation of the area of the circle);
> - Activity 2: Cubature of the sphere (investigation of the volume of the sphere).
>
> Grade 11 and Grade 12 – High school
> - Activity 3: The problem of area (investigation of the area under the curve of the parabola);
> - Activity 4: The problem of volume (investigation of the volume of the paraboloid);
> - Activity 5: The relationship between volumes (investigation of the relationship between the volume of the cylinder, cone, and paraboloid, all with the same base and height).

In this article, we specifically discuss some emerging results from the exploration of Activity 4 by the duo formed by the third graders Janaina and Matheus. The choice of this pair was based on the notion called convenience sampling (Marshall, 1996), which involves the selection of more accessible subjects, which makes it less rigorous. Also, judgmental sampling (also known as purposeful sampling) is the most common sampling technique for qualitative research and allows the researcher to actively select the most productive sample to answer the research question (Marshall, 1996). Finally, we also consider the theoretical sample, which is generally theory-driven to a greater or lesser extent and requires the construction of interpretive theories of emerging data (Marshall, 1996).

Data analysis was significantly based on the video analysis model proposed by Powell, Francisco, and Maher (2004). This model is composed of the following non-linear procedures: (a) Familiarization with the data: watching the video recordings several times; (b) Description: preparation of written records describing the recorded events; (c) Transcription: preparation of records that accurately represent the speech and gestures of students and session participants; (d) Identification of critical events: an event is critical when it represents evidence for the proposed guideline questions; (e) Coding: creation of codes for the diversity of critical moments

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[7] In addition to understanding differential thinking, in Baron's (1985) and Edwards's (1979) productions, this research sought aspects of differential thinking in historical problems involving calculating areas and volumes. Several important problems in the history of integral calculus directly influenced the design of the activities elaborated in this research. In addition, the National Common Curriculum Base was present for the understanding of the scarce work in high school exploring the elements of differential and integral calculus.

that help to identify patterns and units of meaning in the analytical process; (f) Episode and plot creation: refers to the text that compiles the various critical moments and the process of contrast with other sources of data such as field notes; (g) Narrative composition: particular interpretation of the whole using the data as evidence, thus producing a written narrative.

Results and Discussions

The objective proposed in the activity "The problem of volume" is to bring to the three-dimensional (3D) environment the investigation that previously occurred in "The problem of area," but now to determine the volume comprised between the surface of the paraboloid generated by the revolution of the curve given by $y = x^2$, $x \in [-1,1]$ and the plan given by $y = 1$. As in the activities "Quadrature of the circle" and "Cubature of the sphere," this research proposes and recommends that the activities "The problem of area" and "The problem of volume" be worked on sequentially, but this idea is not fixed and allows adaptations by the person who intends to develop it. In the second teaching session, held on November 13, 2019, there was a moment of socialization with GeoGebra (lasting 17'37"), focused on the 3D environment.

The changing of environments (from 2D to 3D), the input (+ Entrada...), and the tools Move ( ), Point ( ), Cylinder (Cilindro), Sphere ( ), Cone (Cone), and Volume (Volume) were some of the elements explored in the socialization. When starting the activity of the teaching session, the teacher-researcher asked the students about the curve of the parabola studied the previous day; then, returning to the 2D Window of GeoGebra, the pair built, again, curve $y = x^2$, $-1 \leq x \leq 1$. With the visual construction completed, students were asked to imagine the curve rotating around the y axis and what the visual result of this action would be. In the first attempt, Matheus associated a "pot" and Janaina an "antenna," revealing elements of the importance of metaphorical/analogical thinking in scenarios focused on mathematical thinking. As the objects mentioned were 3D, this moment was carried out by requesting that the pair, still in that same GeoGebra file and without changing the element present in the 2D Window, open the 3D Window. Then, the students were asked to look at Step 1 (Table 4) of the printed activity they received a little before socializing with GeoGebra and to develop the items in that step.

Table 4.

Step 1 – Instructions for starting the activity "The problem of volume"

1. *Construction of the surface of revolution in the interval* $-1 \leq x \leq 1$ *using GeoGebra.*

a) *Click on "Entrada"* [+ Entrada...]*, type "Função", choose the option* [**Função**(<Função>, <Valor de x Inicial>, <Valor de x Final>)] *clicking on it and, thus, type inside the parentheses "*$x^2, -1,1$*". Press "Enter"* [↩].

b) *Open the 3D view window by clicking on* [icon]*, present in the upper right corner, followed by* [⋮] *and choose the option "Janela de Visualização 3D"* [Janela de Visualização 3D].

c) *Hide the "Janela de Visualização" by clicking on* [⋮]*, then* [✕ Fechar].

d) *Position the Y Axis as vertical. For that click on* [icon]*, then* [⚙] *select the option* [☐ eixo y é vertical]*. Also access the icon* [icon]*, choosing the option* [icon].

e) *Click oo* [+ Entrada...]*, type "Superfície", choose* [**Superf**ície(<Curva>, <Ângulo>, <Reta>)]*, then type inside the parentheses "*$nome\ da\ função\ do\ item\ 1 gerado\ pelo\ GeoGebra, 2\pi, EixoY$*". Press* [↩].

Executing the steps described in the task, the students completed the paraboloid plot in GeoGebra (Figure 1).

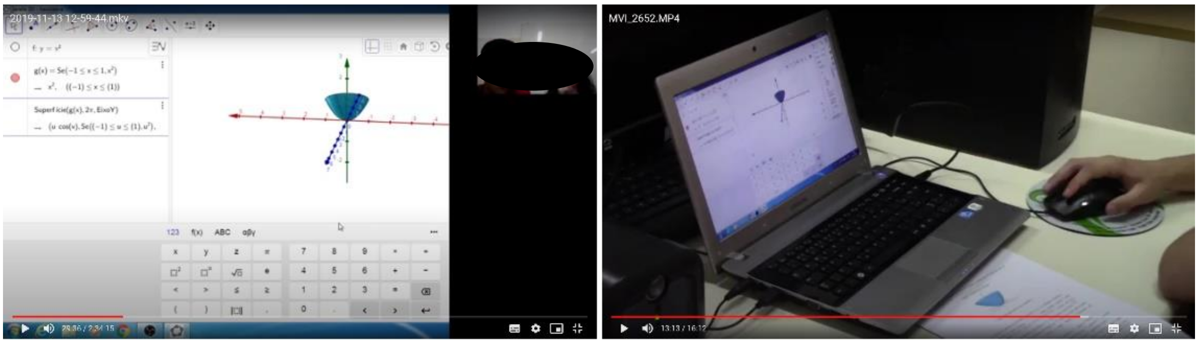

Figure 1.

Construction of the research environment – Janaina and Matheus. Source: Research Data.

After the construction made by the duo, the dialogue went on. The following transcript represents this moment.

> Researcher: So, we are going to calculate the volume of this figure... So, the activity, which is called "The problem of volume," consists of determining the volume of the paraboloid... So, this image has this name, okay?... Paraboloid… Obtained by revolving around the y-axis, the curve $y = x^2$, $-1 \leq x \leq 1$ ... Which was all we had done in terms of construction, okay? And now, how do you think we will be able to calculate the volume for this construction?
> Janaina: Do you have any tools?
> Matheus: Yeah, there's the one with the volume we used...

[Matheus continues to select the Volume 

tool that was previously explored, clicks on the paraboloid image, and the duo realizes that nothing has appeared]
Janaina: I don't think it will...
[Matheus repeats the action done previously, but nothing changes]
Matheus: You try [passing the mouse to Janaina]... Try your luck... [Janaina takes control of the mouse]
Researcher: What do you think is going on?
Janaina: It's not working...
Matheus: It's not going...
Researcher: Why is the volume calculation tool not working?
Janaina: Because it's open... Maybe...
Researcher: Hmm... Yes... Because this image we built... This image we built, in fact, is the shell of our paraboloid... It's the surface, not all the space filled inside. But we are going to use this cone as a basis to calculate the volume that would be inside it. All good? So, what now? If we can't use the Volume tool that GeoGebra brings, which strategy or strategies do you think can be used to solve what is being proposed, which is to find that volume?
Janaina: Ahm, let me see...
Matheus: We can draw another one... The cylinder, something... A square... But I think you'd better see it... [Matheus continues his speech, but we could not transcribe it due to audio problems]
[Janaina and Matheus continue to dialogue, but we could not transcribe it due to audio problems]
Matheus: If we make a cylinder here, it's the same as yesterday, doing it by elimination...
Jane: Uhm...
Matheus: Incredibly, that's it! [Matheus continues his speech, but we could not transcribe it due to audio problems]
(Dialogue between researcher and students, 2019).

In this transcribed excerpt, it is possible to see the agility with which the students brought the strategy of using the approximation to calculate the desired volume to the activity, as was done in the previous activity. Therefore, we consider relevant the recommendation to work the activities sequentially, at least for high school students, since elements of differential and integral calculus are little explored in this level of education. The dialogue between students continued and the first ideas were built with GeoGebra, as can be seen in Figure 2 below.

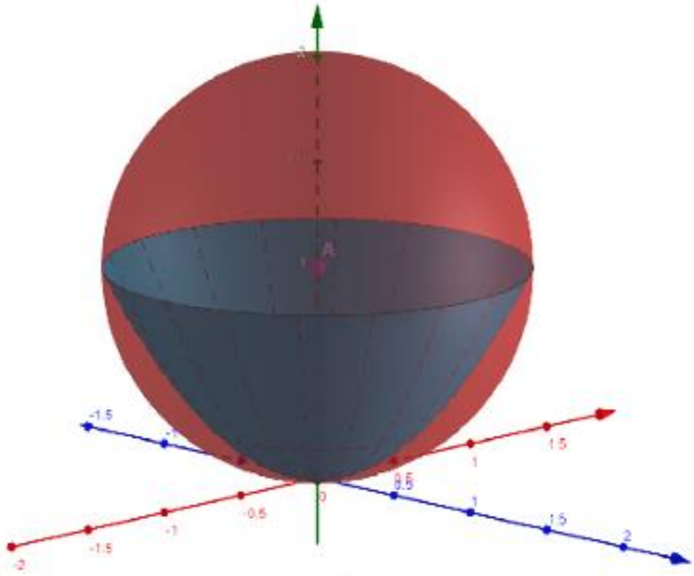

Figure 2.

Construction completed - Paraboloid volume approximation – Janaina and Matheus (1st attempt). Source: Research Data.

After construction, the pair worked with GeoGebra's zoom to zoom in on objects and compare volumes. After a few minutes, a dialogue began between those present.

> Researcher: How much do you think, compared to the sphere?
> Matheus: Half...
> Jane: Yeah...
> Matheus: Or less than half?
> Janaina: A little less than half.
> Matheus: Less than half, because if it were half, it would be all of that... [moving the mouse over the constructed image and indicating a semi-sphere] So it would be around two... two and ten...
> [Janaina commented something, but we could not transcribe it due to audio problems. After the researcher asked the student to speak a little louder, the dialogue continued]
> Janaina: Yeah... I think it's going to be like two and... Yeah, I give two... I take this ninety-five...
> Matheus: I give ten... Two and ten!
> Janaina: I give two! Or less than two, I don't know...
> (Dialogue between teacher-researcher and students, 2019).

In this first attempt at approximation made by the duo, it is possible to perceive the emergence of the following aspects of differential thinking: **Notion of limit and continuity**, for having constructed a sphere circumscribed to the paraboloid; and **Visual-geometric design**, for building an object for comparison. When the students were asked which figure they believed that, when the construction was carried out, the volume would be closer to the one we were investigating, Matheus brought the cone to the discussion, arguing how it would appear visually in the GeoGebra window, while hovering the mouse over some software tools. Another idea the duo had discussed earlier and surfaced then was the possibility of developing the approximation through the cylinder. So, on the second attempt, Janaina and Matheus chose to follow the path of the cylinder, as can be seen in Figure 3, below.

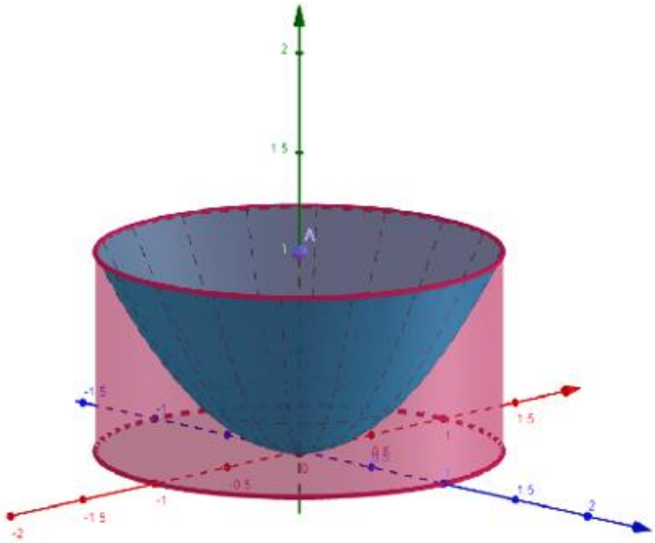

Figure 3.

Construction completed - Paraboloid volume approximation – Janaina and Matheus (2nd attempt). Source: Research Data.

> Researcher: And now, what can you say about the volume of the paraboloid compared to the volume of the cylinder, then? What approximations are you going to make?
> Janaina: The hard part is that there was a lot left over...
> Matheus: Much more!
> Janaina: It's much bigger.
> Matheus: There would be plenty of space then...
> Janaina: Would it make any difference if we took them both out?
> Matheus: What? How do you speak?
> Janaina: So, subtracting the value we found in the cylinder with the cylinder, with this one... Like we didn't do in the other one... I don't think so, right?
> Matheus: You have to think about the following... Hold on, I'm going to turn on the other one [removing the omit option that was applied to the sphere]... What's the diff... Oh... [moving the objects to change the viewing angle]
> Janaina: Hmm... It won't interfere with much...
> Matheus: Because what comes closest here is the sphere, right?
> Janaina: Yes, the sphere is much better.
> (Dialogue between researcher and students, 2019).

At that moment, even if the students did not present a new approximation for the volume of the paraboloid, it is possible to identify the conclusion of an approximation reasoning based again on aspects of differential thinking of “Notion of limit and continuity” and “Visual-geometric conception.” In search of new ideas to explore with GeoGebra and during a conversation between the students, Janaina resorted to the printed activity on the table and, flipping through the pages, identified the word “Cone.” Thus, the pair of students remembered that they had not built this scenario and opted to develop it, as can be seen in Figure 4[8]. The dialog initiated after construction is shown below.

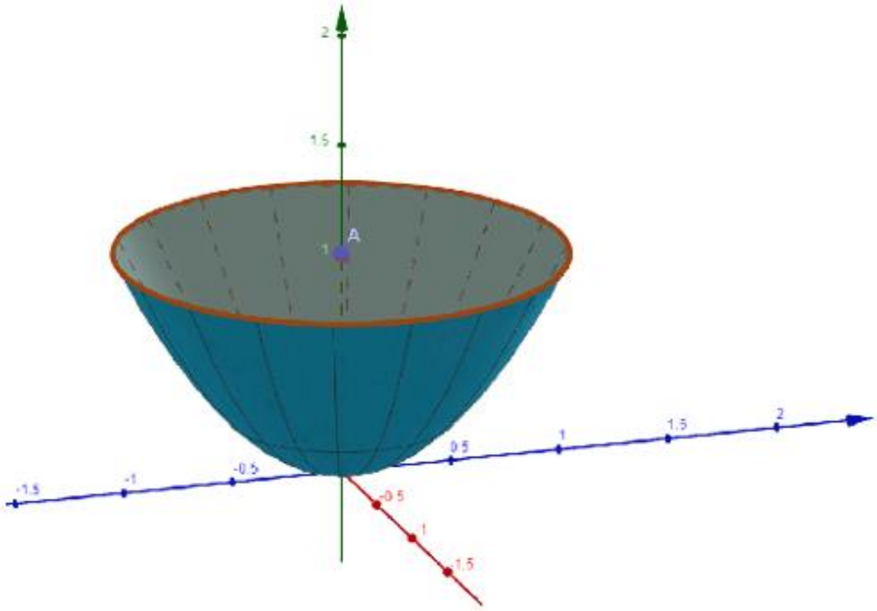

Figure 4.

Construction completed - Paraboloid volume approximation – Janaina and Matheus (3rd attempt). Source: Research Data.

Matheus: My God! It's almost the same!
Janaina: Yeah, it fit! Wow... what a beautiful thing!
[After guiding the students when reducing the transparency of objects built in GeoGebra, and applying what was exposed in the paraboloid, the conversation continued].
Janaina: Look, it's the circumference space! It is not?
Matheus: What?
Janaina: This is the missing circumference, isn't it? No? It will be?
Matheus: So we have to find it... Hold on... [Matheus moves the scroll bar on GeoGebra's algebra window]
Matheus: The cone... The cone has one and five! [referring to the volume of the cone that was presented by GeoGebra: 1.05]
Janaina: One and five...
Matheus: You said two... [referring to the approximate result by Janaina in the first attempt]
Janaina: I said two...
Matheus: So...

[8] Due to the little transparency present in the construction of the paraboloid carried out by the students and, given that the volume of the cone is smaller than the volume of the paraboloid (both with the same base and height), it is not possible to see the body of the cone being explored, only your base.

Researcher: How much do you think the volume of the paraboloid will be?
Janaina: Oh, I'm still between the two anyway...
Matheus: I'll stay... [looking at the image while thinking] One eighty!
Janaina: One and eighty... Between one and eighty-two. It's there!
(Dialogue between researcher and students, 2019).

As in previous attempts, this approach explored by the duo and the entire approach proposal that has been followed so far has shown evidence of the emergence of aspects of differential thinking: **Notion of limit and continuity** and **Visual-geometric design**. Janaina and Matheus kept thinking about how to bring the results they were getting closer to the value of the paraboloid volume. They chose to leave the paraboloid, the cone, and the sphere (built on the first attempt) visible in the GeoGebra 3D Window. The duo followed the strategy of working the difference between the volumes and performed the calculation: $Volume_{semiesphre} - Volume_{cone} = 2{,}095 - 1{,}05 = 1{,}045$. Faced with the result, the pair reflected, trying to interpret the value found. They went back to observing the objects built in GeoGebra, omitted the sphere, changed the viewing angle they were having of the paraboloid and the cone, dialogued, and then concluded that the desired volume would be a value between 1.15 (approximation made by Matheus) and 1.20 (approximation made by Janaina). Even without making new constructions in GeoGebra, but thinking-with-the-software, the students could think and propose a new approach to what they were investigating, with the aspect of differential thinking arising from **the notion of limit and continuity** emerging in this process. When the pair finished the previous reasoning, it was suggested that they start a new strategy to identify a value even closer to the expected one. Faced with the doubt that hovered over the students on how to proceed with the investigation, Janaina and Matheus resumed reading the printed activity and, in view of the figures that were present, decided to follow the same strategy, which consisted of inscribing and circumscribing cylinders to the paraboloid. The fourth attempt built by the students generated the object shown in Figure 5.

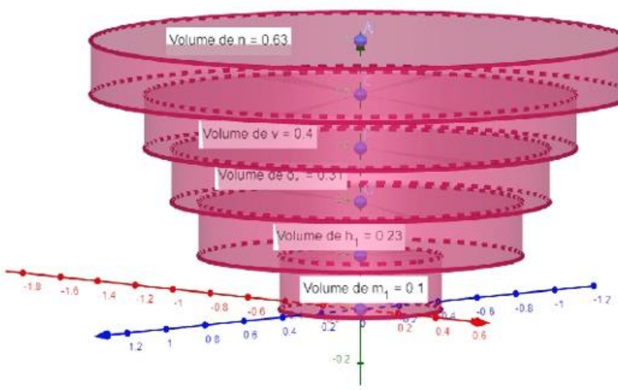

Figure 5.

Construction completed - Paraboloid volume approximation – Janaina and Matheus (4th attempt). Source: Research Data.

> When adding the values of the volumes of the cylinders that were built, the following dialog occurred:
> Janaina: It was one sixty-seven!
> Researcher: Hmm... So, is this thing good?
> Matheus: No...
> Researcher: But do you think it's closer?
> Janaina: Yeah! Much closer...
> Researcher: How much do you estimate it will be then? More than that amount? Less than that?
> Janaina: Less...
> Researcher: How much do you think it will be?
> Janaina and Matheus: It will be less!
> Janaina: So I think it will be one and...
> Matheus: Forty... One and forty to fifty...
> Janaina: Yeah... Between one-forty and one-fifty.
> (Dialogue between teacher-researcher and students, 2019).

From that point on, the students continued with the idea of cylinders inscribed and circumscribed to the paraboloid, but seeking to improve the constructed objects (radius adjustment based on the curve given by $y = x^2, x \in [-1,1]; = x^2$; number of cylinders; among others), as well as the simultaneous use of the scenarios to work on the approximation based on the arithmetic mean between the sum of the volumes of the registered cylinders (V_i) and the sum of the volumes of the circumscribed cylinders (V_c). Figure 6 below shows the development of the fifth attempt made by the duo Janaina and Matheus.

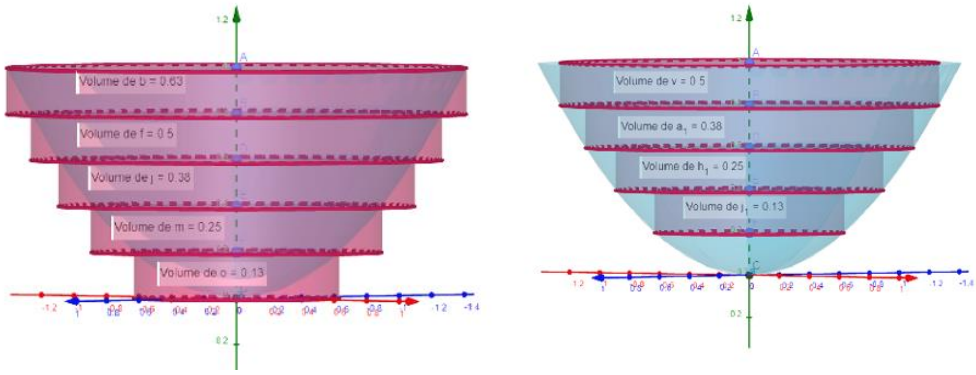

Figure 6.

Construction completed - Paraboloid volume approximation – Janaina and Matheus (5th attempt – Part I). Source: Research Data.

When calculating the arithmetic mean above, the students arrived at a value of 1.59. Mentally rescuing the constructions explored in this attempt, Matheus came to believe that the volume of the paraboloid would be 1.40, and Janaina 1.45. In this sense, at this stage the following aspects of differential thinking are present: Notion of limit and continuity; Notion of infinitesimal; Definite integral concept; and Visual-geometric design. According to the transcript presented after Figure 7, when asked if it was possible to get closer to the desired value, Janaina and Matheus acknowledged it was possible. Then, a new question was asked by the researcher, now seeking to understand which strategy the pair could adopt. The students immediately responded that it would be necessary to reduce the spaces and make more figures, as on the previous day. And so began the sixth investigation attempt by Janaina and Matheus, now building twice as many cylinders in each scenario (inscribed and circumscribed), as shown in Figure 7.

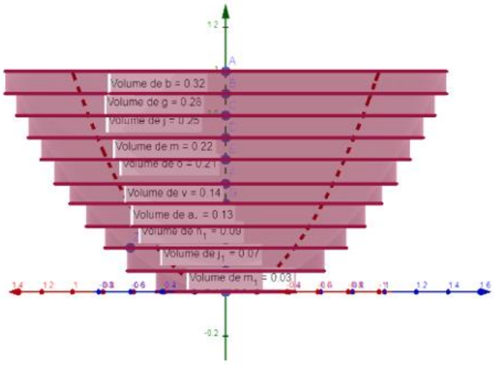

Figure 7.

Construction completed - Paraboloid volume approximation – Janaina and Matheus (6th attempt – Part I). Source: Research Data.

After calculating the arithmetic mean for this attempt, the dialogue transcribed below began.

> Researcher: And now, do you think it is approaching what value is the volume then?
> Janaina: One fifty-five...
> Matheus: Ahm... It's going to be one fifty-eight... five thousand eight hundred something... Almost reaching fifty-nine.
> Researcher: Look, we had five cylinders, our average was one and fifty-nine, right? Then ten cylinders, one fifty-eight. Do you think that if we continue to increase the cylinder, this value will decrease a lot?
> Matheus: No! I think he will keep. A maximum of one and fifty-six, like this...
> Jane: Yeah...
> Matheus: Fifty-seven, bursting...
> Researcher: Okay, beauty! So, ok, it's another way we have to work volume by approximation, equal area, ok? We don't need to have the formula to calculate the volume of the paraboloid... If I know how to calculate the volume of another figure, I can work it several times inside or outside and work the average in our last case, right?
> [Janaina nods her head yes]

Researcher: Ahm... So, what are you going to close on? Bursting what value for our volume?
Matheus: Are you going to take a chance on fifty-seven or fifty...? [asking Janaina]
Janaina: [Janaina comments on something that it was not possible to transcribe due to audio problems] One fifty-seven [Matheus nods in agreement]. One fifty-seven! We closed at one fifty-seven!
(Dialogue between teacher-researcher and students, 2019).

Therefore, given the result that the pair decided to assume as an approximation of the paraboloid volume, the investigation of this activity ended. In this sixth and last attempt made by the students, the emergence of the four aspects of differential thinking is notorious: **Notion of limit and continuity**; **Notion of infinitesimal**; **Definite integral concept**; and **Visual-geometric design**. In all, there were approximately one hour between the beginning of the first attempt and the end of the sixth attempt. As it is a high time count, and trials 5 and 6 follow exactly the same proposal as trial 4, in addition to presenting the emergence of the same aspects of differential thinking, the time resulting from trial 1 to 4 was transformed into the diagram shown in Figure 8, which brings a column for each minute of investigation among the selected attempts and highlights, in green, the aspects of differential thinking that emerged within each of these minutes. The orange color, in addition to highlighting the emergence of the aspect of differential thinking, indicates the change of tool (use) in GeoGebra. Purple, in addition to signaling everything that the other colors represent, indicates a prolonged period of repetition of some action performed by the students.

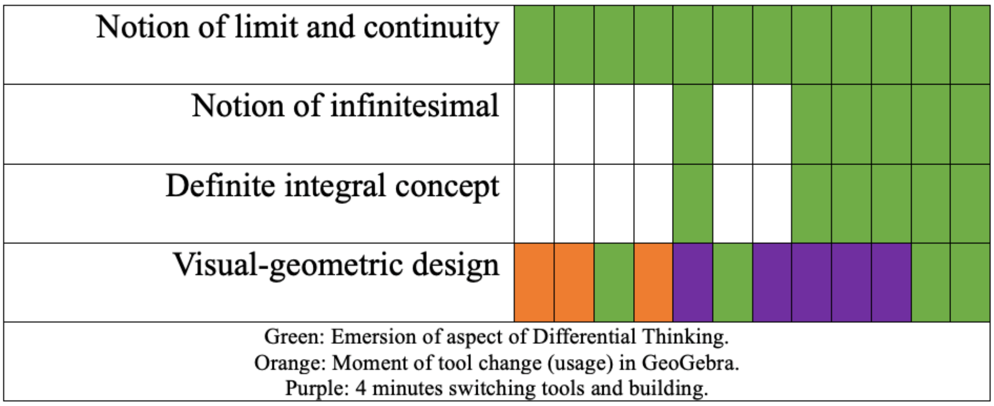

Figure 8.

Aspects of differential thinking emerged in exploring "The problem of volume"– Janaina and Matheus (1st to 4th investigation attempt). Source: Research Data.

From the diagram, we can verify the constant emergence of aspects of differential thinking, **Notion of limit and continuity,** and **Visual-Geometric Conception**. The differential thinking aspects of the **Notion of Infinitesimal** and the **Concept of a Definite Integral** emerged from the fourth attempt made by the pair (it is worth noting that attempts 5 and 6 were not represented in the diagram because they follow the same proposal as attempt 4 and present the emergence of the same aspects of the differential thinking), since the approach was different from those carried out previously and they were directly related. The resources of GeoGebra, in this episode, offered means for exploring all aspects of differential thinking but with emphasis on two of them: the notion of limit and continuity and the visual-geometric conception. The visualization and dynamism of the software that allows simulations and tests to be carried out, in this sense, were fundamental for the development of the students' differential thinking.

Conclusions

This study was initiated with the aim of identifying which aspects of differential thinking emerged in a group of high school students during the investigation of activities involving the calculation of areas and volumes. For this, teaching sessions were developed (Steffe & Thompson, 2000) with high school students from the 1st, 2nd, and 3rd grades. During the teaching sessions, organized in pairs, the students explored the proposed activities using the GeoGebra software. In this context, the question that guided this study was: What aspects of differential thinking and thinking-with-GeoGebra emerge when high school students investigate activities about calculating areas and volumes? To discuss the understanding of differential thinking, we go through other thoughts present in mathematics and base ourselves on the conceptions of Sad (2000), Reis (2001), Wielewski (2005), and Henriques (2010) when affirming that differential thinking is a kind of mathematical thinking and that, even with some particularities, these and other thoughts complement each other. The understanding that this study had regarding aspects of differential thinking, also based on the cited authors, is a manifestation of concepts of differential and integral calculus emerging in a formal and/or intuitive way associated with the idea of thinking-calculus.

Mathematical research (Ponte, Brocardo, & Oliveira, 2016) and thinking-with-GeoGebra (Borba, Scucuglia, & Gadanidis, 2018) were used in this study. In this thinking-with-GeoGebra process, the authors point out that the chosen and employed digital technology has a purpose beyond common use as a tool but is inserted as a basis for the investigation. We chose to work with the GeoGebra software for easy access, exploration, and visualization. For data production, filming the teaching sessions, recording the computer screen and webcam, files generated in GeoGebra, and notes on paper were used.

The study corroborates the idea that developing activities or tasks that integrate the fundamental ideas of calculus through problem solving, “enables students to develop skills aimed at research, formulation, explanations, and arguments, (...), thus constituting an opportunity for the development of higher-order skills” (Siple, Figueiredo, & Herbst, 2023, p. 110). Specifically, in the research reported in this article, GeoGebra was fundamentally important in the visual representation and movement of objects since the software enabled the rapid execution of research proposals made by students and helped discover new relationships and constructions. GeoGebra, associated with the teaching and learning process, proved to be an efficient ally for the cognitive development of high school students. It allowed the production of mathematical knowledge related to elements of differential and integral calculus, bringing subtlety to concepts that are presented and studied in higher education. Thinking-with-GeoGebra emerged during the investigation of activities. As they were unfamiliar with handling the software, the students started the conjectures and verifications by talking and detailing the steps they were performing. This approach also allowed the emergence of aspects of differential thinking but with a longer time interval and the discontinuity of aspects of the Notion of limit and continuity and Visual-geometric conception. Given this, it is possible to conclude that thinking-with-GeoGebra potentiated the emergence of aspects of differential thinking.

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