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From mathematical content to processes of meaning production: a possibility for mathematics teachers education

Del contenido matemático a los procesos de producción de significado: una posibilidad para la formación de profesores de matemáticas

Du contenu mathématique aux processus de production de sens: une possibilité pour la formation des professeurs de mathématiques

De conteúdos matemáticos para processos de produção de significados: uma possibilidade para formação de professores de matemática

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Abstract

This essay focuses on the mathematical education of mathematics teachers. In the first section, we present key findings from several studies that address the topics, establishing a foundation for our subsequent discussions. In the second section, we outline some studies which use the Model of Semantic Fields as their primary theoretical and methodological framework to specifically address this topic. In the final section, we analyze and propose various approaches for the development of alternatives for initial mathematics teacher education. One of such approaches would involve organizing teacher training processes through activities that are centered around ordinary life activities. Teachers and prospective teachers could generate discussions and problematizations by sharing and elaborating on various methods of producing meaning. This process aims to highlight differences rather than replace or prioritize knowledge. Additionally, it enables the inclusion of mathematical content prescribed in curricula.

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Keywords: Mathematics teacher education, mathematics education, Model of semantic fields.

Resumen

En este ensayo presentamos una discusión a respecto de la formación (matemática) del profesor de Matemática. En la primera parte, destacamos anotaciones de algunas pesquisas que tratan del tema, delimitando un terreno para nuestras discusiones. En la segunda, presentamos algunos trabajos que abordan esa temática, los cuales toman como principal referencia teórico-metodológica el Modelo de los Campos Semánticos. En la tercera y última parte, problematizamos y apuntamos algunos delineamientos para la construcción de otras posibilidades de formación para/con Profesores de Matemática. Una de ellas sería una organización de procesos de formación de profesores tomando como referencia actividades basadas en categorías del cotidiano. A partir de situaciones presentadas a profesores y futuros profesores, discusiones y problematizaciones pueden ser producidas por medio de compartimientos y ampliaciones de modos de producción de significados, en tentativas de un detallismo de las diferencias y no de la substitución y jerarquización de conocimientos, lo que no excluye abordajes de contenidos tenidos como matemáticos y prescriptos en currículos.

Palabras-clave: Formación de profesores de matemática, Formación matemática, Modelo de los campos semánticos.

Résumé

Dans cet essai, nous présentons une discussion sur la formation (mathématique) des professeurs de mathématiques. Dans la première partie, nous mettons en évidence des notes issues de diverses recherches qui traitent du sujet, délimitant ainsi un terrain pour nos discussions. Dans la deuxième partie, nous présentons quelques travaux qui abordent ce thème et qui prennent comme principale référence théorique et méthodologique le Modèle des Champs Sémantiques. Enfin, dans la troisième et dernière partie, nous problématisons et indiquons quelques orientations pour la construction d'autres possibilités de formation pour les professeurs de mathématiques. L'une d'entre elles consisterait en l'organisation de processus de formation des professeurs basés sur des activités liées aux catégories du quotidien. À partir de situations présentées aux professeurs et aux futurs professeurs, des discussions et des problématisations peuvent être produites grâce à des partages et à l'élargissement des modes de production de significations, dans des tentatives de détailler les différences plutôt que de remplacer ou les hiérarchiser des connaissances, ce qui n'exclut pas des approches de contenus considérés comme mathématiques et prescrits dans les programmes.

Mots-clés : Formation des professeurs de mathématiques, Formation mathématique, Modèle des champs sémantiques.

Resumo

Neste ensaio apresentamos uma discussão a respeito da formação (matemática) do professor de Matemática. Na primeira parte, destacamos apontamentos de algumas pesquisas que tratam do tema, delimitando um terreno para nossas discussões. Na segunda, apresentamos alguns trabalhos que abordam essa temática, os quais tomam como principal referência teórico-metodológica o Modelo dos Campos Semânticos. Na terceira e última parte, problematizamos e apontamos alguns delineamentos para a construção de outras possibilidades de formação para/com professores de Matemática. Uma delas seria uma organização de processos de formação de professores tomando como referência atividades baseadas em categorias do cotidiano. A partir de situações apresentadas a professores e futuros professores, discussões e problematizações podem ser produzidas por meio de compartilhamentos e ampliações de modos de produção de significados, em tentativas de um detalhamento das diferenças e não da substituição e hierarquização de conhecimentos, o que não exclui abordagens de conteúdos tidos como matemáticos e prescritos em currículos.

Palavras-chave: Formação de professores de matemática, Formação matemática, Modelo dos campos semânticos.

From mathematical content to processes of meaning production: a possibility for mathematics teachers education

In 2001, a report published by Wilson et al. highlighted the lack of clarity regarding the correlation between specialized instruction regarding certain contents during undergraduate studies and teacher proficiency regarding said content upon completing their degree. According to the authors, this finding indicated a pressing need for research that systematically investigates the characteristics of instruction in specific content and how such training manifests in the classroom behavior of teachers. In a way, this concern, and the aforementioned report, also emphasize a deeper understanding of the so-called pedagogical preparation.

The 2001 Curricular Guidelines for mathematics courses mandated a 400-hour workload for supervised internships in undergraduate programs. Additionally, as a curriculum component, there was a requirement for an additional 400 hours of practice focused on the professional demands of mathematics teachers. These prerequisites for formulating political-pedagogical projects for undergraduate mathematics courses introduced additional considerations for the design and implementation of such courses, offering opportunities to deviate from the fixed *3+1 model*. However, whether in the document itself or the underlying expert opinion, there are limited explicit indicators of the precise mathematical knowledge required for educating mathematics teachers, with a focus on their practice in elementary education. There are indications of increased focus on internships and pedagogical knowledge required for the profession. Nonetheless, it is still important for a mathematics teacher to have a solid foundation in mathematical disciplines such as integral differential calculus and real analysis. This suggests that the fundaments of mathematics are primarily learned through academic mathematics courses.

In 2015, new National Curricular Guidelines were enacted contemplating initial preparation at higher education (degree courses, pedagogical training courses for graduates and additional degree courses) as well as continuous education (Brazil, 2015). The new guidelines superseded National Educational Council Ruling number 1 of February 18, 2002 (*Resolução CNE/CP nº 1*) and subsequent amendments, as well as National Educational Council Ruling number 2 of February 19, 2002 (*Resolução CNE/CP nº 2*) and subsequent amendments. However, the workload for internship and practice as a curricular component was not altered. The abovementioned guidelines were recently revoked by National Educational Council Ruling number 2 (*Resolução CNE/CP nº 2*) of December 20, 2019, which defined the National Curricular Guidelines for initial education of elementary education teachers and created the National Core Curriculum for initial education of basic education teachers (*Base Nacional*

Comum para a Formação Inicial de Professores da Educação Básica -BNC-Formação) (Brasil, 2019). Regarding the 2015 recommendations and the curricular guidelines for mathematics courses, the workload for supervised internship and for a discipline renamed as *practice of curricular components,* were kept consistent. Given the overarching objectives of the 2015 and 2019 recommendations, which were to provide teacher education in a broad sense, irrespective of individual qualifications, there was no opportunity for detailed discussions regarding mathematical content.

From this brief chronological outline of legislation which affected the structuring of undergraduate courses in mathematics, we observed that, since the publication of the curricular guidelines for mathematics courses, in 2002, the disciplines of academic mathematics, the so-called *fundaments of mathematics necessary for the mathematical preparation of the teacher who will teach mathematics,* still remain untouched in the official guidelines. Therefore, the structure of undergraduate courses in mathematics still does not deviate from what the legislation recommends, thus establishing a separation between disciplines with specific content and the so-called pedagogical disciplines (Lins, 2004; Gatti, 2010; Moreira, 2012; Viola dos Santos & Lins, 2016; Oliveira et al, 2021). Educational spaces remain predominantly silent, with few details and discussions that challenge the so-called academic mathematics.

There is a limited amount of research in mathematics education regarding the mathematical preparation provided to mathematics teachers by undergraduate courses. Throughout the years, some studies have been conducted, as evidenced by the works of Wilson et al. (2001), Moreira (2004, 2012), Moreira & David (2005, 2008), Linardi (2006), Lins (2006a, 2006b), Oliveira (2011), Viola dos Santos & Lins (2014, 2016), Fiorentini (2005), Fiorentini & Oliveira (2013), Elias et al. (2015), Elias (2018), and Luchetta & Viola dos Santos (2020).

Among numerous possible reasons, two clear factors may be identified to argue in favor of the limited amount of research being conducted. Firstly, discussions regarding mathematical education in undergraduate courses encompass mathematicians and address aspects pertaining to their professional practices. Furthermore, there has been a long-standing dispute regarding allocation of resources withing the educational process, with a clear division between specific content and pedagogical content. Thus, the controversy surrounding the mathematical education of future teachers is inevitable.

The aim of this essay is to explore the (mathematical) preparation of mathematics teachers, initially classified as mathematics. Ultimately, we will give an alternative approach to this education without relying on classifications. The initial section of the essay focuses on summarizing findings from relevant research, establishing a framework for subsequent discussions. The second part presents a discussion of research that focuses on this topic and uses the Model of Semantic Fields (MSF)[4] as the primary theoretical and methodological framework (Lins, 1999; Lins, 2004; Lins, 2012). In the concluding section, following the introduction of various issues, we outline a set of principles for the development of an alternative educational program for mathematics teachers.

Research regarding the education of mathematics teachers

As part of her doctoral research, Oliveira (2011) examined how researchers in mathematics education study the treatment of mathematical content in investigations related to the education of mathematics teachers, focusing on the themes covered in mathematical training. To this end, she conducted a comprehensive national bibliographic analysis of three journals: Educação Matemática em Revista, Boletim de Educação Matemática (BOLEMA), and Zetetiké, spanning from 1997 to 2007. Additionally, she examined dissertations, theses, and books related to the topic. Based on this analysis, Oliveira (2011) provided comments and considerations that aim to develop an understanding of the texts studied. "The main focus was on how authors view/perceive/discuss mathematical content or the so-called specific disciplines in mathematics undergraduate courses" (p. 51). The publications reviewed by that author were organized into six groups: 1) Regarding the teaching and learning of mathematics: its nature and concepts; 2) Attitudes and approaches while dealing with mathematical content; 3) Exploration of mathematical content through the use of methodologies and tools; 4) Approaches and discussion of content in educational activities of mathematics teachers; 5) Mathematical content, representations and meanings; 6) Relevance of certain contents as part of the curriculum.

Based on the discussion offered in this review, we observed that the writers of the work analyzed employ various approaches to address mathematical content in their research. The topics discussed encompass several methodologies, ranging from the fundamental nature of

[4] The Model of Semantic Fields, according to the author, "intentionally seeks to preserve interaction (or communicative spaces)," (Lins, 2008, p. 545, author's emphasis); it incorporates into its constitution elements that enable this preservation to be expressed, including purpose and intention. This can be viewed as a "tool for research and development in mathematics education, as well as for guiding classroom practices and for enabling teachers to produce a sufficiently fine, thus useful, reading of the process of meaning production in the classroom". (Lins, 2001, p. 59).

mathematics to the applicability of specific themes within the mathematics degree program. According to Oliveira (2011), among research that has been published

> [...] specific types of knowledge, separate from those found in academic mathematics, are acknowledged and valued as valid and significant; the significance of street mathematics is examined and categorized, as well as the creation of new meanings, potentially related to mathematics, that will exist alongside such existing street (non-mathematical) meanings, without attempting to supplant them; interrelationships between many branches of mathematics are continuously analyzed, with a focus on the power dynamics inherent in their respective applications; social practice of mathematics teachers is viewed as an articulating element of specific disciplines which the teaching degree encompasses; didactic-pedagogical sequences which problematize concepts and conceptual representations of undergraduate students regarding mathematical content are created, deepening intuitive views within teaching practice; focus on mathematical content is directed towards the meanings it generates, in order to explore student learning and comprehension employing concepts such as mandatory curricular components and mathematical knowledge for teaching, with the aim of enhancing teaching practice (p. 106).

While the aforementioned publications touch on multiple topics beyond mathematics, the majority does not address any mathematical content. Generally, content is widely accepted and considered to be previously established. “Content is therefore content. This exemplifies one of the most deeply rooted beliefs held by teachers: viewing mathematics as a collection of entities existing separately from the actions involved in manipulating such entities” (Garnica, 2008a).

This notion is also evident in mathematical education programs. There is a general acceptance, with little scrutiny, of the inclusion of disciplines including mathematical content in undergraduate courses, based on the premise that it is important for the mathematics teacher to have a strong command of mathematics. Mathematics teachers must possess a clear and evident understanding of mathematics. It is evident that mathematics teachers must know mathematics! However, in what way will this knowledge be constructed? According to Garnica (2008a), it is important to conduct research which supports the development of intervention plans that focus on reevaluating the methods used for teaching mathematical concepts in disciplines that involve mathematical content included in mathematics teaching degrees courses.

We believe that research regarding the mathematical education of mathematics teachers in undergraduate courses is currently in the early stages of developing a new viewpoint on how to prepare mathematics teachers in such courses. The reason is that the indications given are essentially merely simplified guidelines for the required knowledge of teachers and lack a thorough examination of the *reasoning* or *processes* underlying such knowledge.

In a 2005 paper, Fiorentini explores the topic of mathematical and didactic-pedagogical preparation in mathematics degree programs. The article discusses the education of mathematics teachers resulting from didactic-pedagogical subjects as well as mathematical subjects. As per his account:

> The majority of teachers of calculus, algebra, analysis, topology, etc., believe that their role is limited to teaching mathematical concepts and processes. [...] [however they] also teach a manner of being as person and as teacher, that is, a certain way of perceiving and forming connections with the world, as well as with mathematics and its teaching. [...] The prospective teacher not only acquires knowledge of mathematics through them, but additionally assimilates a particular mindset in understanding, handling, and assessing their learning (pp. 110-111).

The author further states that didactic and pedagogical disciplines

> [...] Can [...] contribute to altering the way mathematics is perceived and understood, especially if the focus shifts away from the static, pre-existing knowledge, often presented in teaching manuals, onto dynamic knowledge, in the process of understanding and elaborating meanings, with symbolic language as a mediator (pp. 112)

Fiorentini's arguments seek to reconcile the dichotomy between specialized instruction, which focuses on content, and pedagogical instruction for teaching the same content. This perspective regarding initial preparation presents a promising outlook for the development of teacher education programs that prioritize the professional practice of mathematics teachers, rather than exclusive focusing on mathematics and subsequently on methods for teaching mathematics.

In his doctoral thesis, Moreira (2004) sought to examine the mathematical education provided in undergraduate courses “by analyzing the interplay between the knowledge imparted during the educational process and the knowledge relevant to the challenges faced in professional practice at schools” (p. 1). The research focused on the mathematics teaching degree course at *Federal University of Minas Gerais* (UFMG) and specifically examined the issue of numbers. From a theoretical standpoint that differentiates school mathematics from scientific or academic mathematics, the author asserts that “mathematical knowledge is developed during the education process based on the viewpoint and principles of academic mathematics. This approach disregards significant school-related matters that do not align with such perspectives and values" (p. vii). Considering this, Moreira suggests a reevaluation of mathematical education to effectively reconcile the importance of school mathematics and scientific mathematics in this endeavor.

In 2010, the author published an article titled "Mathematics Education of Elementary School Teachers: which mathematics?". In that article, the author discusses several aspects related to the mathematical education of mathematics teachers. Initially, he introduces a series of concise and insightful assertions:

> Those who want to be mathematics teachers, are going to teach mathematics.
> To teach mathematics, the teacher must know mathematics.
> Therefore, we must teach mathematics to undergraduates (prospective teachers)
> [...]
> Teachers must know more than what they teach.
> Teachers must know academic mathematics in order to have a unified vison of school mathematics. Otherwise, it will turn into a chaotic collection of disjointed regulations and equations.
> The field of scientific mathematics is a notable accomplishment of human civilization and, as such, it should be widely disseminated and taught in schools.
> Developing the scientific spirit in future generations is vital, and to achieve that, schoolteachers must have knowledge of scientific mathematics.
> The primary goal of mathematics education in schools should be the acquisition of knowledge in its abstract and *objective* state. Thus teachers must use academic mathematics as a model. (pp. 681-682)

According to Moreira, if we follow the logic of these arguments, “discussions about the curriculum for educating mathematics teachers tend to concentrate on the internalist aspect of mathematics, which involves mathematics for learning mathematics” (Moreira, 2010, p. 682). The conversations revolve around the extent to which one delves into undergraduate disciplines such as linear algebra and analysis, with academic mathematics as a benchmark. This controversy unveils a disregard for certain aspects of teaching practice, as it is assumed that if a teacher has a deep understanding and proficiency in academic mathematics, they will naturally excel at teaching school mathematics, which is seen as a specific instance of the former.

In an effort to challenge this reasoning, Moreira poses the following question: “Is there a way of understanding mathematical objects that is better suited for elementary education teachers compared to that used by professional mathematicians to comprehend such objects?” (Moreira, 2010, p.685)

To address this question the author proposes actions which would be closely tied to the elucidation of the education of mathematics teachers considering the specific circumstances surrounding their professional activities. Answering this question would facilitate comprehension of the mathematics taught by mathematics teachers. From that it would be possible to establish a comprehensive plan for an introductory education program specifically focused on the profession of mathematics teachers, rather than the career of a mathematician, which has been the traditional approach for years and continues to be so today. In his work, Moreira (2010, p. 691) organizes his arguments by suggesting that “ […] rather than attempting to *integrate* what is conceived separately, we should consider developing a course of study in which mathematical knowledge is inherently integrated with the challenges faced by teachers in their classroom practice”.

In a recent thematic issue of the magazine *Perspectivas da Educação Matemática*, titled "Problematizations of Mathematical Education in Mathematics Degree Courses" (2021), authors spousing diverse theoretical frameworks and varying approaches to analyzing mathematical education engaged in discussions pertaining to a relevant topic to the present article.

Works such as those by Biza et al. (2021), Cyrino (2021), and Ribeiro et al. (2021), published in that thematic issue, are commendable for their thorough examination of the intricacies, scope, aspects, distinctive features, and dynamics (discussions, problematization and production) involved in the practice of mathematics, surpassing the mere acquisition of content and pedagogical techniques for teaching such content. Such works present discussions regarding the nature of mathematical knowledge that do not depart from a pedagogical, as well as psychological and cultural discussion, constituting specific knowledge of the professional who teaches mathematics. The work of Shulman published in the late 20th century, paved the way for mathematics educators to do research aimed at constructing and organizing a knowledge basis for mathematics teachers. Various ideas concerning the knowledge and expertise of mathematics teachers have been developed, such as those proposed by Ball et al. (2008) and Rowland (2013), which are just two well-known examples. These theories have been extensively discussed in the three publications previously mentioned.

The works by Clareto and Rotondo (2021), Giraldo and Roque (2021), Elias (2021), and Oliveira et al. (2021) in the thematic magazine issue, on the other hand, presented critical analyses that challenged the underlying logics and narratives that contribute to the conceptualization of the mathematical knowledge of teachers. The objective is not to create different debates on mathematical knowledge, axes, or methods of organizing mathematics, but rather to critically examine the Eurocentric notion of academic mathematics itself. These works not only challenge the fundamental nature of academic mathematics, but also encourage us to reflect on how mathematicians engage in political, philosophical, and economic discourse within contemporary society, particularly within undergraduate mathematics courses. This thematic magazine issue compiled a diverse collection of works that aimed at investigating the process of mathematical education in undergraduate courses. The works presented a multiplicity of perspectives and employed distinct theoretical and methodological approaches. One notable finding derived from the examination of these many alternatives is that the idea of regarding academic mathematics as a given and positioning it as the foundation for mathematics instruction is not substantiated as a theoretical argument supported by research evidence. The work published in that thematic issue exemplifies the considerations we have made.

In conclusion, we reiterate a statement made by Moreira and Ferreira (2013)

> [...] If we consider that the mathematical knowledge acquired during teacher education is primarily based on the challenges resulting from teaching practice, we can understand this knowledge as specialized for the profession, capable of providing a distinct and *unique* viewpoint (regarding the teacher) of the school mathematics classroom. Viewed from this perspective, it appears that the role of this *mathematics of the teacher* is not limited to its own separate domain, which requires deliberate efforts to be artificially connected to other isolated niches. [...] In this scenario, ideally, at least, various locations and distinct mathematical knowledge would intersect without clashing, resembling the actual practice of mathematics teaching in schools. (pp. 1003-1004)

The perspectives offered by Moreira and Ferreira in this excerpt align with specific findings and suggestions put forth in our recent research on the education of mathematics teachers. In the following section, we will outline the trajectory that our research group has pursued, as well as three works that will aid us in further delineating an *additional* proposal for the education of mathematics teachers.

From content and ways of producing meaning regarding the mathematics of mathematicians (or mathematical education) to the examination of processes of production of meaning (or mathematics education) of mathematics' teachers

Initially, our research in Sigma-t[5],focused on creating syllabuses and methodologies for mathematical subjects in mathematics degree programs, with the aim of preparing future educators. The selection of the initial discipline, linear algebra, was prompted by the interest of certain members inside the group. The purpose of our preliminary discussions was to determine the optimal approach, either geometric or algebraic, to be adopted in this discipline. To this end, in addition to reading and discussing textbooks, we tried to create our own textual content in

[5] At the time, the early 2000, Sigma-t was a research group formed by Professor Romulo Campos Lins and graduate students who were enrolled in the postgraduate program in mathematics education of *Universidade Estadual Paulista* (São Paulo State University) in Rio Claro. Presently, it is a network focused on research and development in mathematics education which brings together mathematics teachers who have an interest in the Model of Semantic Fields. (MSF). Available at http://sigma-t.org.

the form of worksheets. Despite our diligent efforts, we were unable to find a satisfactory answer or comprehend why we consistently found our output to be indistinguishable from preexisting literature.

The turning point occurred when we recognized that this phenomenon was inherent, given that we were operating within the domain of the mathematical categories of mathematicians (Lins, 2004). Within the field of linear algebra, for instance, the concepts of vectors, basis, and dimension are defined in a precise and unambiguous manner. While there may be a few different ways to define basis and dimension, such definitions are always equivalent within the context of the mathematical discipline. What we sought was a set of categories that would enable us to discuss topics beyond the realm of academic mathematics. We aimed to discuss mathematical aspects pertaining to the mathematics educator, particularly, the mathematics of the mathematics teacher.

Silva's doctoral research in 2003, undertaken as a member of Sigma-t, identified *space* as the primary category. We initiated the development of a course titled Space which served as a preparation program in mathematics education specifically designed for aspiring teachers. For instance, instead of separately studying vector spaces within a linear algebra course and metric spaces in a course dedicated to that topic, the central focus of the course would be the notion of *space*, which would be examined from multiple perspectives, including those pertaining to linear algebra, metrics, Euclidean geometry, among others.

At this juncture, the prevailing perspectives remained primarily rooted in the mathematical domain of mathematicians. Despite changing the core, we approached the subject matter of space by considering several mathematical categories such as numbers and measurements and combinations and probabilities that share similar qualities. Furthermore, we observed that our findings aligned closely with the key thematic divisions proposed in the National Curricular Guidelines (*Parâmetros Curriculares Nacionais,* Brazil, 1998), the Standards of the National Council of Teachers of Mathematics (NCTM) (1989), and the British National Curriculum.

Simultaneously, as previously stated, we engaged in formulating a precise, practical and operational definition of the mathematics used by mathematics teachers. The definition should be independent of content, that is, it should not involve explaining or specifying the specific mathematical knowledge which the teacher must possess. Neither should the focus of the discussion be centered on demonstrations, rigor, and language.

Upon integrating this definition with the initial categories that had been developed, we proceeded to advance to a third phase in our collaborative efforts. We realized that the selection of categories in which the courses should be focused "should not be based on the mathematical abilities of mathematician or curriculum guidelines. Instead, they should align with common areas of human activity, such as *Decision Making* and *Measurements*" (Lins, 2005a).

The quest to understand the nature the mathematics used by mathematics teachers led Lins (2006b) to characterize the modes of production of meaning of mathematicians which originated in the early 19th century and were consolidated through the efforts of Bourbaki (*circa* 1930), which he dubbed the mathematics of mathematicians .

> It might seem odd to characterize any *mathematics* in terms of the process of production of meaning, rather than in terms of, say, content (e.g., definitions and theorems) and methods for establishing truth. My primary contention is that, while for the mathematician, or more precisely, the philosopher of mathematics, the challenge is capturing the *essence* of something that is already established and widely accepted as a crucial aspect of a social activity, for the mathematics teacher, this approach is inadequate. This is because, regardless of how much teachers desire their students to think in a specific manner or comprehend a statement in a particular way, they cannot predict how students will interpret and utilize information. My characterization of the mathematics used by mathematics teachers, is not primarily directed at what the teacher thinks about or of mathematics, but rather their ability to *observe* and *comprehend* the mathematical processes occurring when students are engaged in mathematical activities. Most, of the time, these observations take place during interactions where the production of meaning is happening. (Lins, 2006b, p. 2)

The majority of the courses involving the mathematical education of teachers of mathematics in Brazil, as well as in many other countries, are designed and delivered from the perspective of mathematicians.

According to G. H. Hardy (1877-1947), "the mathematics of the practicing professional mathematician" is "authentic mathematics", and this "condition excludes many things which are relatively easy to understand, but which are more closely related to logic and mathematical philosophy" (Hardy, 2000, p. 87). A very peculiar feature of the mathematics of mathematicians is that as soon as something is defined, it is what it is and will be until one decides to change the definitions. This concept can be illustrated by the following scenario:

> [...] if a mathematician states that 'the limit of a function *f* is such and such and such', this is what the limit of a function *f becomes*, and this is not due to a *natural* cause (descriptive definition), but to a symbolic determination (constitutive definition) (Lins, 2004, p. 95, emphasis added)

Therefore, when the mathematician defines an object, it is not appropriate to discuss this definition in other areas (outside mathematics itself). This is solely done to explore its potential impact on other domains of interest or its efficacy in resolving or elucidating previously identified issues. According to Lins (2006b), “there is no other area of human knowledge in which its practitioners have this level of control over the objects they deal with are or are not, as in the mathematics of mathematicians” (Lins, 2006b, p. 14)

In his book "In Defense of a Mathematician", Hardy discusses the concept of “fame in mathematics” as a valuable and reliable investment, “provided one has the means to acquire it.” He emphasizes that no other field has such well-defined and widespread agreement on its standards. According to Lins (2004), due to this particular quality, the mathematics of the mathematician might be described as *internalist*.

Another distinctive characteristic of this mathematics is that it is *symbolic*. This symbolic nature, which is opposed to an ontological nature, means that its objects “are known not in what they are [through their essence as things, as is the case when we say what a bottle is], but solely in their p*roperties, in what can be said about them*” (Lins, 2004, p. 96, emphasis added).

Bicudo (1991), when distinguishing some characteristics of mathematics, presents the following excerpt from the book "Realms of Meaning", by Philip H. Phenix:

> Many mathematics students and teachers never really understand the subject, as they identify it with calculations for practical purposes. Ordinary language is primarily concerned with the adaptation of the community to the tangible reality of objects and individuals. Mathematics, on the other hand, has no such relationship to tangible reality. Mathematical symbolism occupies a world of independent and self-sufficient thought. *They need not represent real things or classes of real things, as do the symbols of ordinary language*. mathematics occupies its own distinct realm. Its domain is that of *pure* symbolic forms, whose applications, no matter how useful, are secondary and incidental to the core symbolic meanings. (Phenix, 1964, p. 71 *apud* Bicudo, 1991, p. 36, emphasis added)

Additional characteristics of this mathematics are examined by Bicudo (1991). For example, the statement “mathematics is given (in part) *a priori*” (p. 34), implies that it is independent of experience. Unlike other disciplines such as chemistry, physics, and biology, the principles of mathematics are not derived from the laws of nature and are not contingent upon them. Another characteristic is that “mathematics is exact” (p. 35) in the sense that all its terms, definitions, rules of inference, etc., have a precise meaning; and a third is that “mathematics is abstract” in the sense of “abstracting everything that is not essential to a given purpose” (p. 35).

Consistent with Lins (2004), we adopt the perspective that the mathematics practiced by mathematicians is *internalist* and *symbolic*. We shall do that because we firmly believe that these two attributes encapsulate the commonly and often informally discussed aspects of the mathematical prowess of mathematicians.

> These two features, internalism and symbolic objects, largely explain the meaning behind the terms *theoretical* or *abstract* when referring to the mathematics of mathematicians. Additionally, these characteristics contribute to the sense of unfamiliarity and disconnection experienced by the general public. (Lins, 2004, p. 96)

For us, the fundamental aspect of the mathematics of mathematicians, whereby concepts are precisely defined, which involves stating the essence of the concept remains unaltered. “Even though the logic used to establish truths may differ, for instance classical, para-consistent, or fuzzy logic, this primarily leads to the development of new fields rather than generating conflicts.” (Laing, 1970, *apud* Lins 2006b, p. 14)

According to Lins (2006b), this mathematics of mathematicians, as seen by professionals in the field, and very ingrained in the culture of teachers who teach mathematical subjects, is the result of a type of *cleansing* that began in the first half of the nineteenth century and was firmly established around 1930 by initiative of the Bourbaki group. In this process, all intuitions that relied on the *physical world* were eliminated, in order to avoid *errors* caused by false *perceptions*.

> Starting with Hamilton, integers became mere constructions, creations derived from well-stablished concepts rather than questionable things. Cantor's introduction of an infinity greater than another infinity definitely impacted the nature of the domain of mathematicians (Lins, 2006b, p.14).

For the past three decades or more, there have been ongoing discussions among mathematics educators regarding the significance of this particular branch of mathematics in the education of mathematics teachers. These discussions have prompted research that explores different approaches and adjustments to its inclusion in undergraduate degree programs. In this scenario, some of the work conducted within our research group, Sigma-t, discussed the (mathematical) education of mathematics teachers. Below, we present some examples, using the Semantic Fields Model as a reference.

Viola dos Santos (2012), in his doctoral thesis explored the possibilities of a mathematical education through teaching mathematics degrees. He conducted theoretical analyses by examining interviews with mathematics educators, as well as other textual sources such as articles, dissertations, and theses, which were relevant to his research. In addition to the Semantic Fields Model, Viola dos Santos (2012) incorporated Oral History as a theoretical-methodological approach (Garnica, 2008b; Garnica et al., 2011).

The legitimacies for mathematical education produced by Viola dos Santos (2012) were explored in textualizations and theoretical analytical publications. Among those are: *The elementary education teacher needs to take a course in which they develop intellectual autonomy* (textualization of the interview with Henrique Lazari); *A teaching degree course in mathematics would have mathematics content* (*calculus, algebra, among others), always starting from problems, establishing relationships with school mathematics* (textualization of the interview with Dona Lourdes); *The professional practice of teachers should be the focal point of teaching degree courses. Choices must be made* (textualization of the interview with Plínio Cavalcanti Moreira); *Experience as an educational opportunity* (theoretical-analytical essay); *For a different mathematical education within the teaching degree course; On the mathematics of mathematics teachers and the mathematics of mathematicians* (Theoretical-analytical essay).

They are referred to as legitimacies, as defined by the SFM, because within its scope, the legitimacy of a belief-affirmation is not established by its truth (what may or may not be said), nor by logical criteria deduced axiomatically, or even by empirical criteria observed in certain situations. The legitimacy of a belief-affirmation is established by believing that we belong to some communicative space (Lins, 1999, 2001, 2012). By engaging in actions aimed at establishing legitimacy, we generate belief-affirmations, as well as justifications, to persuade others in authority to endorse our statements. We engage in storytelling and shape our identities through storytelling.

Thus, the theoretical movements proposed by Viola dos Santos (2012) do not serve as the *definitive* framework for structuring mathematical education in undergraduate courses. Each aims at generating interpretations, viewpoints, settings, and opportunities for potential preservation or alterations in the initial education of mathematics teachers. We should refrain from analyzing them or making any argument that limits their scope. It is important to acknowledge the presence of alternative forms of legitimacy that deviate from the standard framework used in the structure of undergraduate programs for preparing future mathematics teachers.

Educators of mathematics teachers must recognize, comprehend, and actively explore other approaches to mathematical education within mathematics degree programs. Whether that exploration might serve to reinforce the existing model or critically examine and temporarily suspend it.

In her doctoral thesis, Linardi (2006) successfully demonstrated a vulnerability in the model of mathematical instruction that primarily focused on the methods of mathematicians for producing meaning in mathematics, to the exclusion of other approaches (LINS, 2004). The author aimed to detect manifestations of mathematical concepts of mathematicians in the professional methods employed by a mathematics teacher. It was confirmed that the teacher was capable of dealing with the mathematics of the mathematician, including definitional, internalist and symbolic ways of producing meaning (Lins, 2004). However, those methods of producing meaning did not translate into guiding principles of her practice as a mathematics teacher. This finding revealed that mathematics education, as conventionally viewed, (Moreira, 2010), was unsuitable as a teaching approach for that teacher. To a certain extent, such shortcoming partially reflects a deficiency of mathematical programs for mathematics teachers regarding their professional practice.

We believe that there is a certain *naturality* in the education of mathematics teachers concerning the way in which the purported topics of mathematical training are addressed in university curricula. Naturality in the sense that *it has always been that way,* therefore it must remain that way. This may point to the existence of a certain ideology established within the realm of mathematics teacher education.

Linardi's (2006) work highlights that it is not enough for teachers to know only the mathematics of the mathematician, characterized by its own modes of production of meaning. Mathematics teachers must have a deep understanding of mathematics. However, their education should also encompass other important aspects, such as the connection between the mathematics of the mathematician and school mathematics, different applications of mathematical knowledge, the potential to critically examine the social and political implications of mathematical knowledge, as well as the use of various representations to convey mathematical concepts.

However, what is the purpose of including *mathematical* disciplines in the undergraduate curriculum of mathematics courses? By extrapolating arguments rooted in tradition, Lins highlights the significant challenge faced by the community of educators due to the existence of courses, such as calculus, that have mathematical content but are disconnected from the theoretical and practical aspects of the mathematics teaching. (Lins, 2005b, p. 117). The author articulates the inquiries posed throughout that article, regarding the implementation of mathematical education disciplines in mathematics degree programs, as they typically exist. The author challenges the commonly accepted justifications for the importance of these disciplines, namely that they teach “the content to be taught in school” and establish “the true foundations of what the future teacher will teach.” (Lins, 2005b, p. 119, comment added).

He clarifies that the focal point of his intended discourse is that the essence of the professional activity of (mathematics) teachers is “to read students and make decisions about what is happening and how to proceed" (Lins, 2005b, p.120). In this sense, this author argues that the mathematics of the mathematician can be valuable by providing "a distinct chance to encounter the peculiar strangeness[6] inherent in concepts that defy conventional wisdom and everyday experience in all aspects." (p. 121) Infinities of varying magnitudes(?) Events that are theoretically possible despite having zero chance(?) Lins believes that experiences of estrangement provide a compelling rationale for including courses in linear algebra, real analysis, and other subjects in the curriculum of undergraduate programs. According to Lins, "only by becoming aware of this estrangement, by having experienced it as a student-future-teacher, will teachers comprehend the need to always read their students, instead of just comparing them against a map of *what should be*" (Lins, 2005b, p.121).

In order to fully explore the potential of mathematics-related disciplines, it is necessary to substitute the Pedagogy-Mathematics dichotomy and shift to mathematics education disciplines. This transformation will provide future teachers with opportunities to experience unfamiliar concepts.

The purpose of including the works of Linardi (2006) and Viola dos Santos (2012) in the presentation at the end of this section was to highlight some investigations conducted by our research group. Those works are intended to serve as a reference for further analysis, and the development of ideas provided in the following section.

Another legitimacy: everyday categories as a possibility for mathematical development of mathematics teachers

The concept of utilizing everyday categories in the mathematics preparation of mathematics teachers, with the aim of generating and organizing meaningful ideas, emerged and was developed in order to establish a new framework for their education. This was achieved through the implementation of specific actions outlined in the research project titled "Design and Implementation of a Continuing Education Program for Mathematics Teachers" (Lins, 2006a).

When Lins (2006a) suggested utilizing everyday life categories to incorporate educational practices into a teacher preparation course, such categories were / should be applied in a context where other categories, of the mathematics of mathematicians, are prominently featured. The intention in bringing to light categories of everyday life was to emphasize the

[6] This estrangement process can be described as a scenario in which on one hand, "those for whom something is natural, yet unfamiliar, and on the other those for whom [what is being said] cannot be expressed" (Lins, 2004, p. 116, comment added).

existence of alternative knowledge that is generated in the organization of our lives, distinctive from other knowledge involved in the use of mathematical categories by mathematicians.

Any attempt to explain human way of life, which encompasses behaviors, actions and thoughts, and organization of activities, through which men consider themselves complete, finalized, ultimately proves to be innocuous. For example, imposing the identity of mathematics or submitting to it, regarding it as truth, to a certain degree, involves the attempt to explain and perceive the world from a single perspective. This same form of perception also arises when practical information, due to its lack of validation through scientific procedures, is dismissed or perceived as inferior based on an evaluation scale.

When contemplating legitimacies from the perspective of MSF, it makes no sense to hierarchize knowledge *for something inherent to it.*

Within the course of existence, our mundane activities such as waking up, eating, and moving are not influenced by scientific knowledge or complex theories. Connected to these unspecialized actions are what Lins (2006a) refers to as categories of ordinary existence.

Believing in the uniformity of reality, which is the same for everyone, would violate the assumptions of MSF. We are not attempting to suggest that this knowledge is any less intricate or demands less effort from those who create it. We only wish to emphasize it.

Considering everyday life as

> [...] that which is given to us every day (or that we share), overwhelms us day after day, oppresses us, as there is an oppression of the present. Every morning, when we wake up, what we take on, is the weight of life, the struggle of living, or of living in this or any other condition, with fatigue, with desire. Everyday life is what binds us intimately, from within. It is a story partially about us, almost in a state of withdrawal, sometimes veiled. We must not forget this "*world of memory*", as Péguy put it. It is a world that we deeply cherish, olfactory memory, memory of childhood places, memory of the body, of childhood gestures, of pleasures. (Leuilliot, 1977 *apud* Certeau et al., 1996, p. 31)

Perhaps this may assist us in recognizing that in our daily lives, in ordinary life, there is a certain type of non-formalized knowledge that we produce and engage with constantly.

Thus, considering this knowledge, based on everyday life categories, in teacher education can be viewed as a method for *un-veiling* the very processes of production of meaning. In a way, it is an attempt to show that objects are not given a priori, but constituted in processes of production of meanings, which depend on the legitimacies that are at stake in that activity. In a way, this is an effort to demonstrate that objects are not inherently given, but rather are formed through processes of production of meaning, which are influenced by the legitimacies involved in that particular activity.

Lins (2005a) presents a suggestion for the treatment a certain category of daily existence, which he referred to as *space*.

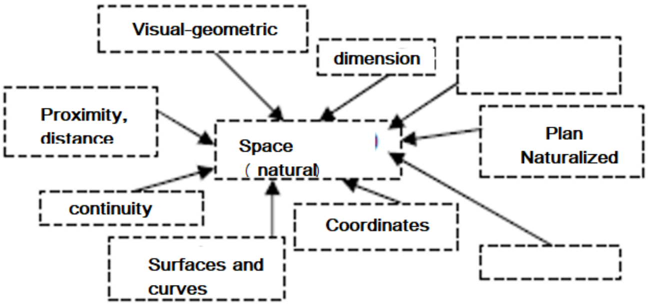

Figure 1.

Space – Center as an everyday life category (Oliveira, 2011, p. 43)

In this proposition, we note that the category *space* focuses on activities that originate from problematizations of space, specifically regarding its natural and physical aspects. Questions such as: "What changes occur in this specific area when we start to identify objects within it?" or "What transformations arise when we start discussions about the distance between the objects within it?" (Lins, 2006a, p. 11), could potentially trigger mathematical concepts and ideas, among others, which could lead to the emergence of possibility of estrangement.

It is important to mention that the text boxes in figure 1 were intentionally represented as dotted lines to indicate that the fields associated with the *space* category are not fixed or limited. They can be changed or expanded. Therefore, this category, like any other in the organization of human activity processes, is not predetermined, even though people have some ideas about it.

Another category of everyday life, called *decision-making*, also proposed by Lins (2005a), was discussed in one of the modules of the extension course "Space, Arithmetic, Algebra and Decision Making: A Professional Development Course for Mathematics Teachers," which was examined in Oliveira's doctoral dissertation in 2011. Within the module, some scenarios served as catalysts to initiate debates among mathematics teachers who attended the course.

While Lins did not propose this category as a replacement for the categories of the mathematics of the mathematician, it serves as an alternative that can be included in the education of mathematics teachers.

The inclusion of that category in that module of the extension course provided opportunities for exploring the concepts of alienation and decentralization in the classroom, as discussed in the work of Oliveira in 2011 and 2012, as well a publication of Julio and Oliveira in 2018. Furthermore, the extension course also imparted additional features, which were emphasized by the remarks of the participating teachers.

The first feature pertains to the handling of commonplace scenarios that, in theory, did not suggest the presence of mathematical elements to be addressed. Each teacher, group of teachers, or the entire class, along with the course lecturer7, had to make decisions regarding various situations. They guided the discussion and selected the most suitable ideas, whether mathematical or not, in order to gain a deeper understanding of the given scenario.

By employing decision-making scenarios and emphasizing the significance of interpreting the production of meaning, the text underscores the inherent inability to predict the conclusion of this process, which is another distinguishing feature of the course. Managing ambiguity, whether it pertains to instructional methods or engaging in activities pertaining to the analysis of a scenario as a student, as was the case for teachers throughout the course, is not a source of comfort. Therefore, experiencing this type of scenario can help teachers deal with the impossibility of anticipating classroom events or what their students may tell them about certain issues. Once again, we stress the significance of implementing decentralization, particularly in these difficult conditions.

During the Decision Making module sessions, we observed a tendency towards embracing the recommendation of Oliveira (2011, pp. 29-30) that emphasizes "the importance of considering the mathematical education of teachers as a process of production of meaning within their own mathematics classrooms, rather than focusing solely on mathematical content."

By presenting teachers with real-life situations that incorporated mathematical concepts, it was possible to engage them in the process of production of meaning and draw their attention to a phenomenon known as estrangement. Embedded within that phenomenon is the concept of decentralization, which is essential for establishing a communicative environment within the classroom.

As initially envisioned by Lins, the paradigm shift accomplished in that course proved to be potentially productive, as it assumed

> [...]as a guideline, it is necessary to conduct the preparation and development of teachers based on categories that they can convey to students. This approach eliminates the need for teachers to be trained within specific categories and then invest in what some authors

[7] Professor Romulo Campos Lins.

> refer to as *recontextualization*, which requires a specific and complex professional competence. Teacher education takes place from *the context of everyday life categories*, so that the *recontextualization* happens from the natural (daily) towards the non-natural (mathematical). Thus, the transition to the modes of production of meanings of the mathematics of mathematician takes place *as an expansion of understanding*, and not as a *true essence of what is said on the street*, nor is it a replacement of intuitive thinking for mathematics (Lins, 2006a, p. 7)

Inspired by the aforementioned extension course, which took place from 2015 to 2017[8], we created initiatives associated with a research project involving teachers and prospective teachers of mathematics. Our aim was to generate and employ teaching scenarios for the professional development of mathematics teachers, focusing on the expansion of the processes of production of meaning through activities grounded in quotidian categories.

In a study conducted by Santana (2017), elementary school teachers participated in a working group where they explored such scenarios. The study explored the potential for ongoing professional development by inviting teachers to engage in alternative activities within dedicated areas for discussion, problematization, and the generation of new approaches for mathematics classrooms. One of the activities was as described below:

> "Thomaz Lanches is a snack bar that employs a unique charging strategy for customers. The snacks are displayed on shelves and the soft drinks are stored in refrigerators to which customers have full access. Patrons help themselves to their satisfaction, and when they go to cash register, the cashier asks them how much they have consumed. Do you think that the owner of Thomaz Lanches incurs losses with this charging strategy?

Initially, some teachers believed that the establishment did not exist, asserting that in Brazil there is a significant number of individuals with questionable moral character, making such a commercial enterprise implausible. Subsequently, other teachers confirmed the existence of the restaurant, in the city of Campo Grande and attested that the Sfiha served there is delicious. The teachers engaged in a discussion aimed at bridging the gap between this setting and their classrooms. They discussed topics such as losses, methods for calculating gains and losses, magnitudes, and proportions. In essence, they sought to apply mathematical principles to daily life.

During the discussions, additional interpretations emerged, suggesting that other

[8] Project conducted with a grant of National Research Council (CNPq), public proposal of 2014, which included five universities (UFMS, UFMT, UNIPAMPA, UFSJ and UNIFESP). A wide range of works were generated in conjunction with this undertaking. In the context of this article, only one master's dissertation will be referenced.

concepts and emotions are also involved in that activity. A teacher emphasized the need for addressing topics such as honesty and corruption within the school environment: *Schools should work on issues such as corruption and honesty in the classroom. If we want our country to evolve a little bit, we have to start discussing it in schools.* Another teacher added*: We can't just transmit content* (Santana, 2017).

Amidst these claims, there is an assertion suggesting that mathematical conversations are influenced by political and ideological perspectives. Hence, it is imperative to establish opportunities within the education of mathematics teachers to engage in debates beyond the scope of mathematical content.

Another issue that arose from this exercise was to *challenge* the notion that profit is inherently connected to a mathematical concept. Why would a merchant's profit be linked to the formation of a new relationship between consumers and a commercial enterprise? The mathematical material in this case was challenging due to the fact that the circumstance presented was not *trapped* within the scope of school curriculum.

In the teacher preparation work conducted by Santana (2017), as well as in other studies conducted within the interinstitutional research project previously mentioned, and the extension course described and analyzed by Oliveira (2011), it was observed that categories of everyday life gave rise to additional processes of production of meaning. These processes facilitated the explanation of how to create formative spaces, where various meanings were generated.

Final considerations

The discussion presented in this article provides alternative approaches for structuring mathematical education in teaching degree programs. They emphasize both the approach conceived and implemented by Romulo Campos Lins with mathematics teachers during the extension course, and the research project that contemplated various public higher education institutions.

The experiences reported, did not involve the mathematics of mathematicians serving as the basis for teachers' knowledge of classroom mathematics. Furthermore, there was no intention of creating hierarchies among various types of information. We aimed to generate scenarios that could be analyzed through various methods for producing meaning, not limited to mathematical contexts. The objective was to create scenarios that directly pertained to the everyday experiences of students and teachers, thus encouraging discussions and critical analysis of the ways meaning is constructed within such contexts.

We appreciate the valuable approach proposed for the education of mathematics teachers, as it involves a variety of situations and actions to promote decentralization and estrangement in mathematics classrooms. These processes are considered essential for the production of meaning from everyday categories, rather than focusing solely on content. This is a perspective which we consider as a possible approach, albeit not unique or definitive, to analyze and question the concept of mathematical education in terms of the production of meaning, which also occurs in the classrooms, with mathematics teachers and aspiring teachers, rather than relying solely on curriculum guidelines.

The reality of Brazilian education is currently constituted according to the National Common Core Curriculum which seeks to promote an even greater standardization of what would be the knowledge of mathematics teachers. The reliance on mathematical content for processes of production of meaning, in line with the national guidelines, offers a possibility for courses dealing with the preparation of mathematics teachers, which contribute to the construction of democratic, plural spaces, and a careful reading of the specificities of the educational contexts of Brazilian schools. Our discussion *points a finger* at an almost untouchable aspect of the discussion regarding mathematical content and the knowledge required for future mathematics teachers, offering not only another guise for disciplines in mathematics degrees, but also a new structure; aiming to produce mathematical education through different, varied and diverse processes of production of meaning, fostering discussions, problematizations and the creation of other possibilities for mathematics teachers, regardless of whether they are in training, or have already obtained a degree.

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