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Mathematical aspects of the n-queens problem and the construction of knowledge by Computer Science students

Aspectos matemáticos del problema de las n-reinas y la construcción del conocimiento por parte de estudiantes de Ciencias de la Computación

Aspects mathématiques du problème des n-reines et construction de connaissances par les étudiants en informatique

Aspectos matemáticos do problema das *n-rainhas* e a construção do conhecimento por alunos de Ciência da Computação

Gerson Pastre de Oliveira[1]
CEETEPS (Fatec Jundiaí) – UNIP (Universidade Paulista)
Doutor em Educação
http://orcid.org/0000-0001-8113-936X

Abstract

This article reports qualitative research, which had as subjects a group of students from a higher education course in Computer Science, with the proposal of solving an issue related to the n-queens problem, a generalization of the original problem, which consisted of having 8 queens on a chessboard, considering different positions, so that the pieces do not capture each other. The specific didactic sequence consisted of proposing a generalization whose application provides the number of diagonals to be considered for solving the problem on any *n-by-n* board, with *n* greater than 3. Based on the assumptions of Didactic Engineering, and having as main theoretical supports the Theory of Didactic Situations (TSD) and the work of Zazkis and Liljedahal on close and distant generalizations, the students developed an autonomous investigative trajectory, based on collaborations, to present acceptable solutions to the proposed problem. The results allow us to infer that the experience around solving mathematical problems is relevant as a learning resource in higher education Computer Science courses, considering a scenario of intensive use of digital technologies.

Keywords: Pattern generalization, N-queens problem, Theory of didactic situations, Didactic engineering, Computer science.

Resumen

[1] gerson.oliveira@fatec.sp.gov.br gerson.oliveira1@docente.unip.br

Este artículo reporta una investigación cualitativa, que tuvo como sujetos a un grupo de estudiantes de la carrera de educación superior en Ciencias de la Computación, con la propuesta de resolver un problema relacionado con el problema de n-reinas, una generalización del problema original, que consistió en tener 8 reinas sobre un tablero de ajedrez, teniendo en cuenta diferentes posiciones, para que las piezas no se capturen entre sí. La secuencia didáctica específica consistió en proponer una generalización cuya aplicación proporciona el número de diagonales a considerar para resolver el problema en cualquier tablero de n por n, con n mayor que 3. Basado en los supuestos de la Ingeniería Didáctica, y teniendo como principales soportes teóricos la Teoría de Situaciones Didácticas (TSD) y el trabajo de Zazkis y Liljedahal sobre generalizaciones cercanas y distantes, los estudiantes desarrollaron una trayectoria investigativa autónoma, basada en colaboraciones, para presentar soluciones aceptables al problema propuesto. Los resultados permiten inferir que la experiencia en torno a la resolución de problemas matemáticos es relevante como recurso de aprendizaje en carreras de Informática de educación superior, considerando un escenario de uso intensivo de tecnologías digitales.

Palabras clave: Generalización de patrones, Problema de n-reinas, Teoría de situaciones didácticas, ingeniería didáctica, Ciencias de la computación.

Résumé

Cet article rend compte d'un projet de recherche qualitative qui a impliqué un groupe d'étudiants d'un cours de licence en informatique, dans le but de résoudre une question relative au problème des n reines, une généralisation du problème original, qui consistait à disposer 8 reines sur un échiquier, en tenant compte de différentes positions afin que les pièces ne se capturent pas les unes les autres. La séquence didactique spécifique consistait à proposer une généralisation dont l'application fournissait le nombre de diagonales à considérer pour résoudre le problème sur un échiquier quelconque de n par n, avec n supérieur à 3. Sur la base des hypothèses de l'Ingénierie Didactique, et avec la Théorie des Situations Didactiques (TSD) et les travaux de Zazkis et Liljedahal sur les généralisations proches et lointaines comme principaux supports théoriques, les étudiants ont développé un parcours d'investigation autonome, basé sur des collaborations, pour aboutir à des solutions admissibles au problème proposé. Les résultats nous permettent de déduire que l'expérience de la résolution de problèmes mathématiques est pertinente en tant que ressource d'apprentissage dans les cours d'informatique de l'enseignement supérieur, compte tenu de l'utilisation intensive des technologies numériques.

Mots-clés : Généralisation de modèles, Problème des n-rangs, Théorie des situations didactiques, Ingénierie didactique, Informatique.

Resumo

O presente artigo relata uma pesquisa qualitativa que teve como sujeitos um grupo de alunos de um curso superior em Ciência da Computação, com a proposta de resolver uma questão relacionada ao problema das n-rainhas, uma generalização do problema original, que consistia em dispor 8 rainhas em um tabuleiro de xadrez, levando em conta posições distintas, de modo que as peças não se capturem mutuamente. A sequência didática específica consistia em propor uma generalização cuja aplicação fornecesse o número de diagonais a serem consideradas para a resolução do problema em um tabuleiro qualquer *n* por *n*, com *n* maior do que 3. A partir dos pressupostos da Engenharia Didática, e tendo por suportes teóricos principais a Teoria das Situações Didáticas (TSD) e o trabalho de Zazkis e Liljedahal sobre generalizações próximas e distantes, os estudantes desenvolveram uma trajetória investigativa autônoma, baseada em colaborações, para apresentarem soluções admissíveis para o problema proposto. Os resultados permitem inferir que a experiência em torno da resolução de problemas matemáticos é relevante como recurso de aprendizagem em cursos superiores de Ciência da Computação, considerando um cenário de uso intensivo de tecnologias digitais.

Palavras-chave: Generalização de padrões, problema das n-rainhas, Teoria das situações didáticas, Engenharia didática, Ciência da computação.

Mathematical aspects of the n-queens problem and the construction of knowledge by Computer Science students

Among the most important characteristics of mathematical knowledge, one of them is the possibility that its appropriation represents for use in different areas of knowledge. In fact, as Ponte, Boavida, Graça and Abrantes (1997) state, it is possible to discuss mathematical activity as an element that integrates human culture, in a general way. Furthermore, more specifically, in the educational sphere, the use of mathematical content frequently occurs in different courses and disciplines, even in other areas, though, for example, transdisciplinary approaches, interdisciplinary projects or even through the overlap that certain topics present in relation to mathematical themes.

In this sense, learning elements related to Computer Science, mainly programming, an activity through which a code is produced in a certain language to solve a problem, can be discussed/planned based on a teaching strategy in which mathematical problems are present. The motivation for this seems clear: problem solving is at the essence of mathematical activity, so that the contextualization of questions to be resolved with the resources of this discipline are often based on problems in relation to which the aim is to demonstrate the validity of developed solutions. Authors such as Hamilton (2007), for example, suggest that mathematics would be a science aimed at solving problems and developing theoretical elements/problem-solving strategies. Therefore, it seems natural to involve computational learning with the use of mathematical tools, from a problem-solving point of view.

Furthermore, the idea that mathematics is intensively present in the curricula of higher education courses in Computer Science can be formally found in the National Curricular Guidelines for undergraduate courses in the area of Computing (BRASIL, 2016), when the legal determination expresses that Graduates of these courses must have “solid background in Computer Science and Mathematics” as an indispensable part of the requirements for the construction of computational solutions in general and for the promotion of scientific knowledge. In this way, computing, mathematics, and problem solving emerge as inseparable in these training contexts.

However, there are pitfalls to consider. For Klang et al (2021), solving mathematical problems is not an easy task for most students. According to the authors, “students may experience difficulties in identifying solution-relevant elements in a problem or visualizing appropriate solution to a problem situation. Furthermore, students may need help recognizing the underlying model in problems” (Klang et al, 2021, p. 2). Likewise, according to Morais et al (2020), “learning a programming language is a difficult task for some students” (cited in Saito, Washizaki & Fukuzawa, 2016). For the authors, this would occur because

> [...] each student has their individual difficulties, their learning pace, their interests and motivations, so teachers need to identify the characteristics of their students and the difficulties they face so that they can provide the necessary support, in order to ensure better performance for each individual and the group as a whole (Morais et al, 2020 apud Kawagush et al, 2019)

In their study, in which they carried out a systematic review of the literature, Morais et al (2020) analyzed 33 articles on learning programming, one of the essential topics in the construction of knowledge in Computer Science, published in 13 countries. More specifically, one of the research topics sought to evaluate the difficulties encountered by students in the programming learning process. The item “interpretation of computational problems and how to solve them” was the most common, representing the greatest difficulty highlighted by students (with 36% prevalence in the works analyzed). Another important piece of information, arising from the analysis, was the affirmation of another obstacle, also of considerable incidence, indicated as “little or low mathematical ability”, with 21% of occurrences among the data consulted.

Therefore, it is possible to assume that problem solving presents an obstacle for students, both from a mathematical and computational point of view, including overlaps between these items. The authors who indicated the obstacles, however, point out possibilities for dealing with them: Klang et al (2021) mention the possibility of engaging students in small discussion groups, in a collaborative approach, which could even resolve some impasses that often occur in group-based teaching experiences. In these spaces, students could explain their proposed solutions, clarify what they think and advance their understanding of the problem in question (cited in Yackel et al., 1991; Webb and Mastergeorge, 2003). In the researchers' view, based on their literature review, this interaction in small groups, which would be “dialogic spaces characterized by openness to each other’s perspectives and solutions to mathematical problems” (p. 3), would allow:

- use language for reasoning and conceptual understanding;
- exchange different representations of the problem in question;
- become aware of and understand the thinking perspectives of group colleagues.

Likewise, Morais et al (2020) indicated that one of the possible actions to overcome the difficulties indicated by students is to bring them together in smaller classes, which would allow a more individualized approach in relation to the obstacles identified.

In these ideas, identified from problems raised, lies the objective of the research[2] that motivated the writing of this article: to discuss proposals aimed at solving problems involving mathematical and computational elements. Here, the task is to understand aspects related to the *n-queens problem*, more specifically the elaboration, by a group of Computer Science students, divided into small teams, of a valid generalization to determine the number of diagonals of an $n \times n$ matrix[3], with $n \geq 4$, a step that can be used in eventual computational solutions to the mentioned problem. In the next sections, the conditions of the problem, the teaching strategy used, the methodological contributions and the interactions produced by students are explored, with the respective analyses.

The n-queens problem: mathematical and computational aspects

The n-queens problem is a generalization of the initial proposal, created by a chess player called Max Bezzel who enunciated the 8-queens problem in 1848. This problem originally consisted of arranging 8 queens on a chessboard considering different positions, in such a way that the pieces did not attack each other, that is, they could not capture each other (Abramson and Young, 1989; Osaghae, 2021). It should be considered that the queen's movements are the broadest in chess: the piece can move in a line, column, or diagonal for any valid number of squares (or until capture another piece).

Just two years later (1850), Nauck proposed the problem of the completeness of n-queens, a generalization probably produced independently of Bezzel's initiative, and which consisted of arranging q queens on a board $q \times q$ without them attacking each other, considering the prior existence of a certain number s of queens on the board, $s < q$ – that is, it would be necessary to insert, if possible, $q - s$ queens, under the conditions of the problem. This proposition makes up a class of problems with greater computational complexity: it is an NP-Complete problem[4] (Gent, Jefferson and Nightingale, 2017). Figure 1 illustrates the proposition:

[2] Investigation linked to the research entitled "Technology and contemporary society: influence of interfaces, devices and computational concepts in everyday life", which takes place within the scope of the EduTec/Unip research group.
[3] Matrix, within the scope of this article, means a two-dimensional array, with the possibility of homogeneous storage of multiple values. Although there are common terms and homologous definitions between this concept and the mathematical concept of matrix, it is important to understand that it is in the computational sense that the term is used here.
[4] The class of problems known as NP is one whose problems have polynomially limited non-deterministic algorithms.

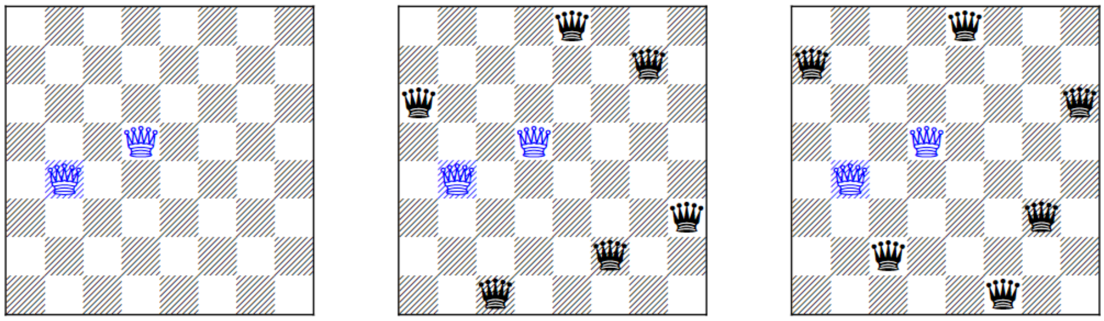

Figure 1

An instance of the 8-queens completeness problem with two possible solutions (the blue queens were previously placed) (*Gent, Jefferson e Nightingale, 2017, p. 844 (adapted)*)

It is important to note, whether in Nauck's generalization or in a proposal to insert q queens in a q x q board without any queen previously positioned, q must be greater than or equal to 4. The board would then be a q x q matrix, with $q \geq 4$, and each position would be represented by an ordered pair (a, b), with $0 \leq a < q$ and $0 \leq b < q$. Figure 2 shows a queen placed in position *(4,2)* of the matrix relative to the board.

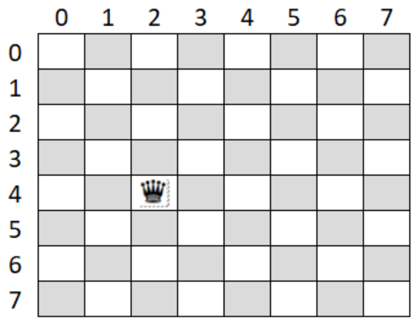

Figure 2

An 8 x 8 "board matrix" with a queen positioned at coordinates (4,2) (*Oliveira, G., 2024 (drawing made by author)*).

Formally, there are 92 different arrangements of queens on the board that represent valid solutions to the original problem with 8 queens, with 12 solutions (Figure 3) so to speak "basic", since the others could be obtained from these through rotation and/or reflection operations (symmetries).

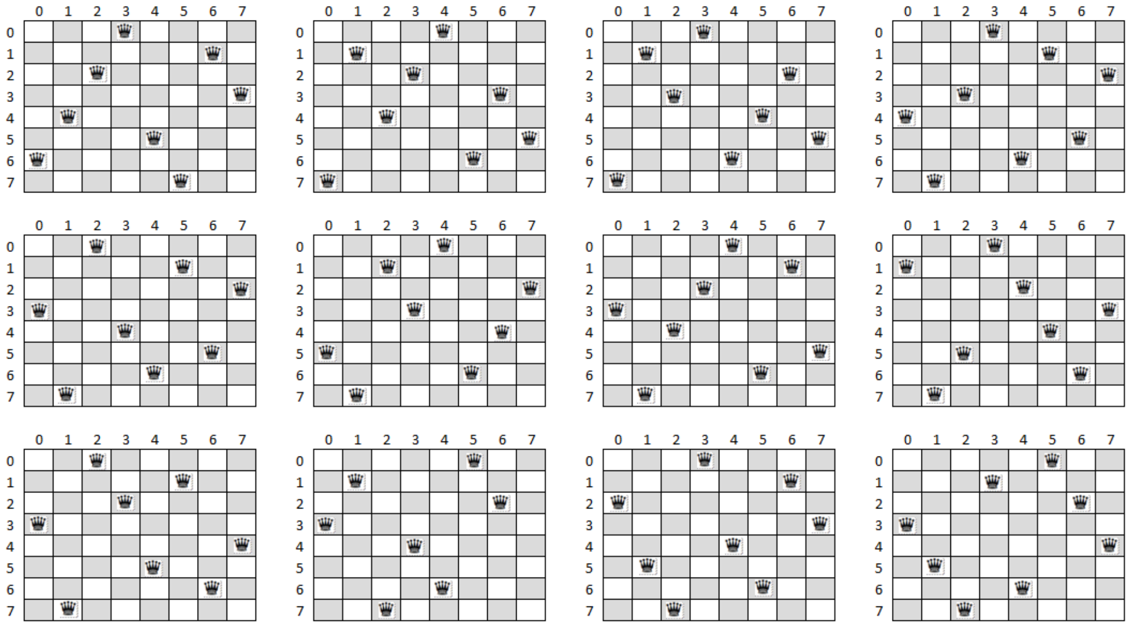

Figure 3.

Basic solutions to the 8-queens problem (*Oliveira, G., 2024 (drawing made by author)).*

In general terms, the problems reported here constitute very common references in the literature linked to Computer Science, particularly in Artificial Intelligence (AI), in the context of which it is possible to find solutions through various techniques, involving, i.e. backtracking[5] and genetic algorithms[6], among other possibilities. When studying these approaches, several activities can be proposed, encouraging problem solving among people who learn the fundamentals of programming for AI in higher education courses, notably those in technology and/or exact sciences. Finding good solutions, which, in computing, involves minimizing computational processing and memory costs as much as possible, constitutes a challenge, even more so for those beginning their studies in this field. Just think, from this point of view, that there are a considerable number of ways in which 8 queens can be arranged on a board with 64 squares (1):

$$(1) \qquad C_{8}^{64} = \frac{64!}{8!\,(64-8)!} = \frac{64!}{8!\,56!} = 4426165368$$

[5] The *backtracking* technique can be seen as a refinement in relation to the technique known as *brute force*: unlike this, the use of backtracking makes it possible to eliminate a series of solution proposals without them needing to be specifically tested. There are numerous algorithms to implement this technique, with different levels of efficiency (Kondrak and Van Beek, 1997).

[6] *Genetic algorithms* represent a class of computing models that are based on procedures that represent metaphors for evolutionary processes. They can be seen as optimization proposals and are based on the successive creation, until an acceptable solution to the problem, of a population of chromosomes, followed by the selection of the fittest individuals through *fitness functions*, reproduction processes to creation of new populations (*crossover*) and random mutations in new populations (Mitchell, 1999).

However, restrictions on placing queens in certain positions so that they do not attack each other contribute to reducing the number of squares on the board in which they could be arranged. In this sense, considering the matrix relative to the board consisting of lines (horizontally) and columns (vertically), it can be concluded that there cannot be two or more queens in the same line and/or in the same column. Thus, placing 8 queens on the board would have a much smaller number of possibilities that would meet the restrictions of the problem, given by 8! = 40320.

Having placed the queens in different rows and columns, it remains to be assessed whether they do not attack each other on the diagonals. Considering that the problem matrix is always square, on an 8 x 8 board, the diagonals would be those shown in Figure 4.

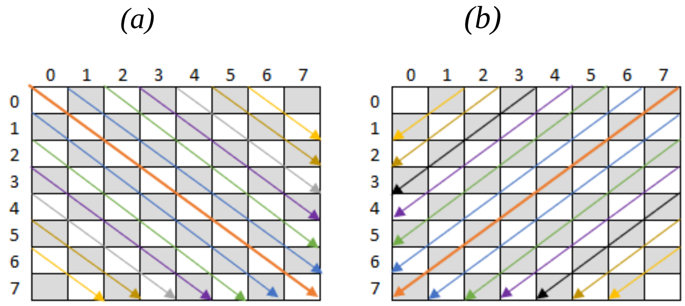

Figure 4.

Diagonals of 8 x 8 square matrix (Oliveira, G., 2024 (drawing made by author)).

As can be seen, there are 26 diagonals in the illustrated case, in which, considered individually, two or more queens cannot be placed[7]: there are 13 diagonals in the same direction as the main diagonal[8] and 13 in the same direction as the secondary diagonal[9].

There are some simple mathematical and computational procedures that allow checking the undesirable conflict between the queens. Thus, if a queen r_1 were in the same line position as a queen r_2, there would be r_1 in (a_1, b_1) and r_2 in (a_2, b_2), with $a_1 = a_2$.

There is also, as mentioned, the possibility of conflicts occurring in the diagonal components of the matrix. In this case, if the conflict occurred on any of the diagonals towards

[7] Diagonals are not considered, within the scope of this problem, to be those that would be formed by just one element, as this condition eliminates the possibility of conflict between queens. These positions are in the "corners" of the matrix, at the upper right and left and lower right and left extremities.

[8] In a square matrix M of dimension $n \times n$, the main diagonal is the one whose elements have equal row and column coordinates, that is, for all $M_{a,b}$, where a represents the row position and b represents the column position, we have that $a = b$.

[9] In a square matrix M of dimension $n \times n$, the secondary diagonal is one whose elements have row and column coordinates such that, for all $M_{a,b}$, where a represents the row position and b represents the column position, we have *that* $b = n - a$. If we consider the Figures used in this article, whose row and column coordinates start at zero, we have $b = n - a - 1$.

the main one (Figure 13a), it would occur that, for r_1 and r_2, respectively, $a_1 - b_1 = a_2 - b_2$. If the conflict were to occur on any of the diagonals towards the secondary one (Figure 13b), it would occur that, for r_1 and r_2, respectively, $a_1 + b_1 = a_2 + b_2$. Thus, for example, if r_1 were in $M_{5,2}$ and if r2 were in $M_{3,4}$, it would occur that $5 - 2 = 3$ and that $7 - 4 = 3$. Therefore, in this situation, r_1 and r_2 would be in conflict in one of the diagonals towards the main one. In turn, queens positioned at $M_{2,6}$ and $M_{5,3}$ would be in conflict on one of the diagonals towards the secondary one, as $2 + 6 = 5 + 3 = 8$.

One possibility of proposing solutions to the n-queens problem is to evaluate the conflicts that exist in any arrangement of the queens on the board, considering, at least, that they would be arranged in different rows and columns, that is, evaluating the conflicts in diagonals . In this way, it would be possible to propose a heuristic that would serve to evaluate each queen arrangement conFiguretion, based on counting the arcs existing on the diagonals, with each arc representing a conflict. This could be a criterion, for example, in a proposal for a simple genetic algorithm, which could discard individuals (boards) whose chromosomal conFiguretion (disposition of queens) presented greater numbers of arches.

Although, in this article, we do not intend to discuss complete computational solutions to the problem, presenting it and exposing its characteristics is an important way of discussing its relevance as a research proposal and its conFiguretion as a mathematical/computational problem. Therefore, we next discuss the specific aspects of one of the problems proposed by the investigation, linked to the ideas exposed in this section.

Theoretical and methodological aspects of the discussion about the problem

The subjects chosen for the investigation reported here were students in the fourth semester of a higher education Computer Science course who agreed to participate in the investigative process voluntarily. In total, originally, there were 10 students, divided into 3 groups, two groups with 3 students and one with four. However, before the procedures began, two students from the group with 4 components withdrew. Given this fact, the students themselves requested that the groups be made up of 4 members, with the relocation of the remaining 2 students. Thus, finally, the subjects formed 2 groups with 4 members[10]. The four experimental sessions, lasting around an hour each, once a week, took place sequentially in the second half of 2022.

[10] To preserve their identities, students are identified from E1 to E8 in this work; group 1 had as members students E1, E2, E3 to E4, while group 2 was made up of students E5, E6, E7 and E8.

The division into groups was intended to facilitate participants' interactions and collaborative work. The researcher did not make prior presentations in order to solve the problem or facilitate the solution, understanding that an approach of this type would not allow students to collaborate with each other. This logic of not anticipating solutions and using problems to lead subjects to construct their own learning is based on the Theory of Didactic Situations – TSD (Brousseau, 1986; Brousseau, 2002). The activity explored in this article considers the notion of didactic situation, inserted in sequences with problems in which students are unable to immediately perceive the teacher's didactic intention. Faced with likely questions from students about how to provide answers to the questions raised, the teacher must make students accept responsibility for constructing conjectures and searching for possible answers – in TSD, this is the idea of *devolution*. In addition,

> [...] the proposition of the problem foresees a material, didactic and theoretical context of an antagonistic nature (the *milieu*), within which the student's investigative process follows three distinct dialectics: action, formulation, and validation. The teacher returns to the didactic nature of the proposal when he proposes to discuss and clarify the status of valid mathematical knowledge, which occurs through the dialectics of institutionalization (Oliveira and Marcelino, 2015, p. 822).

In Brousseau's (2002) view, the relationships between teacher, students and knowledge take place in the context of the milieu, planned to consider an antagonistic logic, representing a source of difficulties, contradictions, and imbalances, from a constructivist point of view. In this way, students are expected to engage in the process of building their own knowledge without directly appealing to the teacher's interventions, but based on retroactions related to the milieu, considering its material, social and objective character.

In a more structured way, the teacher introduces a specific problem for the students to solve, and they must create strategies to approach this situation, in a phase called action dialectics. Students are then encouraged to formulate assumptions related to this problem and share them with their peers, which promotes the development of formulation dialectics. During the discussions that follow, students must organize their ideas around more solid arguments on the subject and persuade their peers that their ideas are valid within a predefined set of rules, thus marking the dialectic of validation. It is important to highlight that, despite the organized description presented here, these phases do not occur in a linear or hierarchical manner; they intertwine and overlap, allowing for several distinct trajectories, characterized by advances and setbacks along the dialectics. Finally, the teacher has the role of externally validating ideas, giving them cultural relevance, in addition to organizing and summarizing the new knowledge acquired, which constitutes the institutionalization process.

Among the methodological assumptions admitted in the qualitative investigation described here, didactic engineering was the approach chosen for this stage. This approach is based on “didactic achievements” and the typical setting is the classroom. In this sense, the researcher proposes a didactic sequence with the aim of enabling students to develop autonomous paths through which they create conjectures and seek to validate them for problem solving. Barquero and Bosch (2015) explain that this is an approach that employs didactic experiments, through phases that would include:

- *Preliminary analyses*: consisting of an examination of the epistemological aspect of the mathematical knowledge involved in the proposal. In this article, clarifications in this regard occurred in the previous section;
- *Construction of situations (design) and a priori analysis*: concerns the design of the study, that is, the constitution of the problems that will be part of the proposed didactic situations, and the evaluation of possible solutions and conjectures of the respondents, including didactic and mathematical analyzes of the questions;
- *Implementation, observation and data collection*: this involves developing the experiment with students, that is, applying the planned sequence, which allows data to be collected and, eventually, observed and interpreted the resolution processes used;
- *A posteriori analysis*: last phase, which presupposes comparison and validation in relation to the hypotheses raised previously, mainly those arising from the a priori analysis.

According to Barquero and Bosch (2015), this approach can be summarized as shown in Figure 5.

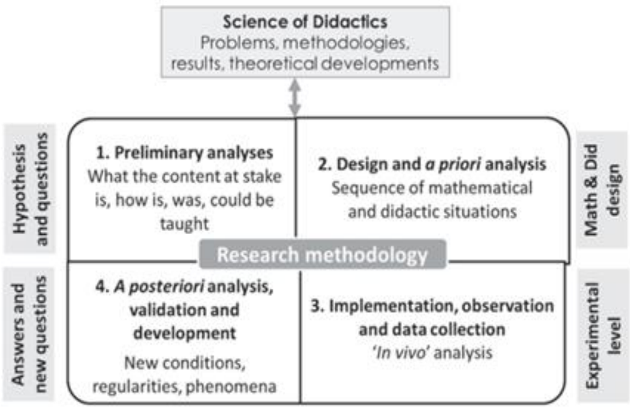

Figure 5

Phases of Didactic Engineering (Barquero and Bosch, 2014, p. 252)

Regarding the procedures, in the first session, the researcher presented the idea and objectives of the research he was conducting, receiving the students' written consent regarding the collection and use of the data produced to solve the problem. The subjects had already been introduced to the principles of Artificial Intelligence (AI) within the scope of the Computer Science course but had not yet studied the n-queens problem. Thus, the researcher made a brief presentation about AI and the specific problem, without mentioning the possibilities for resolution, nor presenting any previously known results; in addition, the students read about the topic and discussed among themselves, with the participation of the researcher. Then, the researcher conducted a semi-structured interview, with open questions, as a procedure to gather the students' impressions about the problem. Finally, he presented an activity that consisted of creating a generalization for the task of finding the number of diagonals of an $n \times n$ matrix, with $n \geq 4$.

Semi-structured interview: a descriptive analysis

The first question of the interview asked students to comment on what they thought about the degree of difficulty of the problem. Even though the responses were accompanied by small comments, it is possible to categorize them as shown in table 1.

Table 1.

Answers to the first question (difficulty degree of the n-queens problem).

Very difficult	**Difficult/challenging**	**Reasonable (requires further investigation)**	**No idea/don´t know**
2	4	1	1

Note. Unique answers

As can be seen, no student indicated that the problem would be easy. This conception seems ideal for the intended scenario: in fact, for Echeverría (1998), it is necessary that anyone who sets out to solve a problem encounters some difficulty in doing so, including in the sense of creating questions regarding how the objective could be achieved.

The second question of the interview sought to find out if the students were interested in studying this problem, considering involved mathematical and computational aspects. All students responded affirmatively. When asked what reasons would motivate such interest, the students presented their answers categorized in table 2.

Table 2.

Reasons why students would like to solve the problem.

Learn about AI and Mathematics	Overcoming an intriguing challenge (personal satisfaction)	Learning differently from “normal” classes	Progress in knowledge aimed at professional insertion	Work in groups and learn from colleagues
5	5	4	3	2

Note. There may be more than one answer per individual.

The data available in Table 2 indicate that all students presented more than one answer, which seems to indicate that they related the process of solving problems with learning specific topics and with the fact that they were challenged to find answers. For Oliveira and Mastroianni (2015), a problem must be designed to encourage the student to solve it, precisely because they perceive it as an obstacle to overcome. Responses also emerged that indicate students' willingness to carry out learning processes differently from those based on traditional classes, marked by exposure and transmission, when they report believing that research would be a possibility for learning and when they express motivation to work in groups. Furthermore, three students indicated that this process could help with their professional insertion. After this moment, the researcher indicated some bibliographic sources and explained that students were free to choose others in order to explore the n-queens problem and its possible solutions.

In the following session, it was possible to determine that all subjects had read on the topic. In the open dialogue that followed, the subjects mentioned techniques linked to AI for dealing with the problem, such as *backtracking* and *genetic algorithms*. Students E2, E5 and E7 also highlighted the importance of determining whether the queens arranged on the board presented conflicts and that this process would be more complex when related to diagonals.

As there were no comments about the perception of conflict in the diagonals through the addition or subtraction of the coordinates of the positions in which the queens were, as already indicated in this article, the researcher did not anticipate this conjecture, only informing that the procedures to solve the complete problem would be left for another time. Considering the opportunity offered by the ongoing discussion, the researcher presented the activity to be worked on in groups by the students.

Group activity

The presentation of the activity that students would do in groups took place in the context of the dialogues established in the second session, around the need to evaluate conflicts between the queens. Student E3 indicated that determining conflicts in row and column would be simpler, computationally speaking; E4 indicated the importance of the distribution of queens being done considering different rows and columns, with which everyone agreed. Regarding the diagonals, student E2 commented that it would be important to have a “formula” or “something generic” to determine the number of diagonals in the matrix, considering, according to him, that “there could be a 127-queens problem” which would be very laborious. The others agreed and asked the researcher how they could obtain this formula (specifically, E1 and E8 requested that the formula be provided). The researcher indicated that this would be a good problem and that it would involve mathematics – and that this was, precisely, the main question at that stage of the investigation. The second session, then, ended here.

In the third session, the researcher formulated the following statement: “create an algebraic expression that allows you to indicate the number of diagonals in which there can be conflict on any board of the n-queens problem. It is very important that the group describes how they arrived at the answer.” Students would have a week to discuss and consolidate a solution to the problem, which would be presented in the following session. The following analyzes are carried out in accordance with Didactic Engineering.

A priori analysis

The problem proposed in the investigation asks the subjects to find an algebraic expression that allows them to generalize the number of diagonals of a board, seen as a square matrix, based on the number of queens available on it (from 4). Students can look for regularities by analyzing individual cases, conjecturing about an algebraic expression that gives the number of diagonals for any number q of queens, $q \geq 4$. This is a problem that can be solved through pattern generalization. This type of activity is extremely important for algebraic thinking, which leads to the indication, in the Principles and Standards for School Mathematics (NCTM, 2000), that patterns form the bases of algebraic thinking; therefore, its exploration would be fundamental to work with students on tasks aimed at identifying relationships and elaborating generalizations (Oliveira and Lopes, 2021). In Mason's view (1996), it is very important to develop skills that allow expressing generalizations, since this learning is fundamental for structuring and advancing the construction of knowledge in algebra.

One way to understand the regularity in the problem is to draw some square matrices from the 4 x 4 matrix, counting the existing diagonals and observing their characteristics. Figure 6 shows some possibilities in this regard.

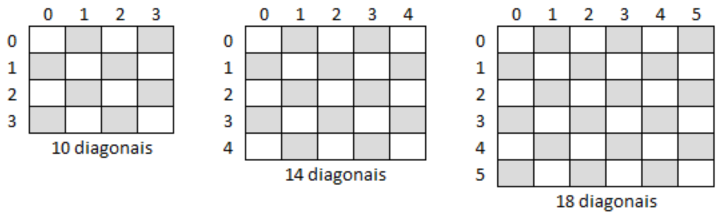

Figure 6.

Number of diagonals in matrices 4 x 4, *5 x 5* and *6 x 6*

From *Oliveira, G., 2024 (drawing made by author)*

Students can notice through this strategy that the number of diagonals increases by 4, at least in relation to the drawn matrices. The initial perception of this relationship conFigures what Zazkis and Liljedahl (2002) call *near generalization*. Such generalizations occur from the observation of cases close to the initial ones – this possibility is called by the authors expansive generalization, in the sense that only the extension of the generalization strategy occurs, without the reconstruction of the original scheme. However, it is expected that students who choose this strategy will be able to resort, at some point, to *far generalization*. For Zazkis and Liljedahl (2002), this "distant generalization" has a reconstructive aspect, is not originated by extensions and is important in obtaining an expression that represents the generic case. In other words, it is important to redo the scheme that constituted the sequence, which would make it possible to answer, without resorting to counting or some other type of extension, what the number of diagonals on the board would be for 253 queens.

In this sense, students could notice, as seen in Figure 4, that there are no diagonals formed in the upper and lower corners of both the left and right sides - that is, one must "discount" these unitary diagonals in which there is no possibility of conflicts. For the three cases shown in Figure 6, it is possible to observe that there are $q - 2$ diagonals in the triangle superior to the main diagonal, as well as $q - 2$ diagonals in the inferior triangle. The same occurs in relation to the secondary diagonal. Finally, to the number of diagonals thus counted, the main and secondary diagonals must be added:

(1) $2(q - 2) + 2(q - 2) + 2 = 4q - 8 + 2 = \mathbf{4q - 6}$

Thus, $4q - 6$ is the expression that indicates the number of diagonals in which there can be conflicts on a board $q \times q$, $q \geq 4$. That is, $2(2q - 3)$, twice the limit of the diagonals on a board $q \times q$, considering both ways, as indicated in El Abidini (2023, p. 4). Evidently, other reasoning towards a *far generalization* would be possible.

Another strategy can be suggested, considering that the number of diagonals on a board is 10 for the *4 x 4* board, 14 for the *5 x 5* board, 18 for the *6 x 6* board and so on. It can be conjectured that the number of diagonals of the matrix $q \ x \ q$ can be determined from an arithmetic progression, whose general term a_n can be found based on the first term a_1 and the common difference *r*, as expressed in (2):

(2) $a_n = a_1 + (n-1).r$

In this case, as the problem's existence condition implies that $q \geq 4$, a_1 has the value 10. In this hypothesis, the common difference would be equal to 4, the difference between the number of diagonals determined in the dimensions of the matrices. Thus, we would have (3):

(3) $a_n = 10 + (n-1).4$

The available information, however, corresponds to a number *q* of queens to be arranged on a board *q x q*. The relationship between *q* and *n* would imply considering that the first valid number of queens is 4, that is, for $q = 4$, there would be $n = 1$; for $q = 5$, we would have $n = 2$, and so on. Thus, it is possible to conjecture, to adjust the number of queens to the equivalent position in the arithmetic progression, that $n = q - 3$, allowing us to write (4):

(4) $a_n = 10 + (q-3-1).4 = 10 + (q-4).4 = \mathbf{4q - 6}$

Considering these conjectures, as recommended in the scope of didactic engineering, without the intention of exhausting the possibilities that could arise in student interactions, the research continued, allowing the analysis of the subjects' productions.

Analysis of the experimental session and a posteriori

As already described, the third session began with the search for the algebraic expression relating to the number of diagonals in which there could be conflict between queens on a board $q \ x \ q$, $q \geq 4$. Whenever the researcher's mediation was necessary, he could be called, but no interventions would be made to resolve the activity, in consideration of the notion of devolution[11] advocated by TSD (Brousseau, 1986). Considering that it would be necessary to prepare a written response based on the conjectures and validations, the researcher indicated that all discussion should take place during that session and that there would be an additional time of up to 40 minutes to prepare the text that would serve as a record of each group's response.

Group 1 was made up of students E1, E2, E3 and E4. Initially, E3 proposed that they draw some matrices and count the diagonals, which was up to him and E1. While they were

[11] According to Brousseau (1986), the term "*devolution*" represents the role of the teacher/researcher within the scope of the mathematical activity of his group of students, preserving the didactic intentionality of the initiative hidden, ensuring that the answers emerge from retroactions in relation to the *milieu* and that the students accept the challenge of solving the problem as their task.

drawing, some hypotheses were discussed, including E2's proposal about seeking the answer based on the number of lines/columns, trying to relate this number to the number of diagonals.

> **(E2)** – What if we tried to multiply row by column and then divide by... I don't know, by 2, maybe?
> **(E4)** – I don't think so; row and column are the same, they are the same. It's the same as the number of queens, too.
> **(E1)** – I drew it here... with 4 queens it gives 10 diagonals without counting those in the corners. In these cases, you can't put two queens.
> **(E3)** – Look here, the one with 5 queens gives 14 and the one with 6 queens gives 18. How much does the one with 7 and 8 give? I think it goes from 4 to 4...

After a while (about 10 minutes), with discussions and new drawings, the group had manually produced representations of the matrices for the cases of 4 to 10 queens, which led them to indicate that the difference between one board and another would be 4 diagonals. The student who proposed the most conjectures was E3, who exposed them to the group so that they could be confirmed or refuted.

> **(E3)** – We have 7 diagonals in this matrix on top of this larger diagonal, counting it too [using a notebook and indicating a drawing in Microsoft Excel of an 8 x 8 matrix]. Down also has 7. It has 7, right, professor?
> **(P)** – What do you think?
> **(E2)** – No, down is 6, you are counting the “big one” twice [main diagonal]... 7+6 = 13.
> **(E1)** – But this one has 26... it's double.
> **(E3)** – It's because on the other side... look, from left to right there are 13 and from right to left there are 13 too! I think we solved it!

A relatively long time was consumed trying to discover whether the conjecture was valid. The students were convinced that it was so when they confirmed it in all the other matrices they had drawn. E4 reminded the others that they needed a “formula”, that is, an algebraic expression.

> **(E3)** – It´s true, but look, this one [8 x 8], is 7 + 6, I was thinking: 8 queens, 8 – 1 = 7 and 8 – 2 = 6. 8 is the n of the formula, so what does it look like... I'll write here.

E3 writes *n – 1 + n – 2 * 2* in a text editor with his computer, using the asterisk as a multiplication symbol, which is common in computing. E1 indicated that they needed it “in parentheses”, otherwise “it would go very wrong”. In fact, after discussing for a few more minutes, they indicated that the expression would be *((n – 1) + (n – 2)) * 2*. This expression is equivalent to *4n – 6* and is correct. Using part of the additional time, the group wrote the solution in one of the group participants' notebooks, as can be seen in Figure 7.

	0	1	2	3	4	5	6	7	8	9
0	9	8	7	6	5	4	3	2	1	0
1	8	9	8	7	6	5	4	3	2	1
2	7	8	9	8	7	6	5	4	3	2
3	6	7	8	9	8	7	6	5	4	3
4	5	6	7	8	9	8	7	6	5	4
5	4	5	6	7	8	9	8	7	6	5
6	3	4	5	6	7	8	9	8	7	6
7	2	3	4	5	6	7	8	9	8	7
8	1	2	3	4	5	6	7	8	9	8
9	0	1	2	3	4	5	6	7	8	9

n = 10

Sentido da diagonal principal

Abaixo da principal: conta de 1 a 9 **(n-1)**

Acima da principal: conta de 1 a 8 **(n - 2)**

Abaixo: desconta a diagonal 0

Acima : desconta as diagonais 0 e 9

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	8
2	2	3	4	5	6	7	8	9	8	7
3	3	4	5	6	7	8	9	8	7	6
4	4	5	6	7	8	9	8	7	6	5
5	5	6	7	8	9	8	7	6	5	4
6	6	7	8	9	8	7	6	5	4	3
7	7	8	9	8	7	6	5	4	3	2
8	8	9	8	7	6	5	4	3	2	1
9	9	8	7	6	5	4	3	2	1	0

n = 10

Sentido da diagonal secundária

Abaixo da secundária: conta de 1 a 9 **(n-1)**

Acima da secundária: conta de 1 a 8 **(n - 2)**

Abaixo: desconta a diagonal 0

Acima : desconta as diagonais 0 e 9

Tanto no sentido da principal como da secundária ficaria ***((n-1) + (n-2))***

Portanto, o número de diagonais de uma matriz ***n x n*** *é* ***((n-1) + (n-2)) * 2***

Figure 7.

Answer to the proposed problem – group 1.

Note. The text below the Figure was originally written in Portuguese by the students.

The group based their conjectures on the search for a generalization based on the pattern they determined through the extension of their discoveries, that is, using what Zazkis and Liljedahl (2002) call *near generalization*. There was no attempt to use other mathematical possibilities, such as arithmetic progressions, for example, to validate the conjectures. Having found a possible pattern and extended it to some cases that they deemed sufficient, they assumed that this proposition would fit any case, even in boards with hundreds of queens, for example. It is not possible to ignore, on the other hand, that the group's dialogues point to an examination of the structure of a group of matrices, which goes a little beyond the mere examination of individual cases – even though the group has formalized the report only with the case $n = 10$, several sketches were produced with other cases. In this sense, we can see an approximation, albeit somewhat incipient, of the *far generalization* proposal indicated by Zazkis and Liljedahl (2002). Furthermore, it is not possible to detect in the group any intention of providing a refinement to the expression through a development that would allow it to be indicated in its canonical form.

In relation to the dynamics of solving the problem, the dialectical movements indicated by Brousseau (2002) were present, from the manipulation of data in a more operational way (action dialectic) to the discussion of conjectures (formulation dialectic). and its subsequent validation.

The members of group 2 were students E5, E6, E7 and E8. The strategy of drawing “small” square matrices (with dimensions between 4 and 6) was also used by this group, which indicates that the generalization sought as an answer to the problem came from expansive procedures, such as extension between the considered versions of the matrices – *near generalization*, in the view of Zazkis and Liljedahl (2002). Dialogues among group members allow us to understand this initial strategy and how it changed based on partial discoveries:

> **(E6)** – [Using his computer and a blank Microsoft Excel spreadsheet] Let's draw the first matrices. Then, we try to understand what changes from one to the other. Let's try 3 x 3 and increase one by one;
> **(E8)** – No, 3 by 3 doesn't work, there's no solution. You have to start with 4, then it's 4 by 4, like a rally car [laughs];
> **(E6)** – It's true, we had read something like that, or the teacher said it;
> **(E8)** – So go on, do a 4 x 4, I'll do it here too. Hey, E5 and E7, help! Did you sleep? [laughs]
> **(E7)** – No, no, I'm following here.
> **(E5)** – You talk a lot! While you were talking, I made some sketches here [also using a computer]. Look, I think 4 x 4 has 12 diagonals.

Initially, the other members of group 2 seemed to agree. Student E5 gave a description of what he called the “shape” of the matrix, or, in his words, “the type of design of each one”.

> **(E5)** –. Look, we can think that each matrix has a type of mirror, an opposite... this larger diagonal on one side [pointing, on your computer, to the main diagonal] and this other one on the other side [secondary diagonal]. It has the bottom and top diagonals of one and the bottom and top of the other. We have to count.
> **(E8)** – True! Then, it will be 12 for the 4 by 4 and 16 for the 5 x 5...
> **(E7)** – The 6 by 6 is 20...

Before continuing with the conjectures, the students in this group realized, like those in group 1, that the unitary diagonals of the upper and lower ends of the matrices did not pose a risk of conflicts between the queens; however, they were in doubt whether or not they should consider them. The researcher was consulted and, once again, he returned the question, asking what the opinion of the group members would be and what reasons would justify such a proposal. After re-reading the problem and spending several minutes analyzing the matrices, they decided that these "corners" should not be counted:

> **(E5)** – Well, if the corners don't count, then each matrix will have another count: 4 x 4 gives 10, 5 x 5 gives 14 and 6 x 6 gives 18...
> **(E7)** – It varies from 4 to 4. Is it because we started counting from the 4 x 4 matrix?

> **(E6)** – I don't think so. If it is the previous one plus 4, would the formula be n – 1 + 4? No, wait a minute, not n – 1, but the result of n – 1, that is, when we are looking at 5 x 5, it is what resulted in 4 x 4 plus 4... do you understand?
> **(E8)** – Yeah, that makes sense, but we would have to have a formula for the first matrix, right? And we still don't know the formula.

E6's proposal was to provide an answer through the use of a sequence defined by recurrence. In this sense, "a sequence is defined by recurrence by explicitly naming the first value (or a few first values) in the sequence and then defining subsequent values in the sequence in terms of previous values" (Gersting, 1999, p. 85) . Although it cannot be indicated that such a proposal is wrong, the problem indicated, albeit implicitly, that the generalization should provide the answer for any predicted case, that is, any number of queens, including 4. Thus, claiming that one of the cases would not be met by the proposal, the group preferred to discard this conjecture.

In the following moments, the members of group 2 defined some assumptions that were important for establishing a common discussion, such as calling the number of queens *N* and relating it to the dimensions of the board, as well as calling *d* the number of diagonals of any *N x N* board. This may seem simple, but it was essential for group members to adopt a common discourse, which allowed them to propose a conjecture that, when refined, produced a valid answer to the activity.

> **(E5)** – We saw that the halves of the matrix are equal... just on the opposite side, inverted... that is, there are the same number of diagonals. So, what's true for one is true for the other...
> **(E5)** – So, I guess that means it's twice something, right? Because they are the same, only the side changes...
> **(E7)** – True! In this 4 x 4, there are 5 on one side and 5 on the other. So that's 10, which is 5 times 2! In others too.

By indicating a logic underlying the constitution of the boards, the students in group 2 employ a reconstructive sense to the pattern, that is, they reconstruct the way in which the matrices are formed, in relation to the diagonals, without resorting to extensions. This is precisely the so-called *far generalization* indicated by Zazkis and Liljedahl (2002), fundamental to obtaining a generalization. After discussing the structure of one of the "sides", which is how they referred to the diagonals in the direction of the main diagonal, the students created a conjecture with texts and drawings (Figure 8) to indicate the conjecture that they ended up validating, as follows:

> Firstly, we take into consideration:
> - On an N x N board, where N is the number of queens
> - d = number of diagonals.
>
> The first step towards creating the algebraic expression was to test 3 matrices by hand. With this, we would see the relationship between them, as the number of sides increased in the matrices. We set up the 4x4, 5x5 and 6x6 matrices.
> Considering the main diagonal and the secondary diagonal, we have 2 directions for counting total diagonals.

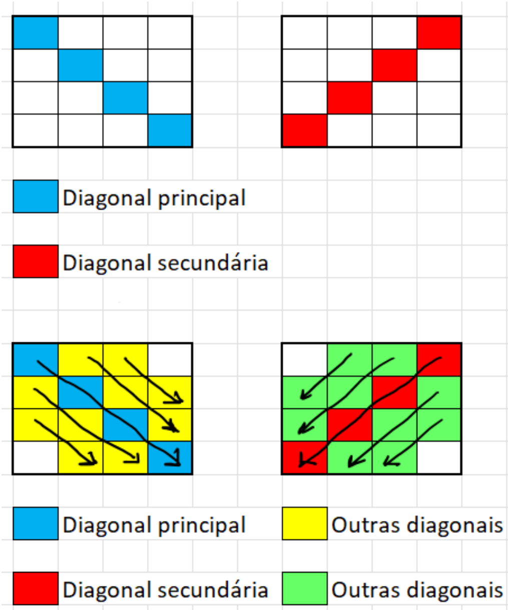

Figure 8.

Representation of drawn matrices – group 2

Note. The text below matrices was originally written in Portuguese by the students.

> Looking at the 4x4 matrix, we realize that it has 10 diagonals, 5 in one direction and 5 in the other – we made a diagram, and each black arrow indicates a diagonal. We decided not to consider the corners because they cannot form conflicts. Then, we made other matrices, up to 9 x 9, and made other discoveries. The 5 by 5 Matrix has 14 diagonals (7 x 2) and the 6 x 6 has 18 diagonals (9 x 2) and so on. We saw that there is always the same number of diagonals in each direction. We also realize that we cannot count the main and secondary diagonal twice, nor the two corners that do not have a conflict. Therefore, it would be N x 2 – 3. As it has two directions, we think we have to multiply by 2, which would give D = (2 * N – 3) * 2.

In the highlighted text, the conjecture presented by the students is obviously valid. However, the students did not consider using the distributive property in the expression D = (2 * N – 3) * 2, which would allow obtaining D = 4N – 6. It can be suggested, in this sense, that the students preferred to maintain the form of the expression that allowed them to deduce it as 2 times N (twice the dimension of the square matrix) minus 3 (removing the main and secondary diagonals, which would be counted twice originally, in addition to the diagonals in which there are no possible conflicts); this result, in turn (from subtraction, originally written in parentheses) is multiplied by 2, as the same reasoning is applied in relation to the "sides" of the main and secondary diagonals.

Final considerations

After the end of the last session, with the presentation of the groups' reports, the researcher promoted a debate, in which he sought to discuss with the participants the results achieved, indicating the correctness of the proposed solutions and the possibility of refining their presentations from a mathematical point of view. The subjects assimilated the ideas and suggested the continuity of the activities, through the construction of computer systems that validated boards based on the arrangement of possible "players" (in terms of the problem studied). Furthermore, they requested to participate in other similar activities, with a list of themes, presented at this final moment, which involved solving problems linked to data structure, artificial intelligence, programming in different paradigms (structured, object-oriented, symbolic) and, quite significantly, mathematics.

Specifically, regarding mathematics, students reported the perception, contrary to the initial survey, that the activity itself would be quite simple, once seen as a problem to be solved and based on the interactions that took place with peers. In this sense, student E3, for example, indicated that he would like to experience other challenges and participate in other study groups or similar research initiatives.

The didactic strategy adopted, based on TSD, seemed quite adequate to deal with the proposed resolution of the problem addressed here. In fact, the elements present in the milieu contributed to the construction of an antagonistic logic. The same can be said about the researcher's stance of non-facilitation, which allowed participants to advance in discussions, based on retroactions that had the milieu as a reference.

Another element deserves to be highlighted: the students primarily used their computers and programs that could offer an appropriate interface to support the construction of conjectures – there was no access to the Internet or any external resources within the scope of the third session. Furthermore, the problem of diagonals had not been presented previously. It is not impossible that students had already come across texts or experiences related to the problem addressed in this session, but it is unlikely, which can be conjectured due to the nature of interactions during problem solving and the scarcity of literature that specifically addresses the issue. . In any case, it can be understood that the use of computer interfaces supported the resolution processes, which may indicate, as Oliveira (2018) points out, that fluency in the use of technologies contributes to subsidizing the thinking of the people involved when building their strategies and conjectures based on the dynamism, interactivity and possibilities for experimentation opened up by the use of mobilized software.

Finally, it is worth noting that the experience surrounding problem solving pointed to the importance of this approach as a learning resource in higher education courses and, in this case, more specifically, in Computer Science courses. Problem solving is a typical activity both in mathematics and in various computational topics, such as programming, for example. Proposing problematizations that occur as challenges and conducting them in a non-directive didactic process seem to create promising trajectories, including interdisciplinary proposals involving, among the components, mathematics.

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