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Evolution of the notion of continuity and reflections on the relationship between discrete and continuum

La evolución de la noción de continuidad y reflexiones sobre la relación entre lo discreto y lo continuo

L'évolution de la notion de continuité et réflexions sur la relation entre le discret et le continu

Evolução da noção de continuidade e reflexões sobre a relação entre discreto e contínuo

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Abstract

Although the beginning of the mathematical treatment of discrete and continuum objects precedes the elaboration of the theoretical notions of continuity and discreteness – the property of being discrete – it is correct to say that the first time in history that the contradiction between the two concepts appeared dates back to ancient Greece, and Zeno's paradoxes are the oldest and clearest example of this contradiction. Despite the changes that occurred with the scientific revolution of the 17th century and the emergence of the notion of function, continuity remained related to the movement of an object from one place to another, although with the work of Descartes began a process of unification between the discrete and continuum aspects of mathematics. In the 19th century, the notion of continuity took on a new form, as the notion of continuity and discrete mathematics began to be approached on the basis of studies of series and motion, which made possible the modern definitions of limit and continuity, which in turn made it possible to establish an intrinsic relationship between the discrete and the continuum.

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After the historical exposition, an attempt is made to show the epistemological and philosophical implications of this process, which are extremely important for the educational process, insofar as the discrete and the continuum are related to language and intuition. The methodology used in this article is a historical bibliographical analysis based on the notion of complementarity as elaborated by Michael Otte.

Keywords: Continuity, Discreteness, Definition, Mathematics.

Resumen

Si bien el inicio del tratamiento matemático de los objetos discretos y continuos precede a la elaboración de las nociones teóricas de continuidad y discreción –propiedad de ser discreto–, es correcto afirmar que la primera vez en la historia que la contradicción entre ambos conceptos apareció se remonta a la Antigua Grecia, y las paradojas de Zenón son el ejemplo más antiguo y claro de esta contradicción. A pesar de los cambios ocurridos con la Revolución Científica del siglo XVII y el surgimiento de la noción de función, la continuidad siguió relacionada con el movimiento de un objeto de un lugar a otro, aunque, con la obra de Descartes, se inició un proceso de unificación entre los aspectos discretos y continuos de las matemáticas. En el siglo XIX se daría una nueva faceta a la noción de continuidad, al iniciarse un acercamiento a la noción de continuidad y a la matemática discreta a partir de estudios de series y movimientos que posibilitaron definiciones modernas de límite y continuidad, que – a su vez – permitió establecer una relación intrínseca entre lo discreto y lo continuo. Luego de la exposición histórica, buscamos mostrar las implicaciones epistemológicas y filosóficas de este proceso, las cuales son de suma importancia para el proceso educativo, ya que lo discreto y lo continuo se relacionan con el lenguaje y la intuición. En este artículo se utilizó el análisis bibliográfico histórico como metodología basada en la noción de complementariedad elaborada por Michael Otte.

Palabras clave: Continuidad, discreción, definición, matemáticas.

Résumé

Bien que le début du traitement mathématique des objets discrets et continus précède l'élaboration des notions théoriques de continuité et de discrétion – propriété d'être discret – il est correct d'affirmer que la première fois dans l'histoire où est apparue la contradiction existante entre les deux concepts remonte à la Grèce antique, avec les paradoxes de Zénon comme exemple le plus ancien et le plus clair. Malgré les changements survenus avec la Révolution scientifique du XVIIe siècle et l'émergence de la notion de fonction, la continuité est restée liée au mouvement d'un objet d'un endroit à un autre, bien que, avec l'œuvre de Descartes, un processus d'unification entre les aspects discrets et continus des mathématiques ait commencé, ce qui, au XIXe siècle, donnerait une nouvelle forme à la notion de continuité, lorsque commence une approche de la notion de continuité et des mathématiques discrètes basée sur les études de séries et de mouvement qui permettront les définitions modernes de limite et de continuité, permettant l'établissement d'une relation intrinsèque entre le discret et le continu. Après l'exposition historique, nous cherchons à montrer les implications épistémologiques et philosophiques de ce processus, qui sont d'une importance extrême pour le processus éducatif, dans la mesure où le discret et le continu sont liés au langage et à l'intuition. Dans cet article, nous utilisons comme méthodologie l'analyse historique bibliographique basée sur la notion de complémentarité telle que développée par Michael Otte.

Mots-clés : Continuité, discrétion, définition, mathématiques.

Resumo

Embora o início do trato matemático de objetos discretos e de objetos contínuos preceda a elaboração das noções teóricas de continuidade e discretude – propriedade de ser discreto –, é correto afirmar que a primeira vez na história em que apareceu a contradição existente entre os dois conceitos é datada da Grécia Antiga, e os paradoxos de Zenão são o exemplo mais antigo e claro dessa contradição. Apesar das mudanças ocorridas com a Revolução Científica do século XVII e do surgimento da noção de função, a continuidade permaneceu relacionada com o movimento de um objeto de um local a outro, embora, com a obra de Descartes, tenha começado um processo de unificação entre os aspectos discreto e contínuo da matemática. No século XIX, seria dada uma nova feição à noção de continuidade, ao se iniciar uma abordagem da noção de continuidade e da matemática discreta com base nos estudos de séries e do movimento que tornaram possíveis as modernas definições de limite e de continuidade, que – por sua vez – permitiram o estabelecimento de uma relação intrínseca entre o discreto e o contínuo. Após a exposição histórica, procura-se mostrar as implicações epistemológicas e filosóficas desse processo, que são de extrema importância para o processo educacional, na medida em que o discreto e o contínuo se relacionam com a linguagem e a intuição. No presente artigo, utilizou-se como metodologia a análise histórica bibliográfica com base na noção de complementaridade tal qual elaborada por Michael Otte.

Palavras-chave: Continuidade, Discretude, Definição, Matemática.

Evolution of the notion of continuity and reflections on the relationship between discrete and continuum

The aim of this article is to present the history of the evolution of the notion of continuity, to reflect on the philosophical and epistemological considerations that this history entails, and to highlight the relationship between continuity and discreteness – the property of being discrete – in terms of the formal and intuitive aspects of mathematics. The method that was used was a historical bibliographic analysis based on the notion of complementarity developed by Michael Otte (Clímaco et al., 2024; Otte, 1994, 2003).

We begin by discussing the emergence of the notion of continuity in Ancient Greece and Zeno's paradox, identified as "Achilles and the Tortoise," to reflect on the incapacity of Greek mathematics to reconcile the qualitative (continuum) and quantitative (discrete) aspects of mathematics.

We then discuss how the relationship between the discrete and the continuum emerged in the 17th and 18th centuries, showing how the Greek heritage was transformed, and its concepts approached from a new perspective – that of the Scientific Revolution – with the growing appreciation of numbers that took shape with the foundation of algebra and the creation of analytical geometry.

We then show how the question of the relationship between the discrete and the continuum was resolved – in a way that is still valid today – by the precise and rigorous definition of continuity in terms of epsilon and delta, on the one hand, and by the establishment of the relationship between the line and numbers, on the other. To do this, we show the fundamental process – which can be considered the continuation of a process that began with Descartes – of the arithmetization of mathematics. A process that revolutionized mathematical knowledge and at the same time promoted an inversion in the way we conceive of the continuum and the discrete.

Before the end, we also explain the importance of the approach taken in the article and why the views of historians who insist that there is no substantial difference between the notions of rigor of the 17th and 18th centuries and those that emerged in the 19th century and continue to this day are incorrect.

In the final considerations, we present some important philosophical and epistemological implications for thinking about the consequences for the educational process of this true scientific revolution – to use the term used by Judith Grabiner, a Cauchy scholar – in mathematics. In fact, little has been done to analyze the educational implications of this revolution, as we do in this article, and for this purpose, we use the notion of complementarity as proposed by Otte (1994, 2003).

The complementarity between the discrete and the continuum is also fundamental because numerous phenomena in the physical world can be modeled using discrete or continuum models, and this has epistemological implications that are not only scientific but also methodological and didactic.

The Emergence of the Concept of Continuity in Ancient Greece

The notions of continuity and discreteness certainly have deep roots in the history of the development of mathematical knowledge and can be found in different times and cultures. Throughout its history, humankind has developed counting systems to deal with quantification from a discrete point of view, on the one hand, and has observed the existence of the dimension of continuum quantities such as length, area, and volume, on the other.

More than 2000 years before a formal definition of continuity of functions appeared, it was common to attribute the property of being continuum to a quantity or phenomenon in nature if it consisted of an uninterrupted whole, without holes; in general, before the 19th century, the most common examples of continuity did not refer to mathematical objects, but to physical concepts such as energy, motion, time, and so on. In mathematics, at least since the 19th century, the exemplary model of a continuum object is the straight line.

In contrast to continuity is the attribute of being discrete. Originally, **discrete** referred to something distinct from something else, denoting a clear separation between elements. Thus, integers numbers, taken as individual entities – ignoring fractions or decimals – appear to be separate, rather than forming a continuum. Similarly, discrete geometry contrasts with continuum geometry in that it deals with geometric structures made up of separate individual points.

The earliest record – referring to the paradoxes of Zeno of Eleia (490-430 BC) – that we have of the existence of a difficulty in properly relating the continuum and the discrete can be found in the work Physics, by Aristotle (2009a, 1013, 4ff), a philosopher who lived between 384 and 322 BC. He refers in particular to the paradox that describes a race between the tortoise and the war hero and Olympic champion Achilles – and which can be stated as follows: in a race between Achilles and the tortoise, in which the tortoise takes the lead, every time Achilles reaches the place where the tortoise was, the tortoise has already covered a certain distance and is at another place; this happens successively, an indefinite number of times, so that no matter how hard Achilles runs, he will never catch up with the tortoise.

The enunciation of this paradox not only showed the inconsistency between physical phenomena and the mathematics of the time, but and more importantly from the point of view of the subject of this article, for the first time showed the difficulty of relating the qualitative and quantitative aspects of mathematics, or between the continuum and the discrete.

In Ancient Greece, there were atomistic philosophers and mathematicians – such as Democritus (460-370 B.C.) – who conceived of a discrete universe, made up of isolated, indivisible parts and used these concepts to successfully calculate the volume of solids and the area of geometric figures.

But the devaluation by Greek thinkers of the practical and utilitarian aspects of knowledge, and the elevation of aesthetic, theoretical, and metaphysical ideals, contributed to a worldview in which the mathematics that was valued – as Plato (2017, 526c) said in *the Republic* – was not that of those who used it for the purpose of buying and selling, but rather that which served the honor of the spirit. This view meant that, despite his great contributions to mathematics, there was a devaluation of the notion of number in Greek mathematics, which led it away from the search for a quantitative, discrete, and static definition of continuity.

Plato and the mathematicians who attended his Academy would form the new generation of mathematicians who, in turn, would decisively influence the mathematics of the Hellenistic period of Greek history – considered by many to be the most fertile period of Greek mathematics, and which included mathematicians of the caliber of Euclid and Archimedes –; they would reject the way of thinking about mathematical objects conceived by Democritus; and they would not continue the atomistic investigations.

The notion of the number line would take centuries to develop. Thus, a geometric and continuum conception of mathematics – which included a qualitative approach – prevailed in Ancient Greece. This dominant conception was not only expressed in Plato's (2017) valorization of geometry and the metaphysical aspects of numbers, but was systematically explained by Aristotle (2009b, pp. 31-34), who formulated a notion of continuity associated with motion and the rejection of the idea that space is composed of a finite or infinite number of points, which would invalidate Zeno's paradoxes:

> The magnitude over which the change takes place is continuum. For suppose a thing changed from C to D. Then if CD were indivisible, two things that have no parts could be consecutive, and since this is impossible [the space between them], it must be a magnitude and therefore be indefinitely divisible. Then this thing makes innumerable changes before it has made any specific change.

Influenced by these ideas, Greek mathematicians and thinkers kept the concepts of discrete and continuum extremely strictly separated conceptually, and even used different terms for each concept: to geometric objects they assigned the term "size", which referred to what is continuum, and they assigned the term "number" to what is discrete[4]. Aristotle had separated quantity and quality as different categories that did not communicate, and so the mathematical legacy remained: there was no definition of a relationship between continuum and discrete quantities, nor any attempt to do so.

The changes that took place in Greek mathematics during the Hellenistic period of Ancient Greece – when Greek culture was on the one hand expanding and on the other hand being influenced by practical and arithmetical problems from the East – did not change this general orientation, and it would be centuries before mathematicians embarked on the path of investigation that would lead to the proper establishment of the relationship between the discrete and the continuum, by means of a clear and rigid relationship between points on the line and numbers.

Continuity in the 17th and 18th centuries

Still in the Middle Ages, scholastics such as Richard Swineshead – or Suisset, whose birth and death dates remain uncertain – and Jean Buridan (1300-1358) resumed studies on the nature of continuity, and the idea of an oriented line appeared, similar to what came to be called the real line in the 19th century, although there was no notion of a set of real numbers. The strengthening of trade and navigation, as well as the resumption of an active commercial life in cities and the increasing introduction of Indo-Arabic numerals in Europe, favored the beginning of a progressive appreciation of numbers and the discrete nature of mathematics (Clímaco, 2011; Sinkevich, 2017). Also in the Middle Ages, Hindus, and Arabs contributed to the development of continuity studies not only through the numerals identified with their names or Euclid's translations, but also by approaching numbers in a way that was less rigidly separated from geometry and by developing the study of equations.

In the 17th and 18th centuries, studies of the solutions of algebraic equations and the Scientific Revolution – especially with the works of Descartes, Cavallieri, Kepler, etc. – gave a tremendous boost to quantitative studies. – gave an enormous impetus to quantitative studies

[4] "Magnitude [μέγεθος (transl. megethos)], in fact, corresponds to one of the two divisions of *quantity* [gr. ποσόν , translit. póson], namely the continuous (such as a line, a surface, or a body), while number (gr. ἀριθμός. translit. arithmos) ... [is related to the] *discrete*" (Heath, 1949, p. 45). Roque's (2012, p. 166) statement that "there was nothing in common between continuous quantities (infinitely divisible) and discrete quantities (made up of indivisible units)" is also important.

to invade all areas of the natural sciences, as well as mathematics itself. Thus, the entire edifice of Greek mathematics was revised under the eyes of mathematicians with practical (scientific, but also financial) and numerical concerns – very different from the Greeks – with the invaluable help of the Cartesian plan and the symbolic notation they had developed.

In this historical period, methods that the Greeks had used – such as exhaustion, coordinates – and their theoretical achievements were taken up again, but for the purposes of calculations and discoveries, the excessive concern with rigor and the search for beauty and harmony that often caused the Greeks to neglect the exploration of other aspects of mathematics were abandoned. In particular, the mathematicians of the 17th century innovated profoundly by incorporating the study of curves expressing motion into geometry, by introducing studies of the operational aspects of numbers and calculations, and by accepting the possibility of making approximations between lines and curves.

Regarding the relationship between the discrete and continuum aspects, we can say that the mathematicians of the Scientific Revolution period, familiar with the mathematical works of the Greeks and using them in a new light, achieved a strong union between these concepts. One of the most important results of this approach to the discrete and the continuum was the creation of algebra (Boutroux, 1992), analytical geometry (with the Cartesian plane), and the notion of function.

It is undeniable that the above-mentioned advances of the 17th century represented a profound change in the nature of mathematics and a significant milestone in the search for an understanding of mathematics in which its discrete and continuum aspects were brought closer together. However, in the 17th century, mathematicians were not yet faced with the question of defining continuity or numbers in terms of numerical sequences, and it was not until the 19th century that the notion of continuity was separated from the notion of motion or transformation of natural phenomena.

Although Isaac Newton (1642-1727) carried out important studies on numerical series, his work *Mathematical Principles of Natural Philosophy* (2012) is still full of geometric demonstrations, which shows that the union between the continuum and the discrete was still relatively fragile in this scientist's work. The English thinker conceived of continuum quantities through continuum motion and explicitly rejected the definition of curves as formed by points – he understood them as generated by the motion of points. The lack of a proper understanding of the relationship between the discrete and the continuum prevented him from giving a correct definition of some of the most fundamental concepts of calculus, such as limits and continuity, which led him to use methods in the calculation of derivatives that, as Berkeley (2010) would show, at the beginning of the eighteenth century, contained important contradictions

Gottfried Wilhelm Leibniz (1646-1716) dealt with the notion of continuity in several works and in his correspondence. In 1702, in a letter to Varignon, he states, "if a continuum transition is supposed to end at a certain limit, then it is possible to form a general argument that also includes the final limit" (Leibniz, 1962, p. 93). In another publication, he states, "nature ... never proceeds by leaps" (Leibniz, 1961, p. 567) or that "no change takes place by means of leaps" (p. 168).

The definition of continuity appears in an even different way – it relates ordinates and coordinates – in Leibniz's letter to Bayle in 1687, in which he presents continuity as a **principle of general order** expressed in the words: "as the data are ordered, so the unknowns is ordered" (Leibniz, 1969, p. 37). In the Latin version *of On the Principle of Continuity*, Leibniz (1904, p. 84) makes an even clearer formulation of the distinction between independent variables (which he calls given) and dependent variables (sought): "a determined order in what is corresponds to a determined order in what is sought".

From the different definitions we have presented, we can deduce that in Leibniz's work the question of mathematical continuity ended up being confused with the **law of conservation of motion** applied to **hard bodies**: when two hard bodies collide, is there a loss of energy or simply a transfer of energy from one to the other? Leibniz maintained that there could be no loss of energy out of respect for the **law of continuity**, since it would not be possible to *"transitionem per saltum"* (cf. Schubring, 2004, p. 182). Throughout the 18th century, there was no significant change in this view, as we can also see in Bernoulli (1727), who postulated the Intermediate Value Theorem as a physical concept – and not a mathematical one.

But even in Leibniz's work, when the author related continuity to the loss of energy, the law of continuity became equivalent to the Intermediate Value Theorem, which for the first time placed this theorem as something fundamental – equivalent to a principle – for mathematics (cf. Schubring, 2004). Obviously, neither continuity nor the Intermediate Value Theorem could be proved.

Although in the 18th century mathematics began to take the form it has today – and mathematical treatises, even those on mechanics, began to contain fewer and fewer drawings and more and more formulas, functions, and series – the criteria for the convergence of series, essential for the definition of the limit, were not studied in depth. To explain concepts such as continuity, the infinitely large, and the infinitely small, mathematicians continued to use explanations that sometimes relied on analogies with nature and sometimes on metaphysical arguments (Bolzano, 1905; Boyer, 1949).

The contradictions in attempts to define the basic concepts of calculus remained unsolved in the 18th century, and only D'Alembert (1723-1790) attempted to define these concepts in terms of limits, but still with certain dubieties.

The practical consequence of this was that infinite series, which appeared more and more frequently in the solution of differential equations, led to completely absurd results, even in the hands of mathematicians of the caliber of Euler (1959), who lived between 1707 and 1783. On the other hand, in a work published in 1748, this scholar attempted to define the continuity and discontinuity of curves according to whether or not they were described by a single law of formation (Euler, 1983), which overall shows how far he was from the central issues for differential and integral calculus in the 19th century, namely the definition of continuity and convergence of series.

Continuity in the 19th century: arithmetization

Arithmetization is the process by which mathematics began to base its most important concepts on arithmetic notions, more specifically on the concept of real numbers. This process revolutionized mathematics by reversing the continuum-discrete relationship. If, until the eighteenth century, the number was seen either as the expression of continuum physical quantities or as a metaphysical principle whose validity was confirmed and legitimized by the workings of nature, with the advent of arithmetic, the number became the basis of the very concept of continuity, through numerical series and the definition of the notion of continuity in arithmetical terms, that is, numerical and static.

If the 17th and 18th centuries had been characterized by the search for new discoveries at the expense of the excessive rigor of the Greeks, the 19th century was a kind of return to rigor, but without the Greek scruples of using numbers and approximations between curves and lines, and with the new symbolism introduced in the previous two centuries.

Bolzano and Cauchy were the mathematicians of the first half of the 19th century who made the most progress in this direction. Bolzano formulated the need to rewrite the foundations of calculus on a **pure** and rigorous basis, but his work was not as well known or had the same impact as that of Cauchy, who developed similar principles in more areas of mathematics than Bolzano and whose work was immediately known by the greatest mathematicians of his time[5].

With the definition we present below – which would be formulated in a similar way by Cauchy a few years later and made even more analytical and rigorous by Weierstrass in the second half of the 19th century – continuity is now treated arithmetically and statically:

> the expression that a function f(x) varies according to the law of continuity for all values of x inside or outside certain limits means only this: if x is any such value, the difference f(x+ω) – f(x) can be made smaller than any given quantity by requiring only that ω can be made as small as we like. With the notation I introduced in Section 14 of the Binomischhe Lehrsatz etc. (Prague, 1816), this is f(x+ ω) = fx + Ω. (Bolzano, 2004, p. 256)

Bolzano also enunciated a convergence criterion for series – prior to Cauchy's – today known as Cauchy's Convergence Criterion and proved the existence of the greatest lower limit (or smallest upper limit) for limited sets, which we call today the least (and supremum). This was rewritten by Weierstrass as **every limited sequence (of real numbers, as we know today) has a convergent subsequence,** and since the end of the 19th century it has been called the Bolzano-Weierstrass Theorem.

A few years later, Cauchy (1821, p. 43) gave the following definition of continuity, very similar to that of Bolzano:

> Let f(x) be a function of the variable x, and suppose that for any value of x between two given bounds, this function always takes a single finite value. If, starting from a value of x between these limits, the variable x is given an infinitesimally small increase α, then the function itself will be increased by the difference f(x+α)-f(x), which depends on both the new variable α and the value of x. That is, the function f(x) will be, between the two limits assigned to the variable x, a continuum function of this variable if, for each value of x between the limits, the numerical value of the difference f(x+α)-f(x) decreases indefinitely with that of α. In other words, the function f(x) remains continuum with respect to x between the given limits if, between these limits, an infinitesimally small increase in the variable always produces an infinitesimally small increase in the function itself. It also says that the function f(x) is, in the neighborhood of a given value assigned to the variable x, a continuum function of that variable whenever it is continuum between two limits of x, even very close ones, which contain the value in question.

Weierstrass, in the second half of the 19th century, presented a definition – not in an article or treatise, but in a lecture note later published by several students who attended his

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[5] Clímaco (2014) and Grabiner (1981) explain Bolzano's isolation, partly because he lived in Bohemia, partly because he never held a chair of mathematics in any institution, partly because he had a scholastic style – quite different from that assumed by the great mathematicians of his time.

courses, and which is very similar to the one found in calculus textbooks today – in which he gives greater precision than the definitions given by Bolzano and Cauchy, and eliminates all reference to notions such as **becoming, approaching, decreasing**, among others, which refer to issues related to motion, infinitely small, or similar.

Here there is a certain inaccuracy in Boyer's statement (1949, p. 287): "Weierstrass defined f(x) to be continuum, within certain limits of x, if for any value x_0 such that for all values in this interval the difference $f(x)-f(x_0)$, in absolute values, is less than ε."

So let's go back to Dugac (1973, p. 64) who, based on the notes of Hermann Amandus Schwarz, talks about the course given by Weierstrass in the summer of 1965:

> By introducing the definition of the infinitely small variation of the variable and the function using δ and ε, Weierstrass introduces a very important notion that will give the definitions of limit and continuity all the precision and clarity they have today. Weierstrass thus gave form to the notion of limit, which until then, after a decisive step by Cauchy, had been expressed by the statement that if h tends to zero, $(x + h) - f(x)$ tends to zero. In fact, he gave the following definition "If it is possible to define a limit δ such that, for any value of h smaller in absolute value than δ, $f(x + h) - f(x)$ is smaller than any arbitrarily small value of ε, then we can say that we are matching an infinitely small variation of the variable with an infinitely small variation of the functions. This is the crucial step towards the current delimitation of the limit, which establishes a functional relationship between δ and ε, expressed by inequalities between the variables and between the values of the function. Moreover, the fact of replacing the intuitive idea of "tends to" with these inequalities leads to a precise analytical expression whose introduction into analysis will have a very great impact. Furthermore, it will replace sequences of points tending to a fixed point with neighborhoods of a point defined by inequalities, which will be one of the origins of general topology. The use of this definition in this course confirms the opinion of Рновити [[90], 25): "It seems that it was Karl Weierstrass who first introduced the notion of the limit of a function as precisely as possible. And the first handbook inspired by Weierstrass' ideas, published by Otto Stolz, gives the current definition of the limit, specifying what is difficult in Weierstrass (Volume 1, 990).

Although Otto Stolz does not define continuity in his work, by establishing the definition of limit using epsilon and delta, he laid the foundation for the definition of continuity found in current textbooks.

Having established the current notation with Weierstrass's work rewritten by Otto Stolz, which, as Circe (2021) said, reveals the intrinsic relationship between didactic needs and the new writing of mathematics in a **pure** way, we think it is important to reveal a part of the history that concerns continuity – more precisely, the relationship between the real numbers and the continuity of the line.

About 70 years after the publication of Bolzano's (2004) definition of continuity in his *Purely Analytic Proof...* – and 23 years after the course in which Weierstrass introduced the concept of the modern limit –, Dedekind (1963, pp. 11-12) stated that the correspondence between the line and the set of real numbers must simply be postulated in such a way that it is obvious to the subject, and thus there is no need to demonstrate this correspondence:

> I find the essence of continuity... in the following principle: If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.”As already said I think I shall not err in assuming that every one will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that by this commonplace remark the secret of continuity is to be revealed... I am glad if every one finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us... from filling up its gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle.

Similarly, the definition of irrational numbers by embedded intervals, elaborated in its current form by Hilbert and Kolmogorov – which we present in the way it was described by Richard Courant (1888-1972), who was Hilbert's student and colleague; and Robbins Herbert (1915-2001) by the statement that "an irrational point is completely described by a sequence of embedded rational intervals of length tending to zero" (Courant & Robbins, 2000, p. 82) – is also nothing more than a form of explicit postulation of something we deduce from the notion of spatiotemporal continuity. For, so conceived, "the existence on the number line (considered as a line) of a point contained in any sequence of nested intervals with rational endpoints" (p. 82) is considered as

> a fundamental postulate of geometry ... We accept it, as we accept other axioms or postulates in mathematics, because of its intuitive plausibility and its usefulness in constructing a consistent system of mathematical thought ... To construct this definition, after having been led to a sequence of embedded rational intervals by an intuitive feeling that the irrational point "exists," is to abandon the intuitive support with which our reasoning proceeded, and to understand that all mathematical properties of irrational points can be expressed as properties of sequences of embedded rational intervals. (Courant & Robbins, 2000, pp. 82-83)

To have a proper understanding of the significance of the profound transformations that took place in the foundations of calculus in the 19th century

Having established in the previous topic the paths followed by the notion of continuity, we will now discuss how the profound transformations that took place in the foundations of calculus – resulting from the correct definition and conception of the notion of continuity – between the 18th and 19th centuries should be properly understood.

The quotations from the eighteenth century made in the previous topic are, in our view, sufficient to show that Leibniz's conception of continuity was far removed from the arithmetical conception of the nineteenth century. This is confirmed by Schubring (2004, p. 176): he states that the authors who considered Leibniz's definition to be equivalent to that of the 19th century "were not aware of an essential conceptual difference ... while the modern concept concerns the continuity of *functions*, Leibniz's version concerns quantities of geometric variables". Thus, even if one can hardly disagree with Roque's (2012, pp. 404-405) statement that "the constitution of the concept of rigor now in force is linked to the history of the analysis of mathematics", his statement about eighteenth-century mathematicians that "we cannot say that their results lacked rigor, as if they had the goal of advancing without concern for the foundations of their methods" (pp. 406-407) should be discussed more carefully.

After all, recognizing the historicity of a certain phenomenon cannot mean abandoning the study of its fundamental characteristics and when they appeared in their most critical aspects.

In this sense, it seems extremely important to understand that the mathematicians of the 17th century began a conscious movement to abandon the excessive rigor of the Greeks in geometry in order to obtain new results, and that this disposition was largely continued by the mathematicians of the 18th century. This movement was associated with what Struik (1989, p. 21) called the "spirit of experimentation". Euler's attitude is an example of this trend. As stated by Boyer and Rusnock, quoted by Clímaco (2011, p. 117), in reference to the mathematicians of the 18th century,

> The objects of these mathematicians' greatest concern were the creation and development of mathematics, and not its foundation. ... If, on the one hand, the greatness of the advances made in mathematics in this century is unquestionable, on the other hand, Berkeley's questioning of the notions of infinity and infinitude remained without a satisfactory answer, and the contradictory nature of the explanations that mathematicians tried to give for these notions was highlighted by the Bishop's criticism, and also by the careless manipulation of infinite series, which led Euler to generalize certain results to the point of affirming equality $\frac{1}{1-x} = 1 + \mathrm{x} + x^2 + \cdots + \mathrm{x}^n + \ldots$ as true for arbitrary values, such as x =-1, x =1, x = 2, etc. (cf. (cf. RUSNOCK, 1997, p. 73).

> Thus, at the end of the 18th century, these concepts (infinity, infinities, and continuity) were explained either in metaphysical terms, or in terms of geometric explanations, or involving the notions of space and time (Boyer, 1949, p. 287).

From the historian's point of view, the key should not be the subjective question of whether Euler and others considered their work to be rigorous or not, but rather to understand the nature of the mathematical work they were doing, how much they devoted themselves to questions of foundations, the value they placed on these questions, and what they understood by rigor and foundations.

We have decided to use a long quotation from Grabiner (1981, pp. 1-2) – a well-known scholar of Cauchy and his rigorization movement – on the historical difference between the notion of rigor in the 18th century and that of the 19th century, which also emphasizes the real revolution that this change caused:

> These two different aspects-use and justificationof the calculus, simultaneously coexisting in the modern subject, are in fact the legacies of two different historical periods: the eighteenth and the nineteenth centuries. In the eighteenth century, analysts were engaged in exciting and fruitful discoveries about curves, infinite processes, and physical systems. The names we attach to important results in the calculus-Bernoulli's numbers, L'Hopital's rule, Taylor's series, Euler's gamma function, the Lagrange remainder, the Laplace transform-attest to the mathematical discoveries of eighteenth-century analysts. Though not indifferent to rigor, these researchers spent most of their effort developing and applying powerful methods, some of which they could not justify, to solve problems; they did not emphasize the mathematical importance of the foundations of the calculus and did not really see foundations as an important area of mathematical endeavor. By contrast, a major task for nineteenth-century analysts like Cauchy, Abel, Bolzano, and Weierstrass was to give rigorous definitions of the basic concepts and, even more important, rigorous proofs of the results of the calculus. Their proofs made precise the conditions under which the relations between the concepts of the calculus held. Indeed, nineteenth-century precision made possible the discovery and application of concepts like those of uniform convergence, uniform continuity, summability, and asymptotic expansions, which could neither be studied nor even expressed in the conceptual framework of eighteenth-century mathematics. The very names we use for some basic ideas in analysis reflect the achievements of nineteenth-century mathematicians in the foundations of analysis: Abel's convergence theorem, the Cauchy criterion, the Riemann integral, the Bolzano-Weierstrass theorem, the Dedekind cut. And the symbols of nineteenth-century rigor-the ubiquitous delta and epsilon-first appear in their accustomed logical roles in Cauchy's lectures on the calculus in 1823. Of course nineteenth-century analysis owed much to eighteenth-century analysis. But the nineteenth-century foundations of the calculus cannot be said to have grown naturally or automatically out of earlier views. Mathematics may often grow smoothly by the addition of methods, but it did not do so in this case. The conceptual difference between the eighteenth-century way of looking at and doing the calculus and nineteenth-century views was simply too great. It is this difference which justifies our claim that the change was a true scientific revolution, and which motivates the present inquiry into the causes of that change.

The words of Lützen (2003 quoted by Roque, 2012, p. 367) as supposed proof that "in more recent texts, however, we can already discern a certain awareness that the implicit concept of rigor in traditional narratives has a retrospective character" simply does not hold up, since neither Lützen gives any indication that the rigor of the 19th century was comparable to that of other centuries – or that it could be surpassed or relativized from one day to the next – nor has the research carried out in the last 20 years (from 2003 to 2023) indicated anything along the lines of what Roque claims, which confirms our thesis that the notion of rigor did indeed undergo profound transformations in the 19th century and that it was not a major concern in the 17th and 18th centuries.

To a certain extent, the mathematicians of the 18th century were aware of the lack of rigor in calculus, to the extent that none of them had yet managed to respond to the objections raised by Berkeley to the foundations of calculus in the same 18th century – nor had they responded in a reasoned way to Zeno's paradoxes. The mathematicians of the 17th and 18th centuries understood that their merit was to make mathematics grow. To do this, they avoided being paralyzed by logical reasons or even by inconsistencies in the foundations, and they believed that the progress of their mathematical work at that time did not depend on such rigor. To say that these mathematicians did not have the same concern or rigor as the mathematicians of the 19th century in no way diminishes their work, it simply helps us to understand the dynamics of growth and transformation of mathematics that are characteristic of each era. Replacing this analysis with the mere observation that all phenomena are historical does nothing to help us analyze and establish the fundamental characteristics of historical phenomena.

This rigor was only achieved when arithmetic was introduced: the foundation of calculus on the notion of limit, the rigorous study of the convergence of series, the definition of each real number by a numerical sequence, the assignment of each point on a straight line to a single real number, and so on. Only this development allowed mathematics to reach the generality it achieved in the 19th century, which gave new impetus to the other sciences and prepared for the great expansion and specialization of mathematics in the 20th century. Finally, with the transformation that took place in the nineteenth century, mathematics ceased to be a science of quantities, magnitudes, and calculations – in the sense of accounting – and began to focus on the notion of proving known results. Moreover, the creation of concepts by mathematicians to demonstrate these results – in particular the concepts of limit, continuity, derivative, and convergence – completely transformed mathematics, leading to what we know today as pure mathematics, something that Roque (2012) himself recognizes.

And the emergence of pure mathematics is not limited to arithmetic, nor is it an isolated phenomenon from what happened in other fields. As Giddens (1991, p. 39) notes, "it is characteristic of modernity ... to reflect on the nature of reflection itself.

Clímaco (2014, pp. 135-136) states that

> one of the most important characteristics of modern mathematics, which emerged in the 19th century, and of both tendencies that tried to define it, is its self-reflexivity, its meta-mathematical dimension. At the beginning of the 19th century, during the Second Industrial Revolution, mathematicians became aware of the need for self-reflection, to reflect on their own concepts. Meta-mathematics then emerged, a way of conceiving mathematics that outlined its own conception of mathematical logic, and with which it, which had always been characterized by quantities and magnitudes, lost this characteristic with the change and breadth of its own ideas, becoming a conceptual discipline.

Concepts such as functions, series, and derivatives – previously regarded as tools of the physical sciences – came to be regarded from the nineteenth century as objects proper to mathematics, to be studied separately or at least far removed from their use in other sciences or from intuitive notions.

From the point of view of the relationship between the continuum and the discrete, the progressive transformations in the notion of continuity in the 19th century led to a reconciliation between the discrete and the continuum: If in Ancient Greece qualitative aspects and a continuum, non-numerical approach to mathematics prevailed, and if the Hindus, Arabs, and Chinese gave priority to the numerical, and therefore discrete, aspects of this science, in the seventeenth century a reconciliation between the discrete and the continuum began, culminating in Dedekind's definition of number (1963, pp. 11-12), which solved, so to speak, what he calls "the mystery of continuity" – the mystery of that unintuitive mathematical concept, the number line, which first haunted the Greeks with the paradox of Achilles and the tortoise, but which even mathematicians like Bolzano and Cauchy could not fully understand.

Finally, we affirm that there is an important connection between the relationship between the discrete and the continuum and that between language and intuition. At the turn of the eighteenth and nineteenth centuries, continuity was transformed from something intuitive, dynamic, and specific to the physical sciences into a static concept written down in language. In this way, the continuum was explained in terms of the discrete. And the relationship between intuition and concept is fundamental to understand any educational debate, since it concerns the relationship between psychological and logical aspects, formal and social, objective and subjective, and so on.

Final considerations

Throughout the article, we have shown that the problem of the relationship between the notions of continuum and discrete goes back to antiquity, and already in the classical period of Ancient Greece, the tension over the nature of the relationship between these notions led to philosophical and mathematical debates that marked an impasse that was not resolved until many centuries later.

With the Scientific Revolution, there were significant changes in the understanding of the relation between continuum and discrete quantities, and in this context, there were significant developments in calculus, initiating a transformation in mathematics that was not completed until the 19th century, when there was a real revolution in the foundations of what is known as differential and integral calculus.

If we examine the historical relationship between the discrete and the continuum, we conclude that they are complementary, in the sense that they cannot be reduced to each other, since they are interrelated. An example of this is the calculus itself, which, although it deals mainly with continuum quantities, could only be fundamental with the use of discrete concepts, such as the notion of limit and the sum of series, so that in order to have a global understanding of its historical development, it is necessary to approach it from the point of view of complementarity, developed above all by Otte (1994, 2003) – and systematized by Clímaco et al. (2024).

Today, this relationship is present in various areas of applied mathematics. In the theory of digital representations, for example, the ability to represent the real line, which is continuum, in discrete computer systems is extremely important, which shows how important the complementarity approach is in making it possible to analyze and understand various phenomena more accurately. However, we recognize that reality is always more complex than we can represent it; some phenomena are better understood from a continuum point of view, while others are better understood from a discrete one. Then there are those more specific phenomena that may not fit perfectly into either perspective or those that reconcile the two perspectives simultaneously – so a complementary approach is fundamental to understanding phenomena in their entirety.

From an educational point of view and in the schooling process, understanding these aspects is important because it provides those involved in the educational process with a more comprehensive, solid and in-depth view of mathematical concepts: numbering systems, graphical representation of functions and the notion of limit, which makes it possible to strengthen learners' ability to reason mathematically – so that they are better able to look at mathematical problems from a broader perspective and to transpose concepts into different forms and contexts of representation.

Understanding the history of the notion of continuity, as well as the complementarity of the discrete and the continuum, is fundamental for the professor to be able to use learning strategies that allow students to explore both continuum and discrete aspects of the mathematical concepts being taught, and to make connections between mathematics and its practical application.

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