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The insertion of the language dimension in the analysis of the teaching problem

La inclusión de la dimensión lingüística em el análisis del problema didáctico

L'inclusion de la dimension linguistique dans l'analyse de la problématique didactique

A inserção da dimensão da linguagem para a análise do problema didático

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Abstract

The purpose of this article is to insert the language dimension into the analysis of the didactic problem with the same degree of relevance attributed to the epistemological, ecological and economic dimensions when studying a mathematical object. We planned a meeting with twelve Engineering and Mathematics students from two public universities in the interior of Bahia and proposed a task with the theme Double Integral of a Quadric Surface, the hyperbolic paraboloid. As theoretical assumptions we take as a contribution the Anthropological Theory of Didactics (TAD) and Peircean Semiotics and the construct elaborated by the authors of the four languages: counterfactual, dictated, in course, in (dis) course. Five of the students solved the task, three from Engineering, who created a question in the context of their degree, the other two, from the Mathematics Degree, focused on developing calculations. The results obstained revealed that future engineers strongly applied counterfactual and (dis)course language, while those in the bacherlor's degree used dictated and in course language; the twelve students experienced difficulties in co-authoring the elaboration, solution and analysis of the activities,

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as they did not find similar models in textbooks and the media. We infer that the language dimension was relevant for the development of abductive reasoning and for the student's co-authorship in the creation and solution of Double Integral question statements.

Keywords: Language dimension, Didactic problem, Double integral.

Resumen

El propósito de este artículo es insertar la dimensión del lenguaje en el análisis del problema didáctico con el mismo grado de relevancia que se atribuye a las dimensiones epistemológica, ecológica y económica al estudiar un objeto matemático. Planificamos un encuentro con doce estudiantes de Ingeniería y Matemáticas de dos universidades públicas del interior de Bahía y propusimos una tarea con el tema Doble Integral, para calcular la medida del volumen de una Superficie Cuádrica, el paraboloide hiperbólico. Como presupuestos teóricos tomamos como aporte la Teoría Antropológica de la Didáctica (TAD) y la Semiótica peirceana y el constructo elaborado por los autores de los cuatro lenguajes: contrafactual, dictado, en curso, en (dis)curso. Cinco de los estudiantes resolvieron la tarea, tres de Ingeniería, quienes crearon una pregunta en el contexto de su carrera, los otros dos, de la Licenciatura en Matemáticas, se enfocaron en desarrollar cálculos. Los resultados obtenidos revelaron que los futuros ingenieros aplicaron fuertemente el lenguaje contrafáctico y (dis)curso, mientras que los de Licenciatura utilizaron un lenguaje dictado y continuo; los doce estudiantes experimentaron dificultades en la coautoría de la elaboración, solución y análisis de las actividades, al no encontrar modelos similares en los libros de texto y los medios de comunicación. Inferimos que la dimensión del lenguaje fue relevante para el desarrollo del razonamiento abductivo y para la coautoría de los estudiantes en la creación y solución de enunciados interrogativos Doble Integral.

Palabras clave: Dimensión del lenguaje, Problema didáctico, Doble integral.

Résumé

Le but de cet article est d'insérer la dimension linguistique dans l'analyse du problème didactique avec le même degré de pertinence attribué aux dimensions épistémologique, écologique et économique lors de l'étude d'um objet mathématique. Nous avons prévu une reencontre avec douze étudiants em ingénierie et mathématiques de deux universités publiques de l'intérieur de Bahia et avons proposé une tâche sur le thème Double Intégrale, pour calculer la mensure du volume d'une Surface Quadrique, le paraboloïde hyperbolique. Comme hypothèses théoriques, nous prenons comme contribution la Théorie Anthropologique de la Didactique (TAD) et la Sémiotique Peircéenne et la construction élaborée par les auteurs des quatre langages : contrefactuel, dicté, em cours, em (dis)cours. Cinq des étudiantes ont résolu le problème, trois em ingénierie, qui ont créé une question dans le cadre de leur diplôme, les deux autres, en mathématiques, se sont concentrés sur le développement de calculs. Les résultats obtenus ont révélé que les futurs ingénieurs appliquaient fortement le langage contrefactuel et de (dis)cours, tandis que ceux du bachelor utilisaient un langage dicté et continu; les douze étudiants ont éprouvé des difficultés à corédiger l’élaboration, la solution et l’analyse des activités, car ils n’ont pas trouvé de modèles similaires dans les manuels scolaires et dans les médias. Nous em déduisons que la dimension linguistique était pertinente pour le développement du raisonnement abductif et pour la co-auteure des étudiants dans la création et la solution des énoncés de questions à double intégrale.

Mots-clés : Dimension linguistique, problème didactique, Double intégrale.

Resumo

O intuito deste artigo é inserir a dimensão da linguagem na análise do problema didático com o mesmo grau de relevância atribuído às dimensões epistemológica, ecológica e econômica ao estudar um objeto matemático. Delineamos um encontro com doze estudantes de Engenharias e de Licenciatura em Matemática de duas universidades públicas do interior da Bahia e propomos uma tarefa com o tema Integral Dupla, para o cálculo da medida do volume de uma Superfície Quádrica, o paraboloide hiperbólico. Como pressupostos teóricos tomamos como aporte a Teoria Antropológica do Didático (TAD) e a Semiótica Peirceana e o constructo elaborado pelos autores das quatro linguagens: contrafactual, dictarizada, em curso, em (dis)curso. Cinco dos estudantes resolveram a tarefa, três das Engenharias, que criaram uma questão no contexto de sua graduação, os outros dois, da Licenciatura em Matemática, se atentaram ao desenvolvimento de cálculos. Os resultados obtidos revelaram que os futuros engenheiros aplicaram de forma acentuada a linguagem contrafactual e em (dis)curso, enquanto, os da Licenciatura, a linguagem dictarizada e em curso; os dozes estudantes sentiram dificuldades na coautoria da elaboração, solução e análise das atividades, por não encontrarem modelos semelhantes nos livros didáticos e nas mídias. Inferimos que a dimensão da linguagem foi relevante para o desenvolvimento do raciocínio abdutivo e para a coautoria dos estudantes quanto a criação e solução dos enunciados de questões de Integral Dupla.

Palavras-chaves: Dimensão da linguagem, Problema didático, Integral dupla.

A inserção da dimensão da linguagem para a análise do problema didático.

Discussing language in a world that represents it in an expanded form and different shades seems elementary to us due to the numerous publications on this topic. However, it is still possible to detect that, in specific areas of knowledge, there are gaps or little-explored space on the power it exercises in human relationships. One of them is the area of science "conceived" as exact, specifically language in mathematics.

Addressing this theme leads us to understand how language is discussed in educational spaces and how it influences the teaching process of a particular mathematical object. The relationships in the didactic triangle proposed by Brousseau (1996), composed of three elements –the teacher, the student and specific knowledge (mathematical object)– do not occur peacefully since the interactions established between the teacher and the student, mediated by an object, go through subjective regulation processes, which can be decisive for the epistemological understanding of knowledge.

This connection is not always transparent, and the lack of assimilation of some nuances of the mathematical object leads to misinterpretations, such as, for example, the algebraic representation of functions defined by f(x) = cos (3x) e g(x) = 3cos(x). Although involving the same mathematical object, it shows differences in the domain and range of the functions, and when explained in high school, teachers often use graphic representations.

However, suppose we associate this issue with the study of linear algebra. In that case, we find that the mathematical object, the cosine function, is not a linear transformation because it does not satisfy the conditions that consider V and W, as vector spaces and a map T: V $\rightarrow W$ is a linear transformation if: (i)T(u + v) = T(u) + T(v), with $\forall u, v \epsilon V$ and V is a vector space; (ii) T(ku) = kT(u), with k ϵR.

Let us study the problem situation of the cosine function as a mathematical praxeology[4]. We can consider task type T and prove that $\cos(kx) \neq k\cos x$ (Table 1) to validate the hypothesis that the functions f(x) and g(x) are different.

[4]The mathematical praxeologies or organisations are theoretical constructs of the ATD that centralise the study in mathematical activities and analysing the tasks. Techniques are the procedures or algorithms used to solve them; technologies are theorems, axioms, and mathematical definitions and theory is the "branch" of mathematics in which the task is situated.

Table 1.

Praxeologies of the cosine function through linear algebra

Tasks	Technique	Technology	Theory
T: Prove that cos (3x) $\neq$ 3 cosx	Considering the two members of the inequality as functions and the second condition for linear transformations: $T(kx) = kT(x)$ with $k\ e\ x\ \in \mathbb{R}$ we get $\cos(3x) = 3\cos(x)$ equality Doing 3x as the sum 3x = 2x + x we have $\cos(2u+u) \overset{?}{=} 3\cos(u)$. equality [a] Applying the technique for the sum of two arcs of the cosine function, we have $\cos 2u \cos(u) - sen2u\ sen\ (u)$ from which the inequality cos(2u)cos(u) – sen(2u)sen(u)$\neq$ 3 cos (u)	θ_1: Linear transformation θ_2:Trigonometric functions	Θ_1: Linear algebra Θ_2: Trigonometry

Note: [a]. We are using this notation because the equality is being hypothesised to verify its truth or not.

From this example, we verify that the cosine function object can be justified by techniques, technologies, and theories unrelated to the student's level of education. To give students a coherent explanation, the teacher (or the textbook) suppresses knowledge not foreseen for that grade, causing gaps in their understanding of the object. In this sense, Bosch, Fonseca, and Gascón (2004) point out:

> The importance of the institutional restrictions that weigh on school mathematics activity and that lead us to situate the incompleteness of secondary school mathematics organisations and the origin of didactic discontinuities between secondary school and university.

This statement refers to Brazilian education, which lacks articulation between levels – high school and higher education– in teacher education. Didactic discontinuity is a theme emptied of discussions in institutions; often, the teacher is not aware of or does not question the knowledge of the textbook, reproducing it without relating it to its origin.

We analysed the topic "calculating the area of a circle" in two textbooks adopted in Brazilian high schools. Table 2 presents the algebraic representation the authors chose, devoid of explanation about its origin and articulation with other pedagogical levels. The above made us question: How does this mathematical object "live" in a higher education institution (HE)?

Table 2.

Reference of the analysed book and the theme

Bibliographic Reference	**Exposition of the theme**
DANTE, L.R. (2000). *Matemática: contextos & aplicações*. (2ªed. Vol.2.) [Mathematics: contexts & applications. (2nd ed., Vol. 2)] Editora Ática. p.264	[...] As the area of the region limited by a regular polygon is given by the product of the half perimeter by the apothem (A = pa), then the area of the circle is: $A = \frac{1}{2}(2\pi R)R \Rightarrow A = \pi R^2$
PAIVA, M.R. (1995). *Matemática*. (Vol. 2) [Mathematics, Vol. 2], Moderna, p.483.	The area of a circle of radius r is equal to πr^2

To answer this question, we can relate measuring the area of the circle to the study of multiple integrals, more specifically, double integrals. Suppose we assume that the circle establishes a relation with two variables, the angle and the radius, and take a general function of two variables defined by $f(r, \theta) = 1$, where r represents the radius, and θ an angle between 0º e 360º. We can obtain the measure of the area of the circle using a double integral in polar coordinates:

$$A = \iint_D f(r, \theta) r dr d\theta.$$

Table 3 describes the mathematical praxeology for calculating area measurements using a double integral.

Table 3.

Mathematical praxeologies for the calculation of the double integral

Tasks	Technique	Technology	Theory
T: Calculate the area of a circle.	t_1: Establish the domain of integration from the radius and the angle expressed by D = $\{0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi\}$. t_2: Replace the domain in the double integral, A = $\int_0^{2\pi}\int_0^R 1rdrd\theta$ t_3: Integrate with respect to the radius r: A = $\int_0^{2\pi}\frac{r^2}{2}\Big\|_0^R d\theta$ t_4: Apply integral properties to a constant: A= $\frac{1}{2}\int_0^{2\pi} r^2\|_0^R\ d\theta$ t_5: Apply the fundamental theorem of calculus: A = $\frac{1}{2}\int_0^{2\pi}(R^2-0^2)d\theta$ t_6: Consider R^2 is a constant, since the integration will be related to the angle θ. A = $\frac{R^2}{2}\int_0^{2\pi}d\theta$ t_7: Integrate with respect to the θ angle A = $\frac{R^2}{2}.\theta\|_0^{2\pi}$ t_8: Apply the TFCT[a]: A = $\frac{R^2}{2}(2\pi-0)$ t_9: Apply to distributive: A = πR^2	θ_1: Polar coordinates θ_2: Fundamental theorem of calculus θ_3: Simple integrals θ_4: Multiple integrals	Θ_1: Differentia l and integral calculus

Note: [a] Fundamental theorem of calculus (FTC)

The articulation between the same mathematical object using different techniques and technologies allows us to know why that object exists and its origin and favours the study of ecological and economic dimensions. It allows us to detect how a specific mathematical object "lives" in different institutions and which laws, rules, and documents guarantee the existence and dissemination of that knowledge.

The incompleteness of school mathematical organisations, highlighted in the previous examples, causes obstacles in understanding knowledge. For example, mathematical language signs are not interpreted and/or represented correctly and may not produce meaning for the teacher and the student.

A critical analysis of the school curriculum and the didactic and mathematical praxeologies in textbooks at institutions allows for identifying the presence of a "monumentalist" paradigm (Chevallard, 2005) in which the student "visits the work"

(Chevallard, 2012) to apply techniques and solve activities without constructing meaning. If we had teaching geared towards research and the construction of knowledge, the articulations we emphasise in the examples could make emerge, enrich the debate and learning, sharpen curiosity, and develop new ideas or knowledge.

Chevallard (2012) points out that the change to the paradigm of questioning the world favours the student to ask questions and try to answer them (mathematically or outside mathematics) using mathematical praxeologies (tasks, techniques, technology, and theory) as tools, expressed in different types of language: written, oral, gestural, imagery, pictorial, among others.

A mathematical object is under intense interpretations, in which the mathematical language adapts to the contexts in which it operates. It starts with the creation of a structure validated by the mathematicians until it gets to the classroom. During this process, the epistemological, economic, and ecological dimensions (Gascón, 2011) directly affect the didactic system (teacher, student, and object). The epistemological dimension studies the mathematical object in its historical development (Almouloud, 2022) and its different forms of conception and approach. The economic dimension encompasses the institutional context, laws, curricula, and adopted textbooks, and the ecological dimension studies the conditions and "transpositive restrictions" (Chevallard, 1991) to which mathematical and didactic organisations are submitted at different stages.

However, there are intrinsically some dimensions called by Gascón (2011) "secondary –for example, the cognitive, personal, ostensive, instrumental dimensions, etc.", in which "the subsequent development of the investigation can originate the formulation and study of other aspects in the other dimensions or their connection with other didactic problems" (ibid, 2011).

The possibility of the existence of secondary dimensions motivated us to add the language dimension, bearing in mind that the registration of a mathematical object occurs through representations, symbols, techniques, technologies, and specific theories that provide uniqueness to the scientific field and when expanding, original and internal signs to mathematics itself are inserted. However, mathematics does not live in isolation and humans, as social beings, mobilise different knowledge to survive and communicate. In this sense, the mathematical language circumscribes areas of diverse knowledge, as in engineering and art.

The significance of the language dimension in the insertion and circumcision movements contributes to the rupture proposed by the questioning-the-world paradigm and to the creative and original formulation of open-ended questions within the framework of the study and research path (SRP). Such an approach favours the movement at the boundary between mathematical and non-mathematical language. In this sense, “the border is defined, then, as a mechanism of semiotisation capable of translating external messages into internal language, turning (non-text) information into a text” (Machado, 2003).

The purpose of what follows is to present the language dimension as a theoretical construct of analysis for the didactic problem, placing it as a fundamental dimension with the same degree of relevance attributed to the epistemological, ecological, and economic dimensions.

Language dimension

The didactic problem is a set of questions asked to the system –teacher, student, knowledge– for teaching a mathematical object. When looking for answers to such questions, it is necessary to verify the conditions and restrictions for the didactic and mathematical organisations and the production of meanings to the several types of knowledge that characterise the epistemological, economic, and ecological dimensions (Gascón, 2011) of the knowledge for teaching.

Such intent requires a system of references that registers the characteristics of mathematical objects with a specific language (symbols, representations), making them manipulative and visible and communicating what the mind wants to express. The ostensive objects (representations, symbols) access a theorem, a definition, and an axiom (non-ostensive objects).

Studying the dimension of language with the degree of relevance it deserves is fundamental to understanding the agreements (suppression and inclusion of terms) not always made explicit in the three dimensions defended by Gascón (2011), such as, for example, the students’ abductive reasoning and the teachers, who lead them to produce something different from what is set. Reasoning as a way of organising thought was defended by Aristotle (2016), who typified it as deductive, inductive, and abductive. Later, Peirce (2005) defined abductive reasoning as spontaneous insights free of judgments and, therefore, original.

In ATD, this reasoning is found in the development of the SRP pedagogy and in the didactic paradigm of questioning the world, as they are based on promoting inquiries around an original question, which may or may not have roots in the scope of mathematics. However, the interpretation, meaning, or representation process is the basis of semiotics, which is concerned with studying language. Santaella (2009) states that “semiotics is the science whose object of investigation is all possible languages, that is, which aims to examine the modes of the constitution of every phenomenon as a phenomenon of production of meaning and sense”.

When the proposed open-ended question is in the exclusive field of mathematics, there is a demand for a specific language, taken by appropriate symbols, representations, techniques, technologies, and theories that are topologically situated within mathematics itself. However, if the question acquires an amplitude external to mathematics, with a problematic of the world in which the inquiries lead to the mobilisation of several scientific objects and from different areas, the language exceeds the mathematical limit.

In this sense, the dimension of language is simultaneously an external and internal field to the didactic problem; it is a “boundary dimension” in which the movements are dynamic, in expansion or in a zone of interrelationships between different means. The language present in the teaching of a scientific field acquires a profile of its own or goes beyond the walls of fragmented knowledge, depending on the intentionality of teaching, didactic organisations, and organisations of specific objects in each area.

Brandão (2021) defends four types of mathematical language that circumscribe and inscribe a didactic problem: dictated, counterfactual, in course and in-(dis)course. For the author, the three components of the didactic system mobilise these languages, depending on the institutionalised didactic problem. The flow between them creates a dependency relationship with one of the paradigms (“visits to works” or questioning the world), resorts to didactic and mathematical organisations, and requests actions for the solutions raised in the problem.

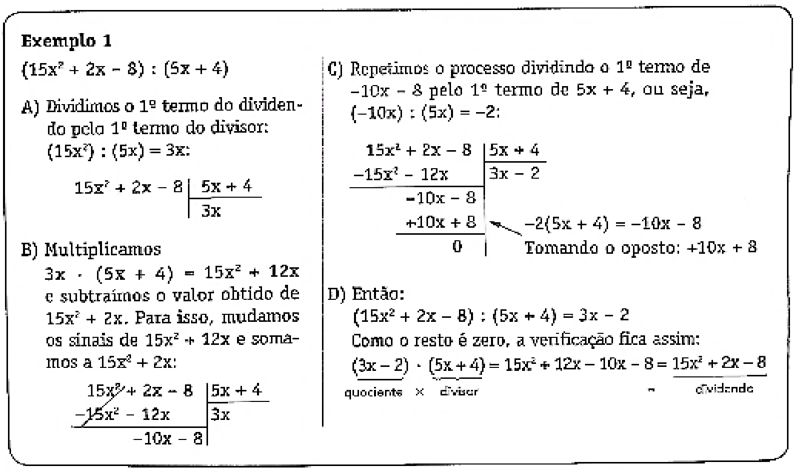
To Brandão (2021), the dictated language is the specific language of mathematics, composed of techniques, technologies, and theories of that field of knowledge. They are symbols, rules, and procedures that are part of the scope of the mathematical canon. Theorems, definitions, proofs, axioms and representations are essential for sharing the mathematical language. Because it is a specific language of knowledge, there is a predominance in its teaching. There are peculiarities inherent to mathematical objects that can cause conflicts in their interpretation and epistemological errors because “in each context, the rule can acquire a different meaning and the student, who should apply the same rule, applies another one in its place” (Silveira, 2015).

This situation can be verified when, in Elementary School II (middle school), students are taught to divide two polynomials through techniques appropriate for learning. However, when this division appears in higher education (HE) in a specific topic, such as a double integral of functions of several variables, in which they should apply the same rule, many students do not associate what they learned in middle school with a problem situation in HE.

Table 4 shows how the division of polynomials is presented in a specific context of middle school and a situation of HE, sometimes as a technology and sometimes as a technique.

Table 4.

Comparison between polynomials in the textbook (Dante, 2008) and a protocol of an engineering student

Division of polynomials as a technology in a textbook
Polynomial division by polynomial Please review the examples carefully. Together with a colleague, try to justify each step of the resolution. We will use the key process described in the table below.  **Exemplo 1** $(15x^2 + 2x - 8) : (5x + 4)$ A) Dividimos o 1º termo do dividendo pelo 1º termo do divisor: $(15x^2) : (5x) = 3x$: $\begin{array}{r\|l} 15x^2 + 2x - 8 & 5x + 4 \\ \hline & 3x \end{array}$ B) Multiplicamos $3x \cdot (5x + 4) = 15x^2 + 12x$ e subtraímos o valor obtido de $15x^2 + 2x$. Para isso, mudamos os sinais de $15x^2 + 12x$ e somamos a $15x^2 + 2x$: $\begin{array}{r\|l} \cancel{15x^2} + 2x - 8 & 5x + 4 \\ \cancel{-15x^2} - 12x & 3x \\ \hline -10x - 8 & \end{array}$ C) Repetimos o processo dividindo o 1º termo de $-10x - 8$ pelo 1º termo de $5x + 4$, ou seja, $(-10x) : (5x) = -2$: $\begin{array}{r\|l} 15x^2 + 2x - 8 & 5x + 4 \\ -15x^2 - 12x & 3x - 2 \\ \hline -10x - 8 & \\ +10x + 8 & \\ \hline 0 & \end{array}$ $-2(5x + 4) = -10x - 8$ Tomando o oposto: $+10x + 8$ D) Então: $(15x^2 + 2x - 8) : (5x + 4) = 3x - 2$ Como o resto é zero, a verificação fica assim: $\underbrace{(3x - 2)}_{\text{quociente}} \cdot \underbrace{(5x + 4)}_{\text{divisor}} = 15x^2 + 12x - 10x - 8 = \underbrace{15x^2 + 2x - 8}_{\text{dividendo}}$
Division of polynomials as a technique in the double integral
$= \int_0^{\pi}\int_0^{1} \frac{-4r^2}{1+r^2}\, dr\, d\theta -$

The first record shows an excerpt from an 8th-grade textbook when the division of polynomials is presented as a technology to be apprehended. The algorithm leads to a *key process* method similar to the integer division technique. The second consists of a protocol by a civil engineering student who, when solving a double integral, needs to divide polynomials by a technique and does not do it, claiming "not to remember the procedure.[5]"

This problem is specific to the dictated language in which the syntax, semantics, and pragmatism (Machado, 1993) of the mathematical object reveal difficulties inherent to this

[5]The speech recorded here occurred during the CDI-III assessment carried out on May 5, 2023.

knowledge. When we refer to mathematical syntax, we mean how the object is expressed; semantics, the meaning attributed to the context and pragmatism to the use in a specific situation. Thus, for the woven argument, it is possible to identify syntactic modifications, as there are differences in how the division appears in the student's protocol and the textbook and pragmatic when used in different situations (one as technology and the other as technique).

Three types of reasoning are used. Initially, the middle school student processes semiosis to learn the object, a new idea presents as abductive reasoning; then, they use the rules of dictated language to deduce the techniques and technologies to be applied. However, when the student is in HE and must mobilise inductive reasoning to move from the particular (the middle school division of polynomials) to the general (the technique needed to solve a double integral), they cannot identify the necessary rules to carry out the division in this context.

Brandão (2021) warns about society's use of dictated language and its use in the classroom because it may communicate or disseminate exclusionary, elitist, and power discourses, with harmful consequences for learning mathematical concepts. The excessive use of that language has "inhibited" the development of other mathematics and/or the discovery of diversified methods of teaching and learning the subject, restricted the dissemination and expansion of the application of its mathematical objects in the scientific evolution (or in everyday life) that goes beyond the school space, and imposed the equability of teaching for mathematicians and non-mathematicians.

We do not defend the exclusion of dictated language because this would deconstitute mathematical science as a body of knowledge. We infer the need to expand the mathematical language to a world of intense changes, in which we require numerical skills that go beyond memorising formulas and mathematical concepts in situations different from when they were created. For Brandão (2021, p.99), counterfactual language:

> [...] has no rules, is unconventional, it just imprints an original quality, an intuition. To characterise it, we borrowed from the Hindu expression: "seeing with the third eye", which conveys the human ability to go beyond physical reality or to see beyond appearances. [...]. The counterfactual language goes through intense transformations; it is self-regulating, self-correcting and, therefore, scientific. [...]. It is circumstantial when the perception of phenomena requires postures, perceptions, and transforming movements, adding original knowledge emerging from other ideas and techniques.

This language arises when mathematicians, artists, and scientists develop new ideas, concepts, and experiences in which creativity is revealed in a way different from what is set institutionally. The student who takes a different path from the teacher to solve a task develops counterfactual language, free of impositions, to open paths for discovery. In this sense, Brandão (2021, p.98) asserts that:

> [...] is not verifiable and concrete facts, but something virtually existing in the human mind; a set of speculations that emerge from a web of irrational relationships, presented as reasoning without nexuses, without concern with the certainty of speculative fiction. That is why they are free and internal to each human being; they are symbolic organisations of the subject, weaving a mesh of relationships, interpretations, meanings, and self-representations with the world.

When the author claims it to be a self-regulating and self-correcting language, she relativises the value judgment in which right or wrong is attributed, allowing the excluded third to exist. There may be a transition phase between dictated and counterfactual language in a continuous flow, with no fixed direction, in which the two transit in a complementary way. Brandão (2021, p.98) explains:

> The two languages –the dictated and the counterfactual– are tools with different purposes: the first, the goal is to externalise and apply the rules, deductions, and inductions recognised by the mathematical community expressed in mathematical and didactic praxeologies. The author takes the instrument as a symbol that serves as a convention to prove something, a habit. The second is a tool that "makes us see" (Herrero, 1988, p.31), contemplate, reflect, create.

Faced with this distinction, we realise that the counterfactual language proposes a "new order" to mathematics, in which creativity guides the paths and "the illuminating insight tends to be seen as a threat of disorder or destabilisation, before being recognised as a contribution, valid in the sense of growth of the singular plurality of men" (Vergani, 2009, p. 180). We can cite a mathematical example when numerical sets expand with the introduction of complex numbers.

According to Brandão (2021), two other languages are constrained to the classroom: in-course and in-(dis)course language. The in-course language is presented in teachers' elaboration on mathematical and didactic organisations to explain a specific subject and when putting into action what they proposed to develop their in-person exposition in a class. This new text, in which teachers write or orally explain their interpretation, is what Chevallard (1991) calls "metatext", which is more difficult to exemplify, as it occurs in the "act of speech" (Austin, 1990) or in the notes made by each teacher. The in-course language is based on the repetition of textbooks and teaching methodologies based on the "visit to works" paradigm.

The author states that the in-(dis)course language is, by default, the in-course language because it is through a broader and deeper knowledge of mathematical organisations that we can establish didactic organisations that enable.

> […] dealing with the same subject from different points of view. [...] takes the concept of relativity of knowledge in the philosophical sense, in which it attributes to knowledge a necessarily limited character as it depends on variable factors, such as the particularity of the subject, the context that is produced, among others. (Brandão, p. 109)

Each of the four languages is made up of knowledge that connects them. Briefly, counterfactual language is linked to abductive knowledge, which plays a fundamental role in sensitive perception, attributed to a transcendental vision that emerges from the creative process in the elaboration of new mathematical and didactic praxeologies in an unconventional and non-standardised way to perceive the mathematical objects. Concerning the dictated language, we opted for two types of knowledge: the first, arising from the epistemological construction of the mathematician and the second, the social, cultural, and historical knowledge that participates in this construction with the posed problems.

For the in-course language, we list two types of knowledge: one, the institutional knowledge originating from textbooks, curricula, and the noosphere that influences the interpretations of the mathematical object and the other, the knowledge taught by the teacher, who carries out the internal didactic transposition and imprints own interpretations on the object. Finally, language in (dis)course establishes the knowledge of subjective and pragmatic interpretations to build an individual interpretation and criticism, which stands before power relations embedded in discourses and institutions. Figure 1 summarises the four types of languages and some of the multiple knowledge that circumscribes them.

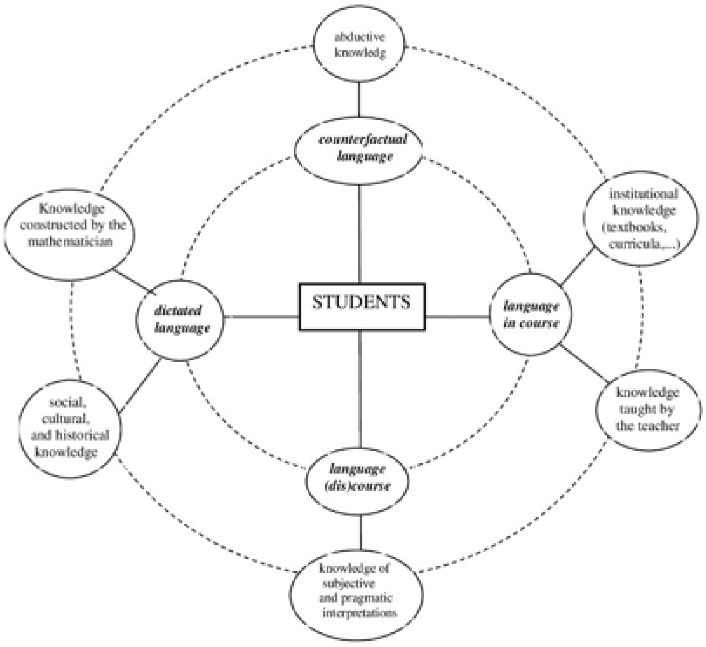


Figure 1.

Overview of the four types of languages and related knowledge (Brandão, 2021)

The knowledge in Figure 1 is personal, cultural, historical, social, mathematical, and abductive, and is involved in students' formative process. In classroom action, the teacher will seek a balance between the four languages. However, Brandão (2021) clarifies that, among the four types of language, there is an inverse proportionality developed in contexts of use that can be analysed in two directions: vertically, the more dictated language is applied, the less counterfactual language is produced; similarly, the more in-(dis)course, the less in-course language. Horizontally, the less dictated, the more in (dis)course language, and the less counterfactual, the more in-course language, as seen in Figure 2.

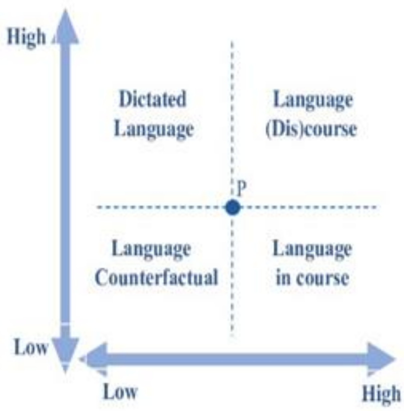


Figure 2.

Proportional movement of the four languages (Brandão, 2021, p.117)

The balance point as the ideal way to use the four languages is marked by point P in Figure 2, where the proportionality levels reach an ideal overlapping level for mathematics teaching and learning. However, by adding a dimension to the didactic problem, some questions arising from the four languages need to be considered as guides for observing the phenomena that emerge in class during the performance of tasks and institutionalisation of concepts. Therefore, we conceived some questions for each language in Table 5.

Table 5.

Overview of the questions formulated for the didactic problem under the language aspect (Brandão, 2021)

Language Type	Characteristics	Asked Questions
Dictated Language	• Symbolic • Rules and norms • Deductive and inductive reasoning	1. Which expressions or words are directly related to the mathematical object under study? 2. What meanings do they generate that facilitate/complicate the understanding of the epistemological object? 3. What is the relevance of mathematical symbols for understanding the mathematical object? 4. How and when to apply a mathematical rule to different tasks?
Counterfactual Language	• Abductive reasoning • Perceptual sensitivity • Creativity	1. What is the collective meaning of teaching a particular mathematical object? 2. From which world is the *raison d'etre* of the claimed mathematical object taken? 3. What can I create with this mathematical object? 4. What already exists in the world that I can use to teach this mathematical object? 5. What records allow me to extrapolate the limited view that the "I" has of mathematics?
Language in Course	• Symbolic interpretations • Internal use context for mathematics	1. What variations of the mathematical object in contexts of use in mathematics can be verified? 2. How are variations in the rules involving the mathematical object under study interpreted?
Language in (dis)course	• Subjective and intersubjective interpretations of the student and teacher • Criticism of power relations • Proposes new mathematical and didactic praxeologies	1. What possible interpretations can emerge from the mathematical object on the part of students and teachers? 2. What activities can be constructed that enable discussion and viable actions to interpret the mathematical object? 3. How can the subjected subject criticise? (Araújo, 2004, p.243); 4. How important is this mathematical object to society? 5. What meanings can be produced from the context of use of this mathematical object? 6. In which "living world" the students to whom I am going to present the mathematical object are inserted? 7. How can the mathematical object contribute to the social development of the "living world" to which students belong?

We emphasise that the action in the classroom, promoted by dialogue between the agents of the didactic system, will enable the rupture of the monumentalist paradigm. In this movement, we must propose tasks that allow students to use abductive reasoning to create questions and seek different answers, which reminded us of an advertisement about science, presenting Einstein's (1955) statement that "it's not the answers that move the world, it's the questions".

Thus, we have developed some tasks in which the four languages can be organised to implement the theoretical constructs created. We also intend to show that the dimensions of the didactic problem have language as a crossing point capable of substantiating all others. In the next section, we will indicate the methodological processes chosen for developing actions that were on the boundaries of the four languages.

Research scenario and participants[6]

This research is qualitative, and the interpretations come from those who attribute meaning to the data collected. For Creswell (2010), qualitative research is "the research process that involves the questions and procedures that emerge, the data typically collected in the participant's environment, the data analysis honours an inductive style and the interpretations made by the researcher about the meaning of the data". The meetings were recorded in audio and video for the reliable collection of discussions held during the proposed activities.

We had the participation of twelve students, six from engineering courses and six from mathematics degree courses from two public institutions in a city in the countryside of Bahia, in an extracurricular face-to-face course called: "Se integre duplamente às superfícies quádricas" [Double integrate over the quadric surfaces], which took place in five meetings on Saturdays, two of them in two shifts, morning and afternoon and the others, only in the morning.

One of the criteria for selecting participants was to have completed the component Differential and Integral Calculus III (the syllabus includes double integrals as one of the topics to be studied), as we aimed to verify whether students produced meanings from previous knowledge and related them to other contexts not covered in textbooks.

[6]The research participants signed the Free and Informed Consent Form (FICF), as this research is part of a larger project by the research group "Processo *de Ensino e Aprendizagem em Matemática – PEAMAT"* [Process of Teaching and Learning in Mathematics] of the Postgraduate Studies Program in Mathematics Education at PUC-SP, which was submitted to the PUC-SP Ethics Committee. PEAMAT. Therefore, the authors of this article explicitly exempt Acta Scientiae from any consequences arising therefrom, including full assistance and possible compensation for any damage resulting to any of the research participants, in accordance with Resolution n. 510, of April 7, 2016, of the National Health Council of Brazil.

To do this, we delivered an original, written task on A4 paper, which excelled by statements different from those presented in the books adopted by the courses of the degrees mentioned. The activity titled “Vaga de emprego” [Job Vacancy] was delivered at the end of the second meeting, and the students would return it up to one day before the last meeting. With this, we could investigate whether the knowledge acquired in students’ educational process enabled them to develop abductive reasoning to formulate their own questions.

An a priori analysis suggested that students found it challenging to use mathematical concepts, considering the degree they were attending, for three reasons: textbooks in libraries do not promote this articulation; the approach to mathematical and didactic organisations developed by teachers has not considered this vacancy, or if they do, there is no dissemination of the material produced; and the lack of website online search for models that can contribute to the execution of the task, which demands creativity to elaborate on and resolve the issue.

After the deadline for task completion, students shared their answers. In the following section, we present the answers to the task, analysing them in light of the language dimension.

Actions in act

The novelty of the task is explained by three motivations: first, the author intended to break with the types of statements in textbooks adopted by the two universities; second, provide a problem situation in which abductive reasoning was present from elaboration to development by students; and finally, construct a web of meanings in which the four types of language were encouraged during execution. With this intention, we present Activity 1 (Figure 3), entitled “Job Vacancy”.

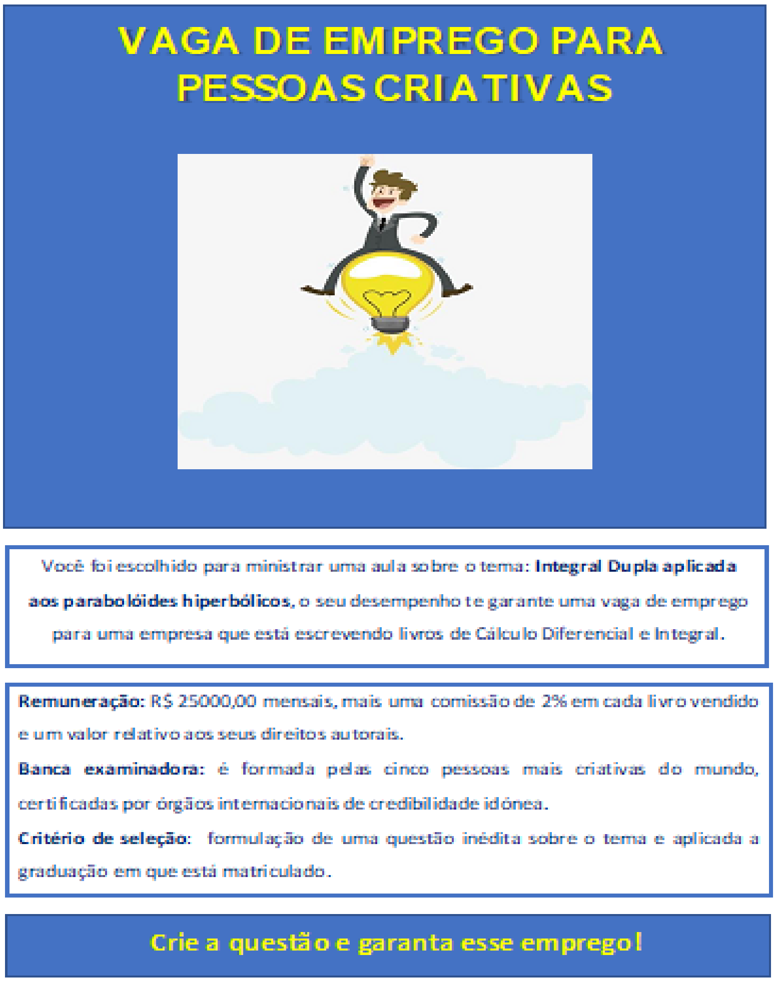


Figure 3.

Activity 1 (Brandão, 2021)

To analyse the collected data, we chose three categories: abductive reasoning, meanings produced, and language dimension. In this sense, we excel in responding to the following didactic problem: Did the double integral mathematical activity enable students to mobilise abductive reasoning and language dimensions to produce meanings in different contexts and within the scope of mathematics?

Of the twelve students who participated in the course, only five (E_2, E_3, E_6, E_8, and E_{11}) answered the task and had their protocols analysed according to the explained categories.

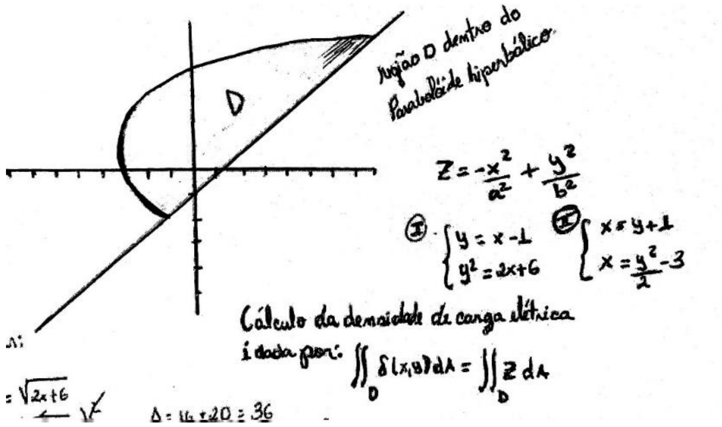
Student E_2 presented the solution (Figure 4) through records of algebraic and graphical representations to express the abductive reasoning in his association between the elements of his undergraduate studies and the mathematical object, which allowed for the articulation of the counterfactual language with the dictated and in-course language. When he uses these languages, we can see his knowledge of technologies, techniques, and theories to be applied in the activity and the semiosis in action when chaining linear procedures expressed by symbols that guide the written representation of deductive reasoning.

Although the student did not present a solution to the problem within the scope of the dictated and in-course language, he succeeded in counterfactual and in-(dis)course language. This makes us infer that when challenged in an unprecedented situation, the student could access elements from his previous experiences to construct a problem creatively.

"Job vacancy" question

Linear, surface, and volumetric charge density is an amount of electrical charge on a line, surface, or volume, respectively. It is measured in coulombs per metre (C/m), coulombs per metre squared (C/m^2), or cubic meter (C/m^3). As there are positive and negative charges, density can also take negative values. Just like any density, it depends on its position. When we have a surface of

$z = \frac{y^2}{a^2} - \frac{x^2}{b^2}$ and a region D bounded by y = x-1 and y^2 = 2x + 6, what is the electrical charge density of region D contained in the hyperbolic paraboloid?



Two ways to solve the problem:

I) $\begin{cases} y = x - 1 \\ y = \sqrt{2x+6} \end{cases}$ $\quad x - 1 = \sqrt{2x+6}$ $\quad \Delta = 16 + 20 = 36$

$(x-1)^2 = \left(\sqrt{2x+6}\right)^2$ $\quad x = \frac{+4 \mp 6}{2} = \begin{matrix} x' = 5 \\ x'' = -1 \end{matrix}$

$x^2 - 2x + 1 = 2x + 6$

$x^2 - 4x - 5 = 0$

$$D = \{(x,y)/ -1 < x < 5;\ \sqrt{2x+6} < y < x - 1$$

II) $\begin{cases} x = y + 1 \\ x = \frac{y^2}{2} - 3 \end{cases}$ $\quad y + 1 = \frac{y^2}{2} - 3$ $\quad \Delta = 4 + 4.8 = 36$

$2y + 2 = y^2 - 6$ $\quad y = \frac{2 \pm 6}{2} = \begin{matrix} y' = 4 \\ y'' = -2 \end{matrix}$

$y^2 - 2y - 8 = 0$

$$D = \{(x,y)/\ -2 < y < 4\ e\ \frac{y^2}{2} - 3 < x < y + 1$$

Figure 4.

Solution protocol of Activity 1 by student E_2[7] (Brandão, 2021, p.360)[8]

[7]All students signed the Informed Consent Form and the documents required by the Ethics Council.
[8]It was necessary to transcribe the solutions from the students' protocols reliably, as the images were unreadable when digitised.

Student E_8, enrolled in the civil engineering course, posed a problem situation (Figure 5) that involved the construction of a hydroelectric plant between the Negro River and the Solimões River, for which the water flow should be calculated.

The Rio Negro and the Rio Solimões are two rivers in the state of Amazonas that constitute the so-called Meeting of Waters. Due to several factors, initially, the waters do not mix, a phenomenon that expands for about 6km. A hydroelectric plant in the shape of a hyperbolic paraboloid was introduced at this Meeting of Waters.
The plant was built so that one side of the hyperbola is the Rio Negro dam, while the other is the Rio Solimões dam. Between the ground and the vertex of the parabola, there is a flow of water that will form a new course for what will become the Amazon River. At this rate, water falls from both rivers, causing the central generators to work, thus generating electrical energy. Calculate the maximum volume allowed for water flow between one of the dams and the power generation turbines using a double integral.

Solution:

Taking a = 1; b =1; c = 1; k = 2 curve 1→ $-x^2$

$F(x,y) = y^2 - x^2$ curve 2→ $-x^2 + k^2$

$$\iint y^2 - x^2 dydx \rightarrow \int \frac{y^3}{3} - x^{2y}y\Big|_{-x^2}^{-x^2+4} dx$$

$$I = (x^4 - 8x^2 + 16) - \left(\frac{-x^2+4}{3}\right) + x^4 - 4x^2 - \left(-\frac{x^6}{3} + x^4\right)$$

$$I = \frac{(-x^6+4x^4+8x^4-32x^2-16x^2+64)}{3} + x^4 - 4x^2 + \frac{x^6}{3} - x^4$$

$$I = \frac{12x^4-48x^2+64}{3} - 4x^2$$

$$I = 4x^4 - 20x^2 + \frac{64}{3}$$

$$\int (4x^4 - 20x^2 + \frac{64}{3})dx \rightarrow \frac{4}{5}x^5 - \frac{20}{3}x^3 + \frac{64}{3}x\Big|_0^1 \rightarrow \frac{4}{5} + \frac{44}{3} = \frac{12+220}{15} = \frac{232}{15} u.v$$

Figure 5.

Protocol for Activity 1 by E_8 (Brandão, 2021)

Student E_8 employed his knowledge of double integrals to calculate the volume or flow of water between one of the dams and the energy generation turbines. This application takes us back to the *raison d'etre* of the mathematical object, which is one of the principles of the epistemological dimension of the didactic problem. There is the use of dictated and in-course languages when technologies, techniques, and theories are used to execute the task and counterfactual and in-(dis)course languages when it articulates other contexts to promote the mobilisation of abductive and deductive reasoning.

To develop the problem situation, student E_8 creates a function F(x, y) and the limits of integration when he identifies each dam as a curve, represented by the equations $y = -x^2$ and $y = -x^2 + 4$ and when he establishes a range of variation for x belonging to the set of reals and expressed by $0 \leq x \leq 1$. We cannot detect those elements if we analyse the data provided in the situation statement. Thus, for the dictated and in-course languages, the statement does not match the calculations, which would be one misunderstanding of the language used and, in mathematics, it would not have any scientific validity. However, if we observe the counterfactual and in-(dis)course languages, it is possible to verify the disruption of the parameters of textbook activities by introducing a very peculiar version in which creativity appears at different moments.

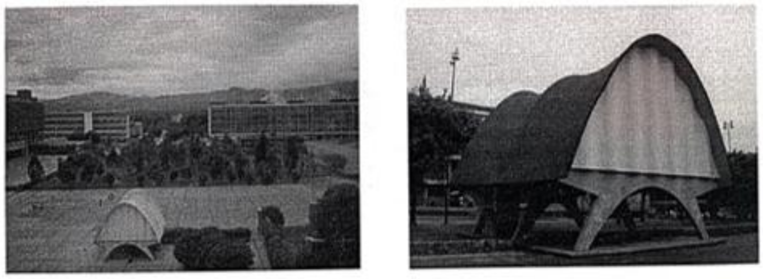
The mathematical model proposed by E_8 is unconventional, free from judgment, and expresses very particularly the abductive reasoning used. It also guides us on how some tasks can be organised around mathematical praxeologies to achieve the paradigm of questioning the world and formulating questions that meet the SRP device. Furthermore, E_8's question made us reflect on how much education has wasted talent and creativity in the classroom by not allowing students to construct and solve problems. We continue to reproduce mathematical models created in a way that does not serve non-mathematicians. When mathematical objects need to be accessed in extra-mathematical contexts, those involved do not feel qualified because they have not been prepared for this in their educational path.

The third solution was by student E_6, who presented a problem situation that involved a civil engineering context in the construction of the roof of the Pavilhão dos Raios Cósmicos [Pavilion of Cosmic Rays] (Figure 6), located in a university city in Mexico, by architect Feliz Condela in 1951. We browsed research sites and identified that it is not a new proposal. However, it can be used as an open-ended question model for the development of an SRP with some adaptations and research that mobilise mathematical praxeologies around the double integral.

An attempt of a question applied to the double integral in hyperbolic paraboloids!

Question: To save reinforced cement, coverings are used to help save money. For such, a roof can use hyperbolic paraboloids, as in the case of the Cosmic Rays Pavilion in 1951, which uses these ruled surfaces. The architect responsible was the Spaniard Felix Condela (1910-1997).

This pavilion, located in the University City in Mexico City, was intended to house a laboratory that required coverage of no more than 1.5 centimetres. The coverage is provided by two hyperbolic paraboloids coupled to a main parabola.



Given this, we want to calculate the amount of reinforced cement used on this surface.

However, I found nothing about the specific equation that governs this coverage, making it impossible to perform the calculation.

Figure 6.
Protocol for Activity 1 by E_6 (Brandão, 2021)

We infer that the student used counterfactual and in (dis)course languages when deviating from the statements of the adopted books, and related Activity 1 with a context outside mathematics. He sought dictated and in-course languages through a “specific equation that governs this coverage” to indicate that it was “impossible to calculate”. This conclusion made us reflect that he could have created or simulated conjectures to apply mathematical praxeologies that “feed” the dictated language.

The two solutions recorded below are from students E_3 and E_{11}, both from the mathematics teaching degree. We identified that the applications are not new because we found published scientific works and similar examples on internet search sites. However, we will highlight students’ adaptations, as we believe they are essential for the analysis of the chosen categories.

Student E_{11} brought as a solution to the activity a model of an industrialised potato chip that has the shape of a hyperbolic paraboloid (Figure 7), sold in cylindrical packaging. It is necessary to point out that during the fourth meeting, the researcher took processed potato chips for a snack, highlighting how similar they are to the hyperbolic paraboloid.

In the analysis of the mathematical praxeologies addressed by E_{11}, the student used abductive reasoning to develop the function defined by f(x, y) to integrate, the integration domains, the algebraic representations to calculate the measurement of the potato surface area, the calculation of maximum and minimum values, and the polar coordinates in the double integral.

Suppose that a Pringles brand potato chip is described by the formula $\rho(x, y) = \frac{y^2}{8} - \frac{x^2}{8}$ (approximately), and that its cylindrical can is 7 cm in diameter at the base and 25 cm in height. The potato chip intersects a 6cm cylinder so that it can move when it enters the can. The container has 5cm of free space between the potato chips and the lid, and the chips are 2 mm thick. How many chips are in that can? What is the total surface area of the potato chips? (Consider just one side of the potato chip).

Solution:

We need to define a global maximum and a global minimum in the function to find the height of each potato chip and, consequently, the number of chips in the package. With the quantity of potato chips, we calculate the area of each potato and the total area the chips have together (in total).

z=f(x, y) = $\frac{y^2}{8} - \frac{x^2}{8}$ cylindre $0 \leq x^2 + y^2 \leq 3$ $A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$

$x = r\cos\theta$; $y = r\sin\theta$; $dA = rdrd\theta$; $0 \leq \theta \leq 2\pi$; $0 \leq r \leq 3$

$$\left(\frac{\partial z}{\partial x}\right)^2 = \left(\frac{-x}{4}\right)^2 = \frac{1}{16}x^2; \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{16}y^2$$

$$A(s) = \iint_s \sqrt{1 + \frac{1}{16}x^2 + \frac{1}{16}y^2 dA}$$

$$A(s) = \iint_s \sqrt{1 + \frac{1}{16}r^2}.rdrd\theta \qquad 1+\frac{1}{16}r^2 = u$$

$$A(s) = \int_0^{2\pi}\int_0^3 \sqrt{u}.8dud\theta \qquad \frac{1}{8}rdr = du \quad rdr=8du$$

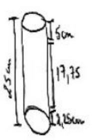
$8.\frac{2}{3}\int_0^{2\pi} u^{\frac{3}{2}}\Big|_0^3 d\theta = \frac{16}{3}.3{,}19.2\pi = 55{,}36\pi m^2$ potato chip surface area (one side)

Geometrically and algebraically, we can verify that in the function $\rho(x, y) = \frac{y^2}{8} - \frac{x^2}{8}$ the function will have a maximum for x = 0 and y = 3, since the coefficient x^2 is negative and will have a minimum for y = 0 and x = 3, since the coefficient of y^2 is positive

Máximo: $f(x,y) = 3 = \frac{9}{8} = 1{,}125$

Mínimo: $f(x,y) = 3 = -\frac{9}{8} = -1{,}125$

Logo a altura da batata é dada por 2,25 cm



$\frac{17{,}75}{0{,}2}$: número de batatas ≈ 88

Área total de superfície =
$88 \times 55{,}36 = 4.913{,}2 \ cm^2$

Figure 7.
Protocol for Activity 1 by E_{11} (Brandão, 2021)

We identified a bridge between dictated and in-course languages and counterfactual and in (dis)course languages, although E_{11}'s emphasis was on dictated and in-course languages, as he approached some mathematical objects in depth and used them to mediate techniques and technologies inherent to the double integral. We infer that the student's undergraduate course justifies such a view.

As for abductive reasoning, it arises from the development of the function of two variables, the association with a hyperbolic paraboloid shape model and the shape that determines the number of potatoes that will be stored in the cylindrical container. But it extends to deductive reasoning, present in the technique, in the mediated symbols and in the image of the represented cylinder.

The fourth solution was E_3's, who approached double integral in an architectural project for a hangar at an airport in Kansas City (Figure 8). We found an example on the internet explored by Fontes (2005), who associated it with the shape of a hyperbolic paraboloid.

Although it was not original, we analysed student E_3's proposal, which demonstrated mastery of the mathematical praxeologies utilised in the solution and used, in a pronounced way, dictated and in-course languages. However, in the comments at the end of the protocol, we identified signs of abductive reasoning and in-(dis)course language, such as noticing that the dimensions are not faithful to the original structure, using the software GeoGebra to get a graphical representation of the mathematical equation, putting the source of the search online carried out, and raising future possibilities of working on the issue discussed.

Double integral activity applied to paraboloids

Image available at: https://groucho-karl-marx.blogspot.com/2011/11/american-airlines-files-for-bankruptcy.html

The largest sep-type hyperbolic paraboloidal (h-p) shell was designed by Milo Ketchum for the TWA maintenance hangar at the Kansas City Airport. This structure comprises the intersection of two saddles, with a gap between supports of approximately 17.5m and a thickness of 10 cm of concrete. This structure allowed the location to be used as an aeroplane hangar, and what is more, the h-p profile allowed the adaption to the volume to be protected. Let us sketch a volume for such a structure. Consider the span of 100m; from the images, we estimate that the building formed has a quadrangular shape. Calculate the volume of the solid below the hyperbolic paraboloid $z = \frac{x^2}{324} - \frac{y^2}{324}$ and above the rectangle R = [-100,100] x [-100,100]

Resolution:

To calculate this volume (volume mentioned in the text), we will use the double integral tricks. Since the paraboloid has a symmetrical shape in the first four octants, we will calculate the integral over the following integration domain, R'= [0.100] x [0.100] and multiply the result by 4.

$$\int_0^{100}\int_0^{100} \frac{x^2 - y^2}{324} dxdy = \frac{1}{324}\int_0^{100}\int_0^{100} (x^2 - y^2)dxdy = \frac{1}{324}\int_0^{100} \left[\frac{x^3}{3} - y^2 x\right]_0^{100} dy =$$

$$\frac{1}{324}\int_0^{100} \left[\frac{100^3}{3} - y^2.100 - 0\right] dy = \frac{1}{324}\int_0^{100} \left(\frac{100^3}{3} + 100y^2\right) dy =$$

$$\frac{1}{324}\left(\frac{100^3}{3}y + \frac{100}{3}y^3\right)_0^{100} = \frac{1}{324}\left(\frac{100^4}{3} + \frac{100^4}{3}\right) = \frac{1}{324}.\frac{2.100^4}{3} =$$

$$\frac{2.100^4}{972} m^3 = \frac{100^4}{486} m^3 = 206.761,32 m^3$$

Since we calculated in the 1st octant, we will multiply the value by 4 and obtain the volume corresponding to the octants corresponding to positive z. Therefore, an estimate of the volume for this structure similar to the one mentioned above is

$V = 823\,045,27\ m^3$

Comentários:

1- Não calculamos o volume do vão.
2- As dimensões não são fiéis ~~oo~~ as originais uma vez que não conseguimos encontrar as medidas oficiais.
3- Poderíamos estimar melhor a altura e assim obter outra equação para a PH.
4- A equação foi conseguida através de manipulações no geogebra.
5- Essa questão tem muito a melhorar.

Figure 8.

Protocol for Activity 1 by E_3 (Brandão, 2021)

The student realises the importance of judgment when he states, “The question must be improved a lot”. This perception comes from the established belief that a good teacher does not make mistakes. This is one of the obstacles to the paradigm of questioning the world because, by asking an open-ended question, the student can bring answers that are not in the teacher’s previous research, something that went unnoticed and that takes one out of one’s “comfort zone” (Borba & Penteado, 2007).

Such a stance before value judgments imposed by society made us reflect on some subjective restrictions in which teachers are immersed, which impair the implementation of didactic organisations of a more liberating and autonomous nature. Challenging school tradition is a path of many stumbles but with innovative and surprising results.

In analysing the five students’ protocols, we identified that they produced meanings associated with their undergraduate courses. Thus, the engineering students created situations external to mathematics to present, with prominence, the counterfactual and in-(dis)course languages, while the mathematics undergraduates focused on the dictated and in-course language.

However, other meanings emerged when the researcher questioned students about why they had not responded to Activity 1. The first reason given was that the activity the teacher put forward required creativity, whereas those in textbooks require technical procedures to solve. We presented the others in clippings of the students’ oral language audio-recorded during the meeting.

> Students E_6, E_7, and E_{11} answered:
>
> E_7: In my opinion, we have a lot of questions in books that are basically to “calculate”. We don’t have to think about how we will apply the CDI. Maybe that’s why I couldn’t ask the question.
>
> E_6: The activity differs when it comes to “job vacancy”, it makes the student elaborate on and relate a question to the real and applicable world, in the textbook, it is just a matter of solving a given equation.
>
> E_{11}: It was a contextualised activity, which leads the student to its creation, editing, and resolution. Books bring mechanical questions, which only require calculation.
>
> **Source:** Brandão, 2021

Students’ comparison evidences the importance the researcher attributed to stimulating the creativity of the course participants to provide a co-authorship activity. Students sometimes feel destabilised and sometimes stimulated by the autonomy caused by this rupture with the task models provided by the adopted books. As for the feelings generated when creating a CDI question related to their undergraduate courses, all mentioned having difficulties. Three students did not respond, five simply mentioned that they had difficulties but did not justify the answer, and four said:

> E_8: It was complicated, but I managed to develop it. That showed me that I'm doing the right course.
>
> E_{11}: At first, I felt difficulties, as I was leaving the comfort zone of just solving problems, but after creating an application, the activity flowed smoothly.
>
> E_7: Well, my case is mathematics. I have to break down some barriers to better see how to apply the entire CDI.
>
> E_{10}: Creating a question is more difficult than answering it, it requires creativity and knowledge.
>
> **Source**: Brandão, 2021

Participants were asked to report the difficulties they felt in resolving the elaborated question. The results were: two students could not find a mathematical model to describe the elaborated situation; one of the course participants found that the sources for using real data were restricted; another described that the association of the hyperbolic paraboloid with electrical engineering was his biggest challenge, and the remaining six participants mentioned they could not remember how to use integration techniques in the task created.

Some considerations

The text proposes the analysis of the didactic problem under a fourth dimension, that of language. From this perspective, we show that it can be considered a frontier field that provides elements external to mathematics to produce meanings different from those found in the books adopted for CDI at universities. We also analysed how the dictated and in-course languages bring epistemological obstacles resulting from cuts made to adapt knowledge to different school levels.

We chose to expose protocols answered by engineering and mathematics students for an activity called "Job Vacancy", in which they were asked to create a double integral question applied to hyperbolic paraboloids. Of the 12 course participants, only five answered, and from the data collected, we identified that the proposed activity mobilised abductive reasoning in its preparation and in the conjectures woven. We also observed that the four languages were presented unequivocally.

As for meanings, students mention the difference between Activity 1 and activities found in textbooks, in which only techniques and technologies are explored with a minimum of applications in other areas of knowledge. In their oral reports, they stated that they did not find research sources that supported them in preparing the questions; they found it difficult to recall techniques and technologies of the double integral and hyperbolic paraboloids, even with the intervention of the researcher at times, reminding them of some techniques and technologies related to the double integral and quadric surfaces; they found it complex to articulate the mathematical objects of the activity with their undergraduate courses, as they are not addressed in this way.

For future research, we suggest that the activities created be analysed and returned to the students so that they can make more precise adaptations and evaluate the use of dictated and in-course languages within mathematics and counterfactual and in-(dis)course languages, in which reasoning abductive emerges more frequently.

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