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Theoretical constructs proposed by Tall for teaching derivatives: reflections on the development of a reference epistemological model

Constructos teóricos de Tall para la enseñanza de la derivada: consideraciones sobre la elaboración de un modelo epistemológico de referencia

Les construits théoriques de Tall pour l'enseignement de la dérivée : considérations sur l'élaboration d'un modèle épistémologique de référence

Constructos teóricos de Tall para o ensino de derivada: considerações sobre a elaboração de um modelo epistemológico de referência

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Abstract

This article aims to contribute to the discussion in this issue, centered around the question "How to develop a Reference Epistemological Model (REM) for the teaching of Calculus?", more specifically considering the teaching of derivatives. The arguments presented here advocate the inclusion of theoretical constructs, such as the ones developed by Tall for the teaching of derivatives, due to their potential to make cognitive and didactical contributions to students and teachers, respectively. The constructs that we refer to in this text were called generic organizer and cognitive root of local straightness by Tall. The authors of this text consider that adding these constructs to an REM may foster integration between theory and practice, which is important for the development of Mathematics teaching. We organized our reflections by linking ideas related to the integration between theory and practice, the conception of an REM, the teaching of derivatives and Tall's theoretical constructs. We concluded the article emphasizing the importance of continuously observing the prevailing epistemology of the concept of derivatives for teaching, aiming to find contributions that support the emancipation of Didactics of Mathematics and promote effective Calculus teaching.

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Keywords: Reference Epistemological model; Calculus teaching; Theoretical constructs proposed by David Tall.

Resumen

Este artículo se propone contribuir a la discusión de este número en torno a la cuestión “¿Cómo desarrollar un Modelo de Referencia Epistemológico (MER) para la enseñanza de Cálculo?”, considerando específicamente la enseñanza de la derivada. Los argumentos aquí expuestos se guían por la defensa de la inclusión de constructos teóricos, como los desarrollados por Tall para la enseñanza de la derivada, debido a las potencialidades que tienen para agregar contribuciones de orden cognitivo y didáctico a los aprendices y a los profesores respectivamente. Los constructos a los que nos referimos fueron denominados por Tall como organizador genérico y la raíz cognitiva de la rectitud local. Para los autores de este texto, la inclusión de estos constructos en un MER puede favorecer la integración de teoría y práctica, importante para el desarrollo de la enseñanza de la Matemática. Organizamos las reflexiones encadenando ideas sobre: integración de teoría y práctica; concepción de un MER; enseñanza de la derivada y los constructos teóricos de Tall. Concluimos la presentación del artículo reforzando la importancia de la vigilancia sobre la epistemología dominante del concepto de derivada para la enseñanza, con vistas a la búsqueda de contribuciones a la emancipación de la Didáctica de la Matemática y el favorecimiento de la enseñanza del Cálculo.

Palabras clave: Modelo epistemológico de referencia; Enseñanza del cálculo; Constructos teóricos propuestos por David Tall.

Résumé

Cet article vise à contribuer à la discussion de ce numéro autour de la question «Comment développer un Modèle de Référence Épistémologique (MER) pour l'enseignement du Calcul?», en considérant spécifiquement l'enseignement de la dérivée. Les arguments présentés ici sont guidés par la défense de l'inclusion de construits théoriques, tels que ceux développés par Tall pour l'enseignement de la dérivée, en raison de leur potentiel à apporter des contributions d'ordre cognitif et didactique aux apprenants et aux enseignants respectivement. Les construits auxquels nous nous référons ont été nommés par Tall comme organisateur générique et la racine cognitive de la rectitude locale. Pour les auteurs de ce texte, l'inclusion de ces construits dans un MER peut favoriser l'intégration de la théorie et de la pratique, ce qui est important pour le développement de l'enseignement des Mathématiques. Nous avons organisé nos réflexions en enchaînant des idées sures: l'intégration de la théorie et de la pratique; la conception d'un MER

; l'enseignement de la dérivée et les construits théoriques de Tall. Nous concluons la présentation de l'article en renforçant l'importance de la vigilance sur l'épistémologie dominante du concept de dérivée pour l'enseignement, dans le but de rechercher des contributions à l'émancipation de la Didactique des Mathématiques et de favoriser l'enseignement du Calcul.

Mots-clés : Modèle épistémologique de référence ; Enseignement du calcul ; Construits théoriques proposés par David Tall.

Resumo

Este artigo se propõe a contribuir com a discussão deste número em torno da questão “Como desenvolver um Modelo de Referência Epistemológico (MER) para o ensino de Cálculo?”, considerando especificamente o ensino da derivada. Os argumentos aqui expostos norteiam-se pela defesa da inclusão de constructos teóricos, como os desenvolvidos por Tall para o ensino de derivada, pelas potencialidades que eles têm de agregar contribuições de ordem cognitiva e didática aos aprendizes e aos professores respectivamente. Os constructos aos quais nos referimos foram denominados por Tall por organizador genérico e a raiz cognitiva da retidão local. Para os autores deste texto a inclusão desses constructos, em um MER, pode favorecer a integração teoria e prática importante ao desenvolvimento do ensino da Matemática. Organizamos as reflexões encadeando ideias sobre: integração teoria e prática; concepção de um MER; ensino da derivada e os construtos teóricos de Tall. Finalizamos a apresentação do artigo reforçando a importância da vigilância sobre a epistemologia dominante do conceito de derivada para o ensino, com vista à busca de contribuições à emancipação da Didática da Matemática com o favorecimento do ensino do Cálculo.

Palavras-chave: Modelo epistemológico de referência, Ensino do cálculo, Constructos teóricos propostos por David Tall.

Theoretical constructs proposed by Tall for teaching derivatives: reflections on the development of a Reference Epistemological Model (REM)

This article aims to contribute to the field of Didactics of Mathematics by examining the development of a Reference Epistemological Model (REM) for teaching Calculus[3]. To do so, we focused our arguments on the concept of derivatives and on the discussion on the use of theoretical elements, such as the theoretical constructs proposed by Tall within the scope of Didactics of Mathematics. We consider that Tall's proposition enriches theoretical and practical relationships and generates cognitive and didactical reflections that favor Mathematics teaching and learning. In order to achieve our goals, we organized this text by presenting elements that integrate theory and practice; the concept of REM; the teaching of derivatives and Tall's theoretical constructs.

Regarding the integration between theory and practice, Jaworski (2006) suggests that Mathematics Education should reflect on this integration and consider how research could contribute to the improvement of Mathematics teaching and learning.

According to the aforementioned researcher, the field of Mathematics Education:

> [...] has become mature in its theoretical propositions. However, according to her, the status of Mathematics teaching remains the same, theoretically, not well configured and poorly developed. While theories provide us with lenses to analyze teaching (Lerman, 2001), the "big theories" do not seem to offer clear perceptions for teaching or ways through which teaching could foster the learning of Mathematics (Jaworski, 2006, p. 188, adapted).

The author understands that it is necessary to promote interaction between theory and practice and between teachers and researchers, because using a theory

> [...] is not able to show us what teaching *should* encompass, but teachers and educators can look for it in order to understand more clearly what teaching *could* encompass, so we learn about teaching with the possibility for developing teaching (Jaworski, 2006, p. 189, author's emphasis).

We agree with Jaworski (2006) that theories play a role in supporting analyses and providing examples, but theories themselves do not offer practical guidance or prescribe specific actions. It is a fact that when a theory is developed and presented, its direct applicability to practical situations is not guaranteed. Thus, like Jaworski (2006), we emphasize the need to

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[3] In this text, we will adopt the term "Calculus" to express that we consider Differential and Integral Calculus of a real variable.

foster the integration between theory and practice in the field of Mathematics Education, more specifically in order to establish a connection between educators and researchers.

To understand the concept of REM, Gascón (2014) says that an REM is a set of ideas, principles and approaches that can be used to form a theoretical or methodological framework for studying a specific area of knowledge; in this case, it is Mathematics teaching. This model provides a conceptual structure so that we can understand how mathematical knowledge is generated and developed and analyze didactical phenomena within an educational context.

An REM plays a vital role in the emancipation of Didactics of Mathematics by enabling didactics research to break free from school codes and the Dominant Epistemological Model (DEM) in educational institutions, which in turn allows us to construct and give visibility to phenomena that remain invisible within the educational context today and that could be revealed by academic research studies. Gascón (2014) understands that emancipation is necessary in two levels: institutional and epistemological.

The first level is characterized when the researcher aims to free themselves from the dependencies that accompany the position of "teacher" (a subject embedded within an institution), of "noosphere" (a subject of the noosphere, who could be, for example, an author of textbooks, curriculum plans, curricular documents, teacher education texts, etc.) and of "mathematician" (a subject of the institution that produces and preserves knowledge). It allows mathematical educators to analyze said models critically and to construct others that could enable the interpretation of didactical phenomena, thus contributing to autonomy in the construction of the object of study of Didactics of Mathematics.

According to Gascón, for epistemological emancipation, it is necessary to consider:

> [...] didactical transposition processes as an object of study; educators should analyze critically the epistemological models of mathematics that prevail in institutions and, thus, free themselves from the uncritical assumption of such models. (Gáscon, 2014, p. 100, our translation).

Gáscon proposes a critical approach to didactics research, emphasizing the need for emancipation from an epistemological model prevailing in an institution. It means that researchers should examine didactical transposition processes in a critical way. By doing so, educators do not only identify the limitations and influences imposed by these models, but also aim to free themselves from accepting uncritically and passively the paradigms proposed by these models. This critical attitude is essential to foster new perspectives and methodologies that are free from prevailing and traditional impositions.

Moreover, using specific models to study didactical phenomena enables researchers to assess, correct and contrast the models with historical and didactical data from teaching practice, contributing to the epistemological emancipation of Didactical Science. Thus, we understand that it is possible to incorporate David Tall's contributions to the development of a Reference Epistemological Model (REM) into the teaching of the concept of derivatives.

About the teaching of derivatives

Research on Calculus teaching and learning has attracted researchers all over the world and shown that the results related to learning have not been good. It has also been demonstrated by the assessment of degree programs in which Calculus is a part of the curriculum. Thus, there are reasons to question teaching methods and their influence on the attribution of meaning to Calculus concepts, particularly to the concept of derivatives.

In a literature review on the teaching of derivatives, Escarlate (2008) identified that this concept is commonly presented through the "derivative / tangent line" analogy, which might result in a type of comprehension that is not compatible with its formal definition. Table 1 presents this analogy.

Table 1.

Analogies between the derivative and the tangent line to a curve (adapted from Escarlate, 2008, p. 34)

DERIVATIVE / TANGENT LINE
The notion of tangency in Basic Education (about the number of contact points) is not enough for the context of Calculus.
An infinitesimal argument (derivative) is necessary to define tangency in all its generality.
In the current structure of Calculus, the concept of derivatives is introduced before that of the tangency of a function at a point.
"The derivative is the inclination of the tangent line." The inclination may be admitted as a definition of the slope of the tangent line at a given point, but not as a derivative at a point.
Difficulties in derivative learning emerge from the mistaken relationship between derivatives and the slope of a tangent line.

The "derivative / tangent line" analogy poses challenges to learning, and they should be considered. One of them refers to the fact that concepts of trigonometric and geometric tangent are introduced in Basic Education, which may potentially limit the understanding of the concept of derivatives. The relationship between a tangent line and a trigonometric function can be established, but there is still an understanding of the tangent line to a curve that lacks rigor, with the interpretation that there is an interception between the line and the graph at a single point. That is to say, the local nature of the intersection is not discussed. Furthermore, according to

Escarlate (2008), the analogy (the slope of the tangent line at a point/derivative at a point) does not favor the knowledge of applications of derivatives, such as the study of increasing and decreasing intervals of a real-valued function. Thus, it is imperative to examine critically and expand the definition of the derivative of a function at a point beyond the aforementioned analogy.

Based on this assumption, it is important to take into consideration a debate on how to introduce the concepts of derivative and limit in initial Calculus courses due to the formal definition of these concepts. An important question raised by Monaghan *et al.* (2023) is whether the definition of the concept of limit is necessary when teaching differentiation. They state that:

> Whether the limit (as a formal concept) is necessary or not when introducing differentiation in an introductory Calculus course is a matter of debate. […], it is not whether limits are important or not (they are important!), but if limits should be formally introduced in the teaching and learning of derivatives in an introductory Calculus course. (p. 92).

Monaghan *et al.* (2023) say that four approaches have been proposed by research studies related to the teaching of derivatives: the rough and ready approach, the limits approach, the differential and infinitesimal approach and the kinematic approach.

We highlight that the rough and ready approach involves working with average rates of change as an intuitive way of introducing derivative concepts. It makes use of functions and their algebraic and graphic representations, emphasizing the use of digital technologies in Calculus teaching. According to the authors, this approach "enables teachers and students to develop directly ideas of derivability at a point without a previous preparation with limits, infinitesimals or the kinematic approach" (Monaghan *et al.*, 2023, p. 92).

The concept of derivatives is defined from the concept of limit of the incremental ratio. According to Monaghan *et al.* (2023), this definition fulfils

> [...] the requirements of mathematical rigor, but, as mentioned before, students have considerable difficulties with limits, thus, should we build an initial approach to pointwise differentiation starting from a mathematical concept that students consider particularly difficult? (Monaghan *et al*., 2023, p. 99).

To give examples of difficulties with the concept of limit, we refer to Cornu's work (1991), according to whom cognitive aspects cannot be derived, exclusively, from a mathematical definition.

The researcher highlights negligence on the part of teachers for considering their students' "spontaneous conceptions", because they encompass a wide range of ideas, images, intuitions and knowledge acquired through their daily experiences. These experiences are not necessarily influenced by formal teaching methods, which leads students to modify and align their individual conceptions with the definition of the concept. It may potentially create difficulties within the learning process. For instance, mistakes might arise when students perceive a limit as an “insurmountable barrier” or consider the limit of a sequence as a stationary value in which the terms of the sequence remain the same for sufficiently large indices (Cornu, 1991, p. 155).

Furthermore, it is fundamental to be aware of the historical development and the conflicts surrounding the concept of derivatives because they may provide valuable information about the nature of the concept and its implications for teaching and learning. In this sense, the author warns us:

> It is difficult to introduce the notion of limit in mathematics because it seems to have more to do with metaphysics or philosophy. Mathematicians are often reluctant to speak about said concepts, from ancient Greece to D’Alembert, who wrote: “One can quite easily do without the rest of all this metaphysics of the infinite in the differential calculus”. Lagrange expressed a similar horror of the metaphysical aspects (Cornu, 1991, p. 161).

This historical resistance to the metaphysics of infinite and to the abstract concepts related to differential calculus reflects the complexity involved in the understanding and teaching of derivatives. The notion of limit, which is essential for the development of the concept of derivatives, is a point of tension between pure mathematics and its philosophical implications. To deal with such complexity, it is crucial for educators to adopt approaches that make these concepts more accessible to students. It may be done through a combination of visual and intuitive methods that could demystify the abstraction of limits and derivatives. The addition of concrete examples and the use of technological tools, such as visualization software, could help to build a more tangible understanding of these concepts, mitigating historical difficulties and providing students with elements to explore and apply the concept of derivatives in a contextualized way. Thus, understanding the historical roots and conceptual challenges surrounding differential calculus may enrich pedagogical practice and foster new approaches to the teaching of derivatives.

According to Ely (2021), the differential and infinitesimal approach to Calculus is based on the use of differentials and infinitesimals as fundamental elements to teach and understand Calculus. This approach aims to restore the notion of direct reference of differential notation, allowing a deep and intuitive interpretation of the mathematical concepts involved.

Infinitesimals can be understood as extremely small numbers that, although not rigorously defined in the 19^{th} century, were reintroduced into mathematics through the development of Non-standard Analysis (Robinson, 1966) in the 1960s. This approach enables a formal definition of infinitesimals and allows Calculus to be rigorously performed based on them.

Non-standard Analysis formally considers infinitesimals within the scope of a set of axioms, offering a sound basis for calculations with ‘extremely’ small numbers. This approach provides mathematicians with the flexibility to choose between Real Analysis and Non-standard Analysis, allowing research and achieving results without compromising mathematical rigor.

According to Monaghan *et al*. (2023), the concept of infinitesimals can be used as an alternative to the introduction of the concept of derivatives in Calculus teaching. The reintroduction of this concept aims to provide a different perspective, allowing the direct use of notations originally conceived for infinitesimals.

In the 19^{th} century, Calculus underwent a significant change when infinitesimals were removed from the curriculum due to the perception that they were not rigorously defined. Then, Calculus was developed in terms of limits, and most Calculus courses today avoid using infinitesimals; they prefer an approach based on limits to define fundamental Calculus concepts such as derivatives, integrals and continuity.

In the 1960s, Non-standard Analysis, developed by Robinson (1966), enabled the formal definition of infinitesimals, which, in turn, allowed a rigorous development of Calculus based on them. Robinson demonstrated that virtually everything that can be done through limit-based Calculus can also be done through infinitesimal Calculus. Researchers like Tall (1980, 1981a, 1981b, 2001) and Ely (2021) conducted studies on the application of this notion in introductory Calculus courses and advocated the benefits of this approach. Terence Tao (2007) also used Non-standard Analysis to avoid the excessively complicated management of epsilons, highlighting that it is not entirely foreign to the Analysis; it is just “an ultrafilter away” from it.

The kinematic approach, indicated by Monaghan *et al.* (2023), explores differentiation through the concept of “derivative at a point”, considering the speed of motion at a specific time interval. It is initially illustrated through uniform motion expressed by the function, but its complexity rises when considering non-uniform motion, modeled by polynomial functions, for example. The conceptual difficulty arises when dealing with the apparent neglect of the time interval, when calculating average speed for a sufficiently short time interval. The conclusion is the definition of the instantaneous velocity function of a moving object at time t_0 as the derivative of the function that models the displacement of the object at t_0.

About Tall's theoretical constructs

The discussion on the potential of theoretical constructs for teaching derivatives should be based on some factors. The first aspect that we consider important in teaching is the need to explore examples that are not usual in the prevailing teaching practice. These examples should highlight the characterization of differentiable and non-differentiable functions at a specific point in their domains, thus enabling the development of the understanding of the concept of derivatives. To do so, it is necessary to expand the exploration of these types of functions, without limiting it to those represented algebraically by a single expression or polynomial functions. This argument is used by Monaghan *et al* (2023) when recommending this diversification. They say it is important to foster

> […] different ways of thinking of a function (equation, graph, diagram); teachers' awareness that students think that functions should be given by a formula and/or be continuous; different families of functions (for instance, linear, polynomial, rational); inverse functions; and domain and range. (Monaghan *et al*, 2023, p. 92)

As an example, let us consider the function $b = b(x)$, called *blancmange*. Part of its graph is expressed in Figure 1.

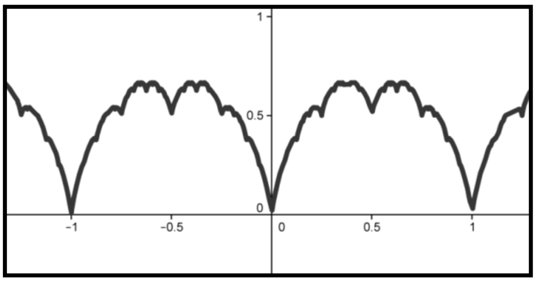

Figure 1.

Graphic representation of the blancmange function, constructed in GeoGebra. (Our production).

This function is represented algebraically by the sum of a convergent series of functions given by $b(x) = \sum_{i=1}^{\infty} f_i(x)$, where f_i: $\mathbb{R} \to \mathbb{R}$, defined as: $f_i(x) = \frac{1}{2^{i-1}} f(2^{i-1} \cdot x)$ and $f(x) = d(x, \mathbb{Z})$[4], with $i \in \mathbb{N}$.

[4] In the metric space ($\mathbb{R}$, $|\ .\ |$), the function d: $\mathbb{R} \to \mathbb{R}$, defined by $d(x, \mathbb{Z}) = inf\,\{|x - z|$, where z is an integer$\}$ is the function that assigns to every x the distance between x and the set $\mathbb{Z}$.

The *blancmange* function does not have a derivative at any point in its domain and it is continuous at all these points. The demonstration of such fact can be found in Tall (1982).

The second example refers to the function $g: [-1, 1] \to \mathbb{R}, defined\ by\ g(x) = \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor}$, where $\left\lfloor \frac{1}{x} \right\rfloor$ denotes the greatest integer of $\frac{1}{x}$, for $x \neq 0$ and $g(0) = 0$.

The function g has a derivative equal to 1 at x = 0 and at the points where the function is continuous, because, in the latter case, the function is constant on each interval $\left[\frac{1}{n+1}, \frac{1}{n}\right[$. The greatest integer function is not continuous at $C = \left\{\frac{1}{n} \mid n \in \mathbb{Z}^*\right\}$. Figure 2 shows the graphic representation of g.

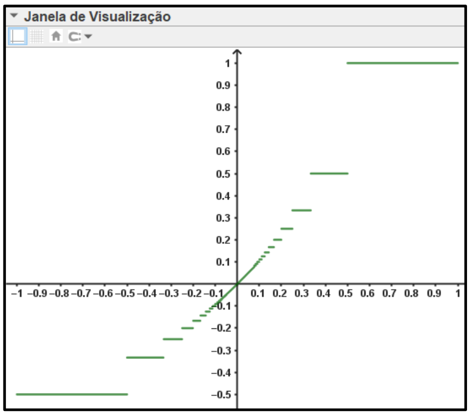

Figure 2.

Graphic representation of the function g. (Our production)

These examples enable the analysis of the relationship between continuity and differentiability, as well as of derivatives of functions that are not defined by a single law of formation. This approach supports the development of formal concepts in Mathematics and expands a prevailing epistemological model in which the functions considered can be defined by a single expression and in which differentiability goes hand in hand with continuity.

It is also viable to explore the notion of rate of change, connecting it to the concept of derivatives, in a manner similar to the rough and ready approach suggested by Monaghan *et al*. (2023). Using advance organizers[5] is mentioned as an effective tool because it provides immediate answers to students' investigations and help them to understand Calculus concepts.

[5] According to Ausubel (2003), an advance organizer consists of "introductory material at a higher level of abstraction, generality and inclusiveness than the learning task itself. The function of the organizer is to provide ideational scaffolding (anchoring) for the stable incorporation and retention of more detailed and differentiated material that follows in the learning passage, as well as to increase discriminability between this situation and the relevant anchored ideas of the cognitive structure. The organizer should not only be explicitly related to the more

To introduce a generic organizer, we are guided by Tall's statement, aiming to introduce how it acts. According to Tall

> [...] an organizer is something that acts in a Piagetian way, first being within an environment in which equilibrium is possible, then it is necessary to find a discrepant property that causes conflict and requires mental reconstruction leading to a new and rich state of equilibrium (Tall, 1986, p. 86-87).

The definition of generic organizer entails the definition of a theoretical construct attached to it, called cognitive root, "a cognitive unit which is (potentially) meaningful to the student at the time, yet contains the seeds of cognitive expansion to formal definitions and later theoretical development" (Tall, 2000, p. 11). Or "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" (Tall, 1989, p. 9).

Examples of cognitive roots stemming from Calculus concepts are local straightness (rate of change/differentiation/differential equations), perceptual continuity (formal concept of continuity); area under a graph (integration).

A function is considered locally straight at a point x_0 within its domain if its graphic representation resembles a straight line when sufficiently magnified. In this case, we say that the function exhibits local straightness.

In a computer environment, Tall devised the generic organizer *Magnify*, which "allows the user to home in on a graph and draw a magnified portion in a second window" (Tall, 2000, p. 11). In Figure 3, the graph of the real function given by the expression $g(x) = sin\ x$ is constructed using this software.

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specific learning situations that follows, but also (in order to be apprehensible and stable) be related to the relevant ideas in the cognitive structure and take them into account" (p. 65 – 66).

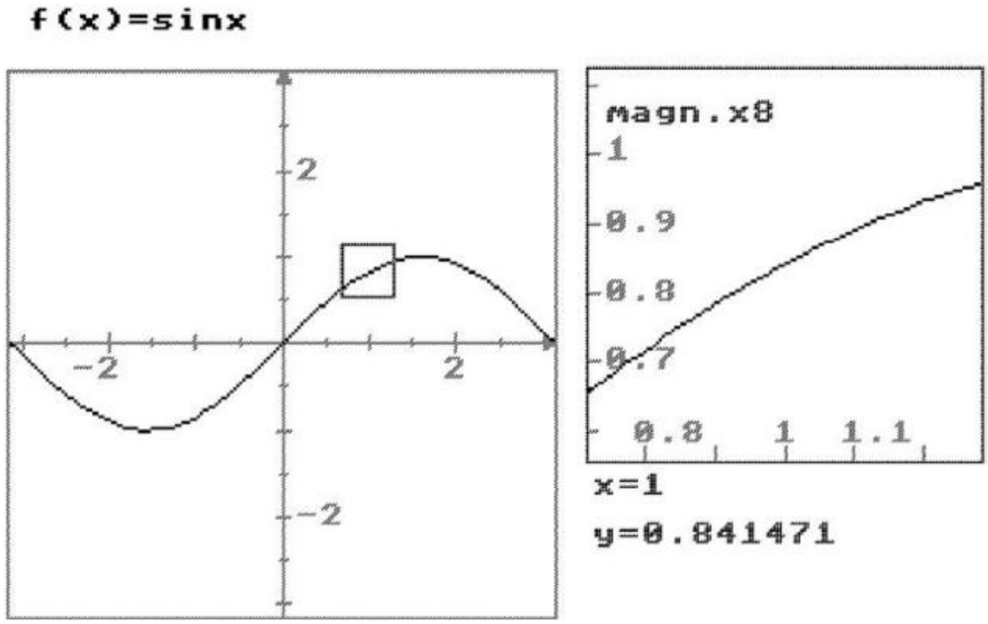

Figure 3.

Magnifying a portion of the sine function graph at x = 1 *(Tall, 2000, p. 1)*

This notion should be considered in the teaching of derivatives for two reasons: the first one is that, according to this notion, by using "the magnification of graphs produced by a computer, it is possible to enable the underlying nature of the limit process to be revealed. It contrasts with the traditional concept of limit, in which the process is explicitly defined. This notion, as stated by Dubinsky and Tall (1991), introduces a new approach that enables a comprehension of limit through the magnification of a portion of the graph in a computer environment. The other reason is the establishment of the following link: f'[derivative function of f] can be viewed as the function that associates each point (x_o, y_0) [on the graph of f] with the value of the slope of the tangent line to the graph at (x_o, y_0)" (Tall, 2000, p. 11, adapted).

According to Tall (2000), this perspective offers valuable information about the nature of the derivative function. Essentially, when examining the graph of a given function, based on the concept of local straightness, it is possible to conjecture at which point differentiability does not occur. When a function is differentiable at a point, magnifying a portion of the graph around this point at a proper level of magnification makes the selected area resemble a line segment, when displayed on a computer.

Moreover, according to Tall, an approach in which the notion of local straightness is used to introduce the concept of derivatives:

> Instead of a symbolic compression that encapsulates a process of limit in the limit object (the derivative), it is an embodied compression that operates on an object (the graph) to provide a new object (the graph of the derivative function). The student can now view the derivative as the function that associates the value of the slope of the tangent line with each point. The new task is to symbolize this view of the derivative function through the calculation of a good approximation, or better still, a perfect symbolic representation (Tall, 2013, p. 303).

This approach may provide students with an intuitive comprehension of the concept of derivative and encourage the symbolization of the concept through arithmetic calculations or refined symbolic representations. Thus, the new proposed task is the transition from a purely symbolic abstraction to a more tangible and visual interpretation of the derivative. We understand that the notion of local straightness may compose an REM for the teaching of this concept.

In Almeida (2017), the generic organizer Magnify was adapted into an application developed in GeoGebra to facilitate work with the cognitive root of local straightness. In Figure 3, we present an application called “MagnifyG” and its functionality for analyzing the local straightness of a differentiable function at a point. The example we chose is the function $y = x^2$ at a specific point.

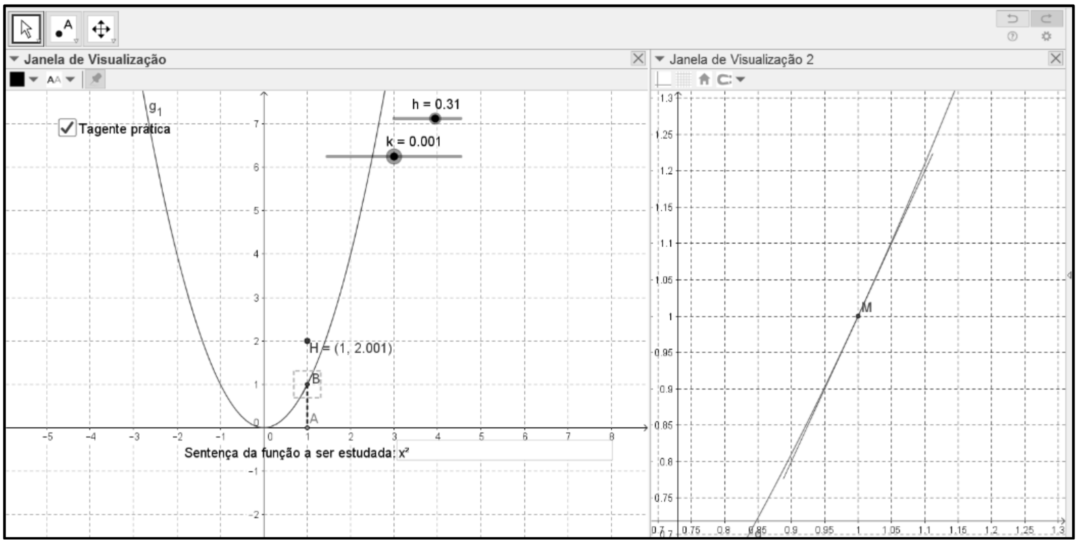

Figure 3.

Use of 'MagnifyG' for the function $y = x^2$.

The "MagnifyG" application consists of two viewing windows. The first window, on the left, contains a text field located in the lower part. In this text field, the user can insert the expression that defines the function, which, in this case, is $y = x^2$. Inside this window, two points are highlighted. Point A lies on the x-axis, whereas point B is on the graph of the function. Point B has a specific condition: it shares the same x-coordinate as point A, and its y-coordinate is the image of the x-coordinate through function f, denoted as (x(A), f(x(A))). Initially, only point A can be manipulated by the user with the mouse. When point A is moved, point B maintains the x-coordinate of A, while its y-coordinate is determined by the function f, thus preserving the original conditions. Furthermore, the first window contains a square in which point B serves as the intersection point of its diagonals. The length of each side of the square is twice the value of the sliding control h. The area of the plane that is shown in the second viewing window corresponds to the magnified portion of the square. The sliding control h is linked to the square.

In the second viewing window, there are two elements: point M and a graph. Point M shares the same coordinates as point B, whereas the graph represents the magnified portion of the graph of the function specified in the text field (in this case, $y = x^2$). The magnified part is highlighted by the square. The elements in this window cannot be altered by the user. This window has some particular characteristics: point M is always positioned in the center of the windows, and axis x and y are adjusted properly. Besides, the dimensions of this window are linked to the vertices of the square built in the first viewing window, thus defining the magnification.

In Almeida (2017), a set of tasks is developed using the notion of local straightness to introduce the concept of derivatives. We highlight here the first set of tasks:

> The first set consists of activities in which the notion of local straightness will be used with the following goals: to observe the existence of a value h to which the magnified portion of the graph resembles a line segment; to calculate the average rate of change of a function at two points belonging to the magnified portion of the graph, where the graph resembles a line segment (Almeida, 2017, p. 138, our translation).

In this set of tasks, there is an investigation about the calculation of the average rate of change of a function at two specific points within the magnified segment of the graph, where the graphic representation resembles a segment of a line r. In these tasks, it is suggested to explore the algebraic representations of the line that best approximates the tangent line t to the graph of this function at the point (x_0, $f(x_0)$), that is, to find an approximation of the form $f(x_0 + k) \approx ak + f(x_0)$, where $a \in \mathbb{R}$ (the slope of the line) and $x_0, x_0 + k \in Dom\, f$, (with x_0 being in the domain of the function f) and k is a variable for approximation, as suggested in the rough and ready approach described by Monaghan *et al.* (2023).

In summary, David Tall's theoretical approach for teaching derivatives, with focus on local straightness and on the exploration of non-polynomial functions, provides a rich and innovative view on the construction of the concept of derivatives. The introduction of examples, such as the blancmange and g, and the use of technological tools, such as the "MagnifyG" application, may provide students with an improved comprehension of the relationship between continuity and differentiability, besides facilitating the visualization and symbolization of the concept of derivatives. The integration of pedagogical practices that approach the rate of change and use the notion of local straightness, as suggested by Tall and complemented by Monaghan *et al*. (2023), may significantly enrich Calculus teaching beyond the prevailing epistemological model. These approaches do not only expand the prevailing epistemological model but may also foster an intuitive comprehension of Calculus concepts, aiming to enable their formalization.

Conclusion

In this paper, we presented contributions by David Tall, a British researcher, that may assist in the development of an REM for the concept of derivatives.

Tall's theoretical contributions, such as local straightness and the analysis of non-polynomial functions, are fundamental for the construction of an REM. Tall's approach expands the current epistemological comprehension of the concept of derivatives by adding functions like blancmange and $g\colon[-1,1] \to \mathbb{R}$, defined by $g(x) = \frac{1}{\left[\frac{1}{x}\right]}$, for $x \neq 0$ and $g(0) = 0$, which challenge current conceptions of continuity and differentiability. These examples do not only enrich the understanding of the concept of derivatives, but they also provide a theoretical foundation that can be integrated into an REM. The notion of rate of change, another meaningful epistemological object, is approached by graphic magnification, enabling an intuitive comprehension of derivatives and their graphic representation.

It is crucial to acknowledge that, while these theoretical concepts are essential for the formation of an REM, practical activities that explore the concepts should be part of the Study and Research Path (SRP) or of Study and Research Activities (SRA). An REM should focus on the definition and exploration of the epistemological objects that underpin the concept of derivatives, while practical and investigative tasks are developed separately, promoting a concrete application of concepts.

In summary, the integration of Tall’s ideas into an REM for teaching derivatives provides a theoretical foundation that enriches the comprehension of the concept. The functions explored and the notion of local straightness are valuable epistemological elements that, when properly integrated into an REM, help to construct a meaningful understanding of derivatives. However, it is important to establish a difference between the epistemological objects that form an REM and the practical tasks that are part of an SRP, thus ensuring a clear and well-structured methodological approach to the teaching of derivatives.

Furthermore, we understand that changing from a Dominant Epistemological Model for the teaching of derivatives may present challenges. Ely (2021) emphasizes we still face several institutional restrictions on the use of infinitesimals for the teaching of derivatives. The researcher says that:

> There are still several major institutional limitations on the teaching of calculus through infinitesimals or differentials. Students may face confusion and opposition from their colleagues and instructors. They might find it more difficult to interact with tutors or learn through standard online resources. Textbooks that use infinitesimals are (still?) not supported by large multimillion-dollar publishing houses. Instructors that use these approaches have reported resistance from some of their peers, especially 30 years ago, when Non-standard Analysis was less popular as a rigorous basis for the teaching of infinitesimals (Pittenger, 1995). These institutional factors may make teaching calculus through infinitesimals and/or differentials seem like swimming upstream. However, if not now, when will he have a good reason to bring about such a change? The benefits to students’ comprehension of calculus, as research has unveiled, make all efforts worthwhile. (Ely, 2021. p. 601).

Thus, as stated by Gáscon (2014), it is necessary to analyze critically the Dominant Epistemological Model in Mathematics teaching, especially when introducing the concept of derivatives. Ely (2021) highlights the existence of institutional limitations to teaching calculus through infinitesimals, pointing out resistance from some peers and instructors. Therefore, we understand that it is necessary to question and reformulate the established models and that theoretical constructs developed within the scope of Mathematics Education could help in this task.

Besides, when a Dominant Epistemological Model is analyzed, as in the case of teaching derivatives with the introduction of the concept through a formal definition, resistance and challenges related to teaching through non-conventional approaches are expected. Ely (2021) reports these challenges and resistance faced by those instructors who choose to teach calculus through infinitesimals, which include opposition from peers, difficulty in interacting with tutors and lack of support from major publishing houses.

We understand that, for the development of an REM, results from research on Calculus teaching conducted by Mathematics Education researchers could be used to contribute reflections on the practice of teachers who teach Calculus-related subjects (functions, limits, derivatives and integrals). We highlight that it is necessary to acknowledge the importance of integrating theoretical results into the creation of educational resources for Mathematics teaching. According to Almeida (2017), another possibility is the development of materials based on theoretical constructs established by other researchers in the field of Mathematics Education, such as "Dubinsky and Sfard, who presented cognitivist theoretical constructs that may be considered in the development of teaching activities" (Almeida, 2017, p. 204, our translation). Thus, we emphasize that an REM is an important tool for a critical analysis of Mathematics teaching, enabling an emancipatory and reflective approach to the teaching and learning process.

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