ISSN 1983-3156
Qualis A1

http://dx.doi.org/10.23925/1983-3156.2024v26i1p523-538

Epistemological study of standard deviation

Estudio epistemológico del concepto de desviación típica

Estudo epistemológico do conceito de desvio-padrão

Étude épistémologique du concept d'écart type

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Abstract

Since ancient times, mathematics has displayed a high level of creativity and impressive dynamism. However, in teaching/learning programs, they appear as relics to be displayed within the walls of the school. To break with this archaism, Statistics has emerged as the part of mathematics that can shed light on its dynamism and societal roots. Here, too, computational, and theoretical aspects have left little room for a clear understanding of the concepts studied. The aim of this article is to show, through the epistemological study of the notion of standard deviation, how the study of the evolution of concepts can help us understand their meaning and serve as a resource for their teaching.

Keywords: Standard deviation, Epistemological study, Interpretation, Statistics.

Resumen

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Desde la antigüedad, las matemáticas han demostrado un altísimo nivel de creatividad y una dinámica impresionante. Sin embargo, en los programas de enseñanza y aprendizaje aparece como una reliquia que debe exhibirse entre los muros de la escuela. Para romper con este arcaísmo, la Estadística ha surgido como la parte de las matemáticas que puede arrojar luz sobre su dinamismo y sus raíces sociales. También en este caso, los aspectos computacionales y teóricos han dejado poco margen para una comprensión clara de los conceptos estudiados. El objetivo de este artículo es mostrar, a través de un estudio epistemológico de la noción de desviación típica, cómo el estudio de la evolución de los conceptos puede ayudarnos a comprender su significado y servir de recurso para su enseñanza.

Palabras clave: Desviación típica, Estudio epistemológico, Interpretación, Estadísticas.

Resumo

Desde a antiguidade, a matemática tem demonstrado um nível muito alto de criatividade e uma dinâmica impressionante. Entretanto, nos programas de ensino e aprendizado, ela aparece como uma relíquia a ser exibida dentro das paredes da escola. Para romper com esse arcaísmo, a Estatística surgiu como a parte da matemática que pode lançar luz sobre seu dinamismo e suas raízes sociais. Também nesse caso, os aspectos computacionais e teóricos deixaram pouco espaço para uma compreensão clara dos conceitos estudados. O objetivo deste artigo é mostrar, por meio de um estudo epistemológico da noção de desvio padrão, como o estudo da evolução dos conceitos pode nos ajudar a entender seu significado e servir como um recurso para ensiná-los.

Palavras-chave: Desvio-padrão, Estudo epistemológico, Interpretação, Estatísticas.

Résumé

Les mathématiques, depuis l’antiquité, ont montré un niveau de créativité très élevé et un dynamique impressionnant. Cependant, dans les programmes d’enseignement-apprentissage, elles apparaissent comme des vestiges à exposer dans les murs de l’école. Pour rompre cet archaïsme, la Statistique est apparue comme la partie des mathématiques qui vienne éclairer son dynamise et son ancrage sociétal. Là également, les aspects calculatoires et théoriques ont laissé peu de place à une claire compréhension des concepts étudiés. Cet article a pour objet de montrer, à travers l’étude épistémologique de la notion d’écart type, comment l’étude de l’évolution des concepts permette de comprendre leur sens et serve de ressources pour leur enseignement.

Mots-clés : Écart type, Étude épistémologique, Interprétation, Statistique.

Clarification of the epistemological study of the concept of standard deviation

Epistemology is the name of the discipline that studies the way we know (Fourez and Larochelle, 2002). For Dorier (1997), the word "epistemology" covers a variety of conceptions, but in mathematics, it plays the role of mediator between historical and didactic work. While the history of mathematics recounts the events of the past that marked the creation of mathematical objects, the didactics of mathematics studies the process through which mathematical objects are transmitted and acquired and explains the links between the teaching and learning of mathematical objects. But to do this, didactics needs epistemology to study how mathematical objects are created, the obstacles encountered in creating these objects and the interactions between different mathematical objects. For the didactician, epistemology meditates on historical culture, to facilitate its restoration as a learning object and a teaching tool.

Dorier (1997) draws on the different conceptions of the word "epistemology" to define the adjective "epistemological" in the context of mathematics (p.16). For him, an epistemological study is a study of the evolution[4] of mathematical knowledge. Thus, to better understand a mathematical teaching object, the researcher must go back to the sources of knowledge and analyze the process that leads from the production of this object in scholarly knowledge to its constitution as an object of knowledge to be taught.

Thus, in Statistics, the epistemological study of the concept of standard deviation describes the evolution of the concept, the conditions of its creation, its uses, the process of its passage from scholarly knowledge to knowledge to be taught, and the distance that exists between the mathematical object, the object of teaching and the object taught. However, in this text, we limit ourselves to describing the context in which the concept of standard deviation was constructed, to deduce an interpretation.

Appearance, origins, and persistence of observation errors

In the fields of Astronomy[5] and Geodesy[6], problems of objectivity of measurements carried out on the same celestial or terrestrial object were often raised by scientists. We think of a measurement as the determination of the number of elements that make up a finite set of objects (e.g., measuring the number of particles in the universe), or as the association of a physical quantity with a number by comparison with a reference unit of the same kind (e.g.,

[4] In this context, evolution means progress, stagnation, or decline.
[5] Astronomy is the science of the stars and the universe.
[6] Geodesy is the science of determining the shape of the earth and measuring its dimensions to draw maps.

measuring the volume of a pile of sand, measuring the temperature of a body, measuring the length of a stick, measuring the area of a surface). In the first case, the result of the measurement is a number without a unit, and in the second case, the result of the measurement is a number expressed in the chosen reference unit.

Eighteenth-century astronomers and their predecessors noted a glaring disparity between the results of measurements of a quantity related to an astronomical or geodesic object. Indeed[7], measurements made by several different observers on the same object, or the same quantity produced different results (fifty people each measuring the dimensions of the same table produced the same number of results); several measurements made by the same observer on the same quantity also produced different results (a person weighing an object a thousand times to determine its mass may obtain a thousand different masses). The same was true of the results obtained when several objects of the same kind were measured by the same observer (a pediatrician taking the body temperature of twenty-five-year-olds would end up with twenty different values).

The astronomers of the Age of Enlightenment, convinced that a measured magnitude of any object has only one "true value" (exact value of the magnitude), attributed the disparity in measurement results to observational errors. They thus qualified all measurements as subjective, since they were tainted by random errors, i.e., errors linked to chance from an unknown source. Random measurement errors are classified into three types: systematic errors, accidental errors, and statistical dispersion.

Systematic errors are most often caused by the measuring device (manufacturing defect, adjustment error, etc.). They can be avoided by the observer but are generally difficult to detect.

Accidental errors occur by chance. For example, when an observer measures an object several times with an optical instrument (direct observation[8]) to determine its true value, he obtains various results with errors, despite having taken all the necessary precautions to make an accurate measurement. These persistent errors are unavoidable and may be due to the observer's dizziness when taking measurements, to anarchic variations in atmospheric refraction, to various vibrations, etc. All these accidental errors may be due to the observer's inability to make an accurate measurement. All these accidental errors will have an impact on the accuracy of the measurement, i.e., because of these unavoidable errors, the measurement

[7] Here, we use examples from contemporary society to help the reader understand.
[8] Direct observations are those made directly on a quantity. Example: measuring the length of a table, the mass of a stone, ...

made will never be accurate. The observer can repeat the same measurement ten times, a hundred times, a thousand times, etc., as many times as he likes, and still obtain ten, a hundred, a thousand, etc., different values: this is **statistical dispersion**. The situation is even more complicated in the case of indirect observations[9], where the result is obtained by combining several equations involving several unknowns (Bru, 2006).

The errors noted in direct and indirect observations of astronomical and geodetic phenomena were an obstacle to the existence of Astronomy and Geodesy as robust, objective, and universal natural sciences. Indeed, a universal natural science cannot be based on errors and subjectivity. Accidental errors in astronomical and geodetic measurements presented considerable challenges, some of which we identify in the next section.

The challenges of astronomical and geodetic errors

Measurement errors presented major challenges for society, as well as for scientific knowledge. The first challenge was linked to the development of maritime trade, and therefore to the need to provide navigators with techniques that would enable them to accurately determine their position at sea, thanks to the observation of the stars (Noel and Tilleuil, 2005). These errors were at the root of many maritime disasters (Armatte, 2004). The second challenge is linked to the existence of Astronomy and Geodesy as universal sciences. In the field of geodesy, the problem of the shape of the earth and the degree of flatness of the globe were posed. If the first problem was solved by expeditions, the second remained a matter of speculation for a long time. In fact, the evaluation of the earth's flattening coefficient gives rise to an alarming disparity depending on the pairs of measurements taken, which highlights the interplay of errors that disqualifies each determination (*Ibidem*).

Because of the challenges posed by astronomical and geodetic errors, scientists tried to eliminate them by looking for systematic errors in measuring equipment, identifying other sources of error that could be eliminated and increasing the accuracy of measuring instruments, but they were unsuccessful. Despite all their efforts, there were still irreducible residual errors. So, they called on geometers to find solutions to the problem of measurement variability by modeling irreducible errors. This gave rise to the theory of observation errors, the aim of which

[9] Indirect observations are those made on several quantities linked by an equation with fixed but unknown coefficients. (Example: to determine the volume V of an object in the shape of a cone of revolution, you need to measure the height of the cone h and the radius of the base disk r. In fact, $V = \frac{1}{3}\pi r^2 h$). Determining the volume of a cone of revolution is an indirect observation.

is to model measurement errors to estimate the unknown "true value" of a quantity measured several times. The causes of residual (accidental) errors will never be known, because "in its metaphysical nature, the accidental error is an epistemic error" (Armatte, 2010).

Modeling observation errors

In the previous section, we explained that measurements derived from direct and indirect observations always contain unavoidable errors from unknown sources. The challenges posed by these accidental errors led astronomers to call on geometers to model errors with the aim of reducing them. The measurement chosen is that which contains the smallest possible error, and which contains the essential information contained in the various observations made.

Here, we explain the contribution of mathematics to solving the problem of modeling observation errors. The question mathematicians had to answer was: how can several measurements of the same quantity of the same measurable object be combined to minimize the final error on the "true value" of that quantity?

To address this issue, mathematicians committed themselves, around 1750, to modeling astronomical and geodetic errors under two conditions and for three main reasons. To model observational errors, mathematicians required that these be made **under roughly** the same conditions, with the same precautions, and be independent. Furthermore, the errors to be modeled must be irreducible by physical means, and their sources must be unknown. Mathematicians are therefore committed to modeling only accidental errors. The three main reasons why mathematicians decided to model errors were:

1. the tools needed to calculate measurement errors were already available at the time: probability calculus and differential calculus.
2. measuring equipment was more efficient, so there was no longer a huge difference between the different values found. They were approximately equal.
3. certain problems, such as the shape and dimensions of the earth, had not yet been solved.

At this level, mathematicians were seeking **to estimate the true value of a quantity of an object for which several measurements had been taken, yielding different results**.
To make this estimate, they first had to make a choice. Indeed, scientists were torn between two ideas: (1) make a single "good measurement" or (2) make several measurements and take their middle ground.

In the early days of research, mathematicians, who were not necessarily practicing astronomers, preferred to choose a single "good measurement" rather than make several measurements and take their middle ground. They were unable to justify their choice, but it seems that the marked improvement in measuring instruments had guided their decision.

Another reason why these mathematicians opted for a single "good measurement" was that some of them were convinced that, in the case of mainly indirect observations, errors would increase as a function of the aggregation of equations resulting from measurements.

Alongside mathematicians who favor a single "good" observation, there are others who have opted for the second choice, i.e., multiplying measurements and taking their middle. Scientists who disagreed with the choice of a single good measurement proposed other methods of reducing accidental errors, the most common of which was the search for an error "medium".

Many scientists opted for a middle of several observations as the value likely to represent the true value of a quantity, but they hadn't specified whether this middle was the arithmetic mean, the harmonic mean, the geometric mean or the median. Nor did they provide convincing explanations; most of them were guided by intuition. For example, Tycho Brahe (1546-1601) implicitly used the arithmetic mean to eliminate observational errors in a set of data on planetary motion. Roger Cotes (1682-1716) also advocated the arithmetic mean indirectly. In fact, in an appendix to his Harmonia Mensurarum, published posthumously in 1722, Cotes proposes the use of a weighted average whose coefficients are inversely proportional to the "dispersion" of observations, to determine the exact position of a point for which he has 4 observations, not all of which are equally reliable (Droesbeke and Tassi, 1990). Ruggero Boscovich (1711-1787) and Thomas Simpson (1710 -1761) were also in favor of a mean of observations or mean of errors, but the mathematical form of this mean or of the best medium that would represent a set of measurements had still not been identified. The problem of finding an appropriate medium remained unresolved.

The search was on for an error distribution law that would enable us to select the optimum medium containing the smallest possible error. Scientists are then led to explore such an error distribution law and determine the average error over a set of measurements.

Standard deviation: the average error over a set of measurements

In this section, we focus on the reasoning of Carl Frederich Gauss, which led to the discovery of the standard deviation. At the start of his work, Gauss put forward the thesis that all measurements of physical quantities obtained during observations are inevitably marred by errors of varying magnitude, despite all the care that the experimenter may take in making the observations.

What's more, these errors result from several generally distinct sources. Thus, he classifies measurement errors into two categories according to the nature of their sources: regular or constant errors, and irregular or fortuitous errors. Regular errors are those generally produced in observations of the same nature, i.e., observations in which the same object is measured several times. For each observation, either the same error is committed by the observer, or the error committed depends on an imperfect subdivision of the measuring instrument or other identifiable circumstances. In short, these errors, if not identical, depend on circumstances essentially linked to the results of the observation. In this case, the source of the error may be known from the results obtained. Errors originating from these identifiable sources are called constant errors. These are the systematic errors defined above. For example, if we repeatedly measure the current intensity in an electric circuit using the same ammeter for each observation, and we record as many results as observations, then we can say that the ammeter is faulty.

The errors recorded in this case are regular errors linked to the faulty ammeter, and to eliminate them we need to repair or replace the device. Irregular errors are those made in observations of different natures or of the same species. Their sources depend on variable circumstances that are independent of the observer. It is impossible to accurately identify the sources of irregular errors from observation results. Indeed, as Gauss states:

> Certain causes of error depend, for each observation, on circumstances that are variable and independent of the result obtained: these errors are called irregular or fortuitous, and like the circumstances that produce them, their value cannot be calculated. Such are the errors that arise from the imperfection of our organs, and all those that are due to irregular external causes, such as, for example, the trepidations of the air that make vision less sharp; some of the errors due to the inevitable imperfection of the best instruments belong to the same category. Examples include the roughness of the inner part of the level, lack of absolute rigidity, etc. (Gauss, 1855, p. 9).

Errors that Gauss describes as irregular or fortuitous are unavoidable accidental errors. Like his predecessors, Gauss opted to model accidental errors rather than systematic ones. Accidental errors cannot be eliminated, but Gauss asserts that their influence can be reduced as much as possible by a skillful combination of observational results. Gauss's method for reducing accidental errors is known as the method of least squares.

Gauss considers the "true value" of a measured quantity, which we call G, as a function with several unknowns. In other words, the true value V of a physical quantity is a function that can be written as *ap* + *bq* + *cr* + *ds* + ..., where *a*, *b*, *c*, *d*, ... are observables (numbers that vary from one measurement to another, but are known) and *p*, *q*, *r*, *s*, . . . are unknowns: *V (p*, *q*, *r*, *s*, . . .) = *ap* + *bq* + *cr* + *d s* + …. This "true value" is a theoretical value.

Parameters *p*, *q*, *r*, *s*, … are *m* ($m \in \mathbb{N} \setminus \{0;\ 1\}$).

Considering a natural number n as large as possible, Gauss assumes that he has n quantities G_1, G_2, . . ., G_n of the same kind and true values V_1, V_2, . . ., V_n respectively. It also assumes that the observations of these quantities respectively give the measured values M_1, M_2, . . ., M_n and that the realization of one observation has no influence on that of another. Each of the n measured values M_1, M_2, . . ., M_n depends on the parameters *p*, *q*, *r*, *s*, ...

Gauss is interested in errors, which are the variations between the true values V_1, V_2, . . ., V_n and the measured values M_1, M_2, . . ., M_n. If the errors are denoted Δ_1, Δ_2, . . ., Δ_n, where Δ_i is the error committed on the measurement of true value V_i for $i = 1, \ldots, n$, then we can write: $\Delta_1 = M_1 - V_1$, $\Delta_2 = M_2 - V_2, \ldots$, $\Delta_n = M_n - V_n$.

Gauss distinguishes three situations that can arise when measuring quantities: (1) the number of functions is strictly less than the number of unknowns (n < m). (2) the number of functions is equal to the number of unknowns (n = m). (3) the number of functions is strictly greater than the number of unknowns (n > m). In other words, the number of observed quantities is either strictly smaller than, equal to or strictly greater than the number of unknown parameters on which the values V_1, V_2, . . ., V_n depend. He is particularly interested in the latter situation. In the first case, the problem is indeterminate and therefore has no solutions. In the second case, the problem is determinate, i.e., it admits at most a single solution. In the third case, the problem is overdetermined, and it is to this problem that scientists are struggling to find a theoretically and empirically justifiable solution.

Now that the problem has been identified, Gauss associates with the error Δ a discontinuous function, unknown in practice, which he calls φ and such that $\varphi(\Delta)$ is the probability that the error Δ is made. Next, he imposes natural conditions on this function. Indeed, he describes the following:

> First, let's assume that the situation in all the observations is such that there is no reason to regard any one of them as more accurate than another, i.e., that equal errors in each

of them must be regarded as equally probable. The probability of an error Δ being made in one of the observations will be a function of Δ, which we will call $\varphi(\Delta)$. Although this function cannot be assigned in a precise manner, we can at least state that it must become maximum for Δ = 0, have in most cases the same value for equal values of Δ and of opposite signs, and, finally, and vanish when Δ is given a value equal to or greater than the maximum error ; $\varphi(\Delta)$ must therefore, strictly speaking, be assigned to the class of discontinuous functions, and if, for ease of calculation, we allow ourselves to substitute an analytical function, the latter must be chosen in such a way that it tends rapidly towards 0 from two values of Δ, one greater, the other less than 0, and that outside these two limits it can be considered as zero. Now the probability that the error lies between Δ and a quantity $\Delta + d\Delta$ that differs infinitesimally little from it, will be expressed by $\varphi(\Delta)d\Delta$, and, consequently, the probability that the error lies between D and D′ by $\int_{\mathrm{D}}^{\mathrm{D}'} \varphi(\Delta)\mathrm{d}\Delta$. This integral, taken from the largest negative value of Δ to its largest positive value, or more precisely from $\Delta = -\infty$ to $\Delta = +\infty$, will necessarily have to be equal to 1. So, we have $\int_{-\infty}^{+\infty} \varphi(\Delta)\mathrm{d}\Delta = 1$.

By the above quotation, Gauss means that the function φ verifies the following conditions:

- $\varphi(\Delta)$ reaches its maximum at $\Delta = 0$ and decreases from 0. This means that $\varphi(\Delta)$ is maximum if the observed value is exactly the true value of the quantity G. Furthermore, small errors are more numerous than large errors, or it is more common to make small errors than large ones.
- $\varphi(\Delta) = \varphi(-\Delta)$. φ is even and the observed values are symmetrical with respect to the true value of quantity G, i.e., the errors are symmetrical with respect to 0. Two errors with opposite signs and the same absolute value have the same chance of occurring - same probability of occurrence.
- φ is a positive analytical function for values of Δ within the error limits, and zero for values of Δ outside these limits. If $\Delta \in]D;\ D'[$ then $\varphi(\Delta) > 0$; otherwise $\varphi(\Delta) = 0$.
- $\int_{-\infty}^{+\infty} \varphi(\Delta).\, d\Delta = 1$. In an orthogonal reference frame, the area of the part of the plane bounded by the x-axis, the curve of φ and the straight lines of equations $y = D$ and $y = D'$ is equal to unity.

Under these conditions, the probabilities that measurements on $G_1, G_2, \ldots, G_n$ give $M_1, M_2, \ldots, M_n$ are respectively $\varphi(\Delta_1), \varphi(\Delta_2), \ldots, \varphi(\Delta_n)$.

Observations such as $\Delta_1 = M_1 - V_1,\ \Delta_2 = M_2 - V_2, \ldots,\ \Delta_n = M_n - V_n$ are considered mutually independent events, so the probability of the intersection of the events is equal to the product of the probabilities of the different events. This translates into:

$$\mathbb{P}(V_1 = M_1, V_2 = M_2, \ldots, V_n = M_n) = \mathbb{P}(V_1 = M_1) \times \mathbb{P}(V_2 = M_2) \times \ldots \times \mathbb{P}(V_n = M_n).$$

The probability of the intersection of the events is denoted Ω. Then we have:

$$\Omega = \mathbb{P}(V_1 = M_1) \times \mathbb{P}(V_2 = M_2) \times \ldots \times \mathbb{P}(V_n = M_n).$$

Now, $\mathbb{P}(V_i = M_i) = \varphi(\Delta_i)$ so, $\Omega = \varphi(\Delta_1) \times \varphi(\Delta_2) \times \cdots \times \varphi(\Delta_n)$.

At this point, Gauss confirms that "the most probable system of values of p, q, r, s, … corresponds to the maximum of Ω".

So,

$$\frac{d\Omega}{dp} = 0, \qquad \frac{d\Omega}{dq} = 0, \qquad \frac{d\Omega}{dr} = 0, \ldots.$$

Ω depends on the parameters $p, q, r, s, \ldots$ so we can equate it with the likelihood function. It's simpler to work with the logarithm of this function than with the function itself, since on the one hand, we're looking for the parameter values at which the likelihood function Ω reaches its maximum, and on the other, both it and its logarithm admit the same maximum. Then:

$$\ln(\Omega) = \ln[\varphi(\Delta_1) \times \varphi(\Delta_2) \times \cdots \times \varphi(\Delta_n)].$$

$$\ln \Omega = \ln[\varphi(\Delta_1)] + \ln[\varphi(\Delta_2)] + \ldots + \ln[\varphi(\Delta_n)]. \qquad (2.1)$$

Deriving this last equality with respect to the variables $p, q, r, s, \ldots$, we obtain the following equations:

$$\frac{\varphi'(\Delta_1)d\Delta_1}{dp} + \frac{\varphi'(\Delta_2)d\Delta_2}{dp} + \frac{\varphi'(\Delta_3)d\Delta_3}{dp} + \cdots + \frac{\varphi'(\Delta_n)d\Delta_n}{dp} = 0$$

$$\frac{\varphi'(\Delta_1)d\Delta_1}{dq} + \frac{\varphi'(\Delta_2)d\Delta_2}{dq} + \frac{\varphi'(\Delta_3)d\Delta_3}{dq} + \cdots + \frac{\varphi'(\Delta_n)d\Delta_n}{dq} = 0$$

$$\frac{\varphi'(\Delta_1)d\Delta_1}{dr} + \frac{\varphi'(\Delta_2)d\Delta_2}{dr} + \frac{\varphi'(\Delta_3)d\Delta_3}{dr} + \cdots + \frac{\varphi'(\Delta_n)d\Delta_n}{dr} = 0$$

....

At present, the problem could be solved by elimination if the nature of the function φ' or that of φ were known, but since the function φ cannot be determined a priori, Gauss approached the question differently by looking for "a function tacitly accepted as a basis, by virtue of a simple and generally accepted principle". It was at this point that he turned to the arithmetic mean. Indeed, he describes it as follows:

> It is customary to regard as an axiom the hypothesis that if a quantity has been obtained by several immediate observations, made with the same care in similar circumstances, the arithmetic mean of the values observed will be the most probable value of this quantity, if not in all rigor, at least with a large approximation, so that the safest thing is always to stop there.

So, the problem now is to choose the function φ such that the arithmetic mean is the representative value of the observed values. Gauss poses:

$$V_1 = V_2 = \cdots = x \text{ et } x = \frac{M_1 + M_2 + \cdots + M_n}{n}.$$

$\Omega = \varphi(M_1 - x) \times \varphi(M_2 - x) \times \ldots \times \varphi(M_n - x)$. From equality (2.1), Ω is maximum if:

$$\frac{[\varphi(M_1 - x)]'}{\varphi(M_1 - x)} + \frac{[\varphi(M_2 - x)]'}{\varphi(M_2 - x)} + \cdots + \frac{[\varphi(M_n - x)]'}{\varphi(M_n - x)} = 0. \qquad (2.2)$$

Let's ask $\Delta_i = M_i - x$ and $F(\Delta_i) = \frac{[\varphi(M_i - x)]'}{\varphi(M_i - x)} = \frac{[\varphi(\Delta_i)]'}{\varphi(\Delta_i)}$, for all $i = 1, \ldots, n$.

Furthermore, $[\varphi(M_i - x)]' = (M_i - x)' \times \varphi'(M_i - x) = -\varphi'(M_i - x)$ i.e.

$[\varphi(\Delta_i)]' = -\varphi'(\Delta_i)$. So, $F(\Delta_i) = -\frac{\varphi'(\Delta_i)}{\varphi(\Delta_i)}$.

According to (2.2), $F(\Delta_1) + F(\Delta_2) + \cdots + F(\Delta_n) = 0$. (2.3)

$F(-\Delta_i) = \frac{[\varphi(-\Delta_i)]'}{\varphi(-\Delta_i)}$. Since the function φ is even, then $\varphi(-\Delta_i) = \varphi(\Delta_i)$ and,

$[\varphi(-\Delta_i)]' = [\varphi(x - M_i)]' = \varphi'(-\Delta_i) = \varphi'(\Delta_i)$.

Then, we have, $F(-\Delta_i) = \frac{\varphi'(\Delta_i)}{\varphi(\Delta_i)} = -F(\Delta_i)$. It appears that F is an odd function.

Furthermore, $M_1 + M_2 + \cdots + M_n = nx$.

This implies that $\Delta_1 + x + \Delta_2 + x + \cdots + \Delta_n + x = nx$.

Hence, $\Delta_1 + \Delta_2 + \cdots + \Delta_n = 0$. We deduce that $\Delta_2 + \cdots + \Delta_n = -\Delta_1$.

So, $F(\Delta_2 + \cdots + \Delta_n) = F(-\Delta_1) = -F(\Delta_1)$.

According to (2.3), $F(\Delta_2) + \cdots + F(\Delta_n) = -F(\Delta_1)$.

$F(\Delta_2 + \cdots + \Delta_n) = -F(\Delta_1)$ et $F(\Delta_2) + \cdots + F(\Delta_n) = -F(\Delta_1)$ then we deduce that $F(\Delta_2 + \cdots + \Delta_n) = F(\Delta_2) + \cdots + F(\Delta_n)$. Hence F is a linear function.

Linear functions that verify the equality $F(\Delta_2 + \cdots + \Delta_n) = F(\Delta_2) + \cdots + F(\Delta_n)$ are of the $F(\Delta_i) = k\Delta_i$ where k is a constant.

$$F(\Delta_i) = k\Delta_i \Leftrightarrow \int F(\Delta_i) d\Delta_i = \int k\Delta_i d\Delta_i.$$

$$\Leftrightarrow \int -\frac{\varphi'(\Delta_i)}{\varphi(\Delta_i)} d\Delta_i = \int k\Delta_i d\Delta_i.$$

$$\Leftrightarrow \int \frac{\varphi'(\Delta_i)}{\varphi(\Delta_i)} d\Delta_i = \int -k\Delta_i d\Delta_i.$$

$\Leftrightarrow \ln \varphi(\Delta_i) = -\frac{1}{2} k {\Delta_i}^2 + \ln k_1$ où k_1 is a positive constant.

$$\Leftrightarrow \varphi(\Delta_i) = \exp\left(-\frac{1}{2} k {\Delta_i}^2 + \ln k_1\right)$$

$$\Leftrightarrow \varphi(\Delta_i) = \exp\left(-\frac{1}{2}k{\Delta_i}^2\right) \times \exp(\ln k_1)$$

$$\Leftrightarrow \varphi(\Delta_i) = k_1 \exp\left(-\frac{1}{2}k{\Delta_i}^2\right).$$

With the function φ determined, it now remains to determine Ω. We have:

$$\Omega = \prod_{i=1}^{n} \varphi(\Delta_i) = \prod_{i=1}^{n} k_1 \exp\left(-\frac{1}{2}k{\Delta_i}^2\right).$$

$$\Omega = (k_1)^n \times \exp\left(\frac{1}{2}\sum_{i=1}^{n} -k{\Delta_i}^2\right).$$

$$\Omega = (k_1)^n \times \exp\left(-\frac{1}{2}k\sum_{i=1}^{n} {\Delta_i}^2\right).$$

Ω is maximal if and only if k is positive, i.e., if $\sum_{i=1}^{n} {\Delta_i}^2$is minimal. And, in this case, *x* is the arithmetic mean of the M_i measurements.

If the measurement errors Δ_1, Δ_2, . . ., Δ_n are made by performing measurements of respective values M_1, M_2, . . ., M_n then the squared least error to be feared from these observations is $\frac{\sum_{i=1}^{n} {\Delta_i}^2}{n}$. The least error to be feared then becomes $\sqrt{\frac{\sum_{i=1}^{n} \Delta_i^2}{n}}$.

In 1893, during a lecture he gave at the Royal Society in London, Karl Pearson gave the mean error to be feared the name standard deviation and denoted it by the Greek letter σ (Dodge, 2010).

Conclusion

The standard deviation, formerly known as the mean error, to be feared, was born during research in which it was not central. In fact, it was discovered between the 18th and 19th centuries during the search for a law of error distribution. The aim of this research was to make Astronomy and Geodesy objective and viable sciences. This shows that what was an accident of history has become an effective tool for studying the notion of dispersion, and an object of teaching.

At the end of this study, we were able to show that mathematical concepts are not epistemological absolutes. They are human constructs, and their teaching should take this into account, restoring the vicissitudes that marked their moment of discovery. The study showed that the arithmetic mean only makes sense if we want to estimate the true theoretical value of a physical quantity. When we have a physical quantity whose true value we don't know, and for which we have a set of measurements that are not very far apart, then we can take the arithmetic mean of the measurements as an approximate value of the true value of this quantity. In this case, we lose information about the true value of the quantity. The information lost is the distance between the arithmetic mean of the measured values and the true value of the quantity from which the measurements were taken. It represents the precision, uncertainty, or margin of error for which the arithmetic mean is the approximate value of the "normal value" of a quantity. It is called the standard deviation.

The error contained in the arithmetic mean of a set of measurements is smaller than the error contained in each measurement of the set.

What research has not yet shown is that epistemological study is a potentially effective tool for teacher training. It would enable teachers to identify the historical development of the concepts to be taught, the losses and additions in the process of creating and transposing the concepts to be taught. A historical and epistemological study would also provide resources for both teachers and teacher trainers.

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