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# A discussion on the definition of the limit of a sequence

**Discusión sobre la definición del límite de una sucesión**

**Discussion sur la définition de la limite d'une suite**

**Uma discussão sobre a definição de limite de uma sequência**

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## Abstract

In this article we present a study on difficulties in the process of learning the definition of the limit of a sequence. It is a qualitative study whose aim was to analyze the actions of a subject when dealing with a situation involving this definition. To do this, we discuss the concepts involved in the conceptual field of this definition, along with the analysis of a proposed situation. The data was produced through written and oral production, collected from the activity resolution sheets and audio and video produced during the session. The analyses show the difficulty of disengaging from graphic representations used to deal with particular situations, in the case of converging sequences, even when the subject is confronted with a study of the conceptual elements involved in the formal definition.

**Keywords**: Limit of sequence, Formal definition, Representations, Higher Education.

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## Resumen

En este artículo presentamos un estudio sobre las dificultades en el proceso de aprendizaje de la definición de límite de una sucesión. Se trata de un estudio cualitativo cuyo objetivo fue analizar la actuación de un sujeto ante una situación que implicaba esta definición. Para ello, se discutieron los conceptos implicados en el campo conceptual de esta definición, junto con el análisis de una situación propuesta. Los datos se obtuvieron a través de la producción escrita y oral, recogida de las fichas de resolución de actividades y de audio y vídeo producidos durante la sesión. Los análisis muestran que es difícil desprenderse de las representaciones gráficas utilizadas para tratar situaciones particulares, en el caso de secuencias que convergen, incluso cuando el sujeto se enfrenta a un estudio de los elementos conceptuales implicados en la definición formal.



***Palabras clave***: Límite de secuencia, Definición formal, Representaciones, Enseñanza superior.



## Résumé

Dans cet article, nous présentons une étude sur les difficultés dans le processus d'apprentissage de la définition de la limite d'une suite. Il s'agit d'une étude qualitative dont l'objectif est d'analyser les actions d'un sujet face à une situation impliquant cette définition. Pour ce faire, nous avons abordé les concepts impliqués dans le champ conceptuel de cette définition, ainsi que l'analyse d'une situation proposée. Les données ont été produites par le biais de productions écrites et orales, recueillies à partir des fiches de résolution des activités et des productions audio et vidéo réalisées pendant la session. Les analyses montrent qu'il est difficile de se détacher des représentations graphiques utilisées pour traiter des situations particulières, dans le cas de séquences convergentes, même lorsque le sujet est confronté à l'étude des éléments conceptuels impliqués dans la définition formelle.



***Mots-clés*** : Limite d’une suite, Définition formelle, Représentations, Enseignement supérieur.



## Resumo

Neste artigo apresentamos um estudo sobre dificuldades no processo de aprendizagem da definição de limite de uma sequência. É uma pesquisa qualitativa cujo objetivo foi analisar as ações de um sujeito ao lidar com uma situação envolvendo essa definição. Para isso, trouxemos uma discussão sobre os conceitos envolvidos no campo conceitual dessa definição, juntamente com a análise de uma situação proposta. Os dados foram produzidos por meio da produção escrita e oral, coletada pelas folhas de resolução da atividade e por áudio e vídeo produzidos durante a sessão realizada. As análises evidenciam a dificuldade em se desvincular de representações gráficas mobilizadas para tratar situações particulares, no caso de sequências que convergem, mesmo quando o sujeito é confrontado com estudo dos elementos conceituais envolvidos na definição formal.

***Palavras-chave:*** Limite de sequência, Definição formal, Representações, Ensino Superior.

## A discussion on the definition of the limit of a sequence

In this article, we discuss and analyze a situation observed during the event Integrated Studies - Calculus in Secondary and Higher Education, held at the Federal University of Mato Grosso do Sul, in a session where the definition of the limit of a sequence was presented and discussed. This event is related to a research project approved by the CNPq 2021 general call for proposals. One of the objectives of this project is to study the Epistemological Reference Model - ERM (Chevallard, 1999) in different institutions in Brazil to understand how the concept of limit, in the *ecological* sense (Chevallard, 1994), lives in a given institution. To this end, different activities were applied and discussed, introducing the concept of limits, with the aim of building a MER around this concept.

Several studies have been carried out on the difficulties associated with the definition of the limit using quantifiers, whether of a function or a sequence (Bloch, 2017; Burigato, 2019; Fernandes, 2015). Some show that there are important epistemological obstacles related to the concepts involved in constructing the definition, such as the notion of infinity, function, etc. (Cornu, 1991), or to other concepts that appear in this definition, such as inequalities involving modulus, quantifiers, and the notation *f(x)* (Doumbia, 2020). This shows how some concepts that appear in the definition of limit can cause problems in understanding this concept.

In this sense, we agree with Vergnaud (1990) on the importance of considering the conceptual field involved in the construction of a given concept when we are interested in its learning. A concept is not presented in isolation; the author argues that presenting a definition is not enough for the student to understand all the aspects involved in the construction of a concept. It is important to consider the set of situations in which this concept is involved or presented to the student, the *operative invariants* related to the situations, in this case mathematical knowledge, as well as the set of representations used in the situations.

In the case of the situation, we are discussing in this text, we are going to discuss the importance of studying each concept involved in the definition presented to the students and how these elements can influence their understanding of the concept. For example, identifying which links are most relevant or which breaks are necessary to move forward in this understanding.

Below, we provide some theoretical background and a discussion of the mathematical elements involved in two proposals for introducing the definition of the limit of a sequence that were mobilized in this study.

## Theoretical elements mobilized to study the situation

For Vergnaud (1990), the conceptual field is made up of a set of situations involving the concept to be taught, together with a set of other concepts involved in its construction.

Another important theoretical element is the *schema*, which, for Vergnaud (1990) is the invariant organization of the subject's action when dealing with a set of situations that have a certain similarity. Thus, by recognizing this similarity with other situations he has faced, he identifies the objective and the sub-objectives, and through rules in action "if I do this... then I will have...", he selects the relevant concepts and the associated mathematical properties, which are the *operative invariants*, in this case the *concepts in action* and the *theorems in action*, the latter of which may not be correct from a mathematical perspective. At the same time, he has control mechanisms over the actions being carried out as well as inferences about what is going to happen; these actions are carried out autonomously with comings and goings.

So when the student has to deal with a situation involving the construction of a new concept, or even involving new aspects of a concept in development, he will mobilize a schema that has been effective in dealing with situations that he has somehow recognized as similar to an activity that he has had to deal with and that he has been successful at. In other words, they have managed to solve the problem. For example, when introducing the concept of the limit of a function at a point, it is common to use an intuitive definition involving algebraic notation $\lim_{x \to p} f(x) = L$ and, in natural language, expressions such as "*if x tends to p, then f(x) tends to L*". Sometimes limits are first solved by calculating values of *f(x),* using tables, for values of *x* close to *p*. When a beginner is faced with cases like this $\lim_{x \to 2} \frac{x^2-4}{x-2}$, they will mobilize a scheme they are developing to construct the new concept. They will probably start by calculating images of the given function, choosing points close to two, to the right, and to the left of two because they think they need to find out what happens to the values of the function "*when x tends to p*", in which case *p=2*. We can infer that he is using the theorem in action: *If I need to find the limit of a function, then I need to calculate the values of the images near the point of investigation of the limit.* This theorem may be effective for this situation if he doesn't do the math wrong, but we know that this knowledge is not enough to solve or deal with all limit situations. On the contrary, it can lead to important errors in the construction of this concept.

Indeed, Job and Schneider (2016) argue that students have great difficulty when they have to make the transition from the informal aspects of calculus to understanding the more formal aspects. The didactic choices proposed in teaching to introduce the concept of limits can make this transition very difficult, such as the use of *ostensives* to illustrate particular cases of studying limits, the mobilization of some graphical representations, or the use of tables.

In this case, to understand the concept and the teaching object of the situation, the student will have to make what Vergnaud calls breaks with previous knowledge that has been mobilized in their schemas to deal with other activities. As in the case of ostensible, the study of limits is sometimes linked to representations that serve as "models" of limit cases in which the student must identify the limit through graphical observation. Thus, to deal with a situation involving the formal definition of limit, students must break away from the idea that it is enough to "remember" the representations they have studied in order to solve the situation because occasionally the graphical representations they have had were particular cases that did not cover all the aspects involved in the formal definition of limit. In this case, we refer to the definition of limit using quantifiers.

In the following, we outline the conceptual field of the notion of the limit of a sequence considered in our study. In addition, we present important elements for the definition of this concept through a discussion of two definitions, A and B.

## Elements of the conceptual field involved in defining the limit of a sequence

The definition of the limit of a sequence using quantifiers involves several concepts that are studied throughout teaching, and, with this, professors can be led to believe that these concepts are well constructed and understood by the students, and that they will be mobilized for construction and understanding when this definition is presented. However, many important aspects are not always understood by the students and need to be revisited and expanded, such as the notion of approximating a number.

Irrational numbers do not have a finite decimal or periodic representation. When we write $\sqrt{2} = 1{,}414213562$ ... we mean, $1 < \sqrt{2} < 2$, 1,4< $\sqrt{2} <$ 1,5, $1{,}41 < \sqrt{2} < 1{,}42$, etc. Every irrational number can be approximated by rational numbers, since every open interval on the real numbers contains both rational and irrational numbers. We can perform the approximation process empirically, but it is desirable to do it algorithmically. For example, the expression

$a_{m+1} = \frac{1}{2}\left(a_m + \frac{2}{a_m}\right),\ m \geq 1$, gives us a sequence that approximates the value of $\sqrt{2}$.

The sequence $a_m$ is a first-order recurrence, i.e., the term $a_{m+1}$ is defined using the term $a_m$ and is fully defined when we assign a value to $a_1$ . Fixing, $a_1 = 2$ we get $a_2 = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2}$ and $a_3 = \frac{1}{2}\left(1{,}5 + \frac{2}{1{,}5}\right) = \frac{17}{12}$. Note that $a_m$ is a rational number for all $m \in N$, and the values of $a_m$ are close to the value $\sqrt{2}$, and the larger *m* is, the closer $a_m$ it is to $\sqrt{2}$.

Recalling the meaning of the word convergent: *heading towards the same point as another*, we can say that the sequence $a_m$ converges to $\sqrt{2}$. One of the demonstrations that the sequence $a_m$ is close to $\sqrt{2}$ strongly uses the fact that $a_m$ it is a Cauchy sequence (Lima, 2013), i.e. its terms get as close to each other as desired when *m* becomes large enough.

The behavior of the sequence $a_m$ can be translated into mathematical language with the notion of a sequence limit, let's look at the definition.

**Definition of sequence limit (A):**

$$\lim_{m \to +\infty} a_m = a \leftrightarrow \forall\, \varepsilon > 0, \exists\, n \geq 1 \;\; \forall\, m > n : |a_m - a| < \varepsilon \qquad \text{(A)}$$

This definition (A) means that the sequence of real numbers $a_m$ has a limit *a*, and we say that the sequence converges to *a* if, given any $\varepsilon > 0$, there exists $n \in N$ such that the terms of the sequence whose indices exceed *n* are at a distance less than $\varepsilon$ from $a$.

Since the value of $\varepsilon$ is arbitrary, we can also say that the terms of the sequence $a_m$ are as close to *a* as you like, starting from some index.

For example, if we set $\varepsilon = \frac{1}{10^2}$, the terms $a_m$ belong to the interval $(a - \varepsilon, a + \varepsilon)$ to $m > n$, i.e., if we approximate the number *a* by any $a_m$, we make an error smaller than $\varepsilon$, which in this case is equivalent to an accuracy of two decimal places. The smaller $\varepsilon$, the greater the precision of the approximation of *a* by $a_m$ . Another fact is that the set of indices of terms that are not at a distance from $a$ smaller than $\varepsilon$ forms a finite set, namely $N_n = \{1,2, \dots, n\}$. The number *n* depends on $\varepsilon$, so if $\varepsilon_1 < \varepsilon$, then it is to be assumed that the natural $m_1$ that makes the inequality $|a_m - a| < \varepsilon_1$ true for $m > m_1$ does not belong to $N_n$.

In other words, we can expect that the larger *m* is, the closer $a_m$ it is to $a$.

This is because there are two possibilities for the sequence $a_m$:

i) there exists $p \in N$ such that with $x_p \neq a$ with $p > n$, or

ii) $x_m = a$ for all $m > n$.

In case i), consider $\varepsilon_1 = \left|a_p - a\right| > 0$ . By definition (A), there exists $m_1 \in N$ such that $|a_m - a| < \varepsilon_1$for all $m > m_1$ . Note that $m_1 > p > n$, because $\left|a_p - a\right| = \varepsilon_1$.

Case ii) means that the sequence is constant for $m > n$.

The limit of a sequence, if it exists, is unique, since two distinct real numbers are not as close as one would like, i.e. if $\lim_{m \to +\infty} a_m = a$, then none $b \neq a$ can be the limit of $a_m$ . In fact, fixing $\varepsilon = \frac{|a-b|}{2}$, there exists $n \in N$ such that $|a_m - a| < \varepsilon$ for all $m > n$ . Therefore,

$$
\begin{aligned}
|a_m - b| &= \left|(-b + a) - (a - a_{m)}\right| \\
&\geq |-b + a| - |a - a_m| \\
&> |-b + a| - \frac{|a - b|}{2} \\
&= \frac{2|a - b| - |a - b|}{2} = \varepsilon
\end{aligned}
$$

for all $m > n$ e $a_m$ cannot converge to $b$.

Every convergent sequence is also a Cauchy sequence. In fact, given $\frac{\varepsilon}{2}$, there exists $n \in N$ such that $|a_m - a| < \frac{\varepsilon}{2}$ for all $m > n$. Therefore, $\left|a_p - a_m\right| \leq \left|a_p - a\right| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $p, m > n$, the interpretation of which is that the terms $a_p$ and $a_m$ are as close as you like for *m* sufficiently large.

**Suggested definition of sequence limit (B):**

$$\lim_{m \to +\infty} a_m = a \leftrightarrow \forall\, \varepsilon > 0, \exists\, n \geq 1 \colon |a_n - a| < \varepsilon \qquad \text{(B)}$$

Definition (B) means that a sequence $a_m$ has limit *a* when, given any $\varepsilon > 0$, there is some index *n* such that the term associated with this index is at a distance less $\varepsilon$ than $a$.

This definition only requires that there are terms in the sequence that are as close to *a* as you like. If a sequence satisfies (A) it will also satisfy condition (B).

In mathematical language, (A) implies (B). However, (B) does not imply (A) and it is important to note that these definitions are not equivalent.

Just consider the sequence $a_m = (-1)^m$, which takes the value 1 in even naturals $N_p$ and -1 in odd naturals $N_i$.

The sequence $a_m$ is not convergent, since it oscillates between two values, but it satisfies the condition of definition (B), since $|a_m - 1| = 0 < \varepsilon$ for any $m \in N_p$ and $|a_m - (-1)| = 0 < \varepsilon$ for any $m \in N_i$. In other words, there are infinitely many values that are close to 1, infinitely many values that are close to -1 and there are not only infinitely many values that satisfy the definition (B).

What's more, the terms of the sequence $a_m$ don't approach each other for *m* sufficiently large.

These concepts are important for discussing the situation presented in this article. The following are the methodological approaches chosen for this study

## Methodological choices and integrated studies

Our research is qualitative in nature, we are interested in intentions and situations "[...] it is about exploring ideas, discovering meanings in actions, and social interactions from the perspective of the actors involved in the process." (Coutinho, 2011, p.16). In particular, we will describe and analyze the facts experienced by a subject participating in an event, trying to identify and analyze elements of the schemes mobilized by him when dealing with the situation involving the definition of the limit of a sequence.

The event - Integrated Studies - lasted five days, and the session we are analyzing took place on the second day. The participants were students in their final year of the mathematics graduate course, as well as master's and doctoral students in a postgraduate program in mathematics education. Thus, everyone involved in the situation we will present already has experience with the concept of function limits and sequences.

The subject of our study is a professor of mathematics who has worked in elementary and higher education and who was at the time a doctoral student in a mathematics education program. In this article, in order to preserve his identity, we will refer to him as Pedro and use his written production and the video discussions. The activity was solved individually on a printed sheet of paper, and then each participant discussed their choices with another colleague, and at the end they all presented their ideas in the diagram, leaving it up to the participant to decide whether to present them in the diagram. Pedro was chosen because he wanted to present and discuss his choices about the situation with all the groups.

The activity had a context and some guidelines, which we present below:

**The context**

In this question, you have the opportunity to experience a realistic situation in which you are asked to explain something to a student who has difficulty with the concept of limits. This situation is realistic in the sense that the student's lack of understanding is found in the field. It's not just a figment of our imagination. So thank you for taking this seriously because it's part of your job as a professor to learn how to deal with this kind of situation.

Suppose a colleague introduced the notion of the finite limit of a sequence of real numbers to the students, giving the following definition:

$$\forall \varepsilon > 0 \; \exists n \geq 1 \; \forall m > n : |a_m - a| < \varepsilon \; (*1)$$

A student who has followed this teaching questions you and doesn't understand why we need the part $\forall m > n$ in the definition. It seems to him that taking

$$\forall \varepsilon > 0\ \exists n \geq 1 : |a_n - a| < \varepsilon\ (*\,2)$$

would be sufficient because this expression $(*\,2)$ clearly indicates that you can “get as close as you want,” which is, according to him, the basic idea of the limit of a sequence.

**Your work**

1. What does $(*\,1)$ mean in natural language? E $(*\,2)$? What distinguishes the two?

2. Write a text that explains your position in detail to the student. Do you agree with his argument? Can't we settle for $(*\,2)$ instead of $(*\,1)$?

**Details of the work to be done**

- Your explanation must be able to be read independently by the student, without intervention on your part, and be *independent*.
- The text is therefore intended for the student.
- The student doesn't know what's in your head. So you have to be very explicit in your explanation and consider that they've just had their first lesson on limits. It's a new concept for them. He's still a long way from mastering all aspects.
- This is an exercise in simulating the work of a professor to test your ability to produce an explanation that gets straight to the point, without unnecessary details that might confuse the student.
- You are not allowed to access the Internet to answer this question.

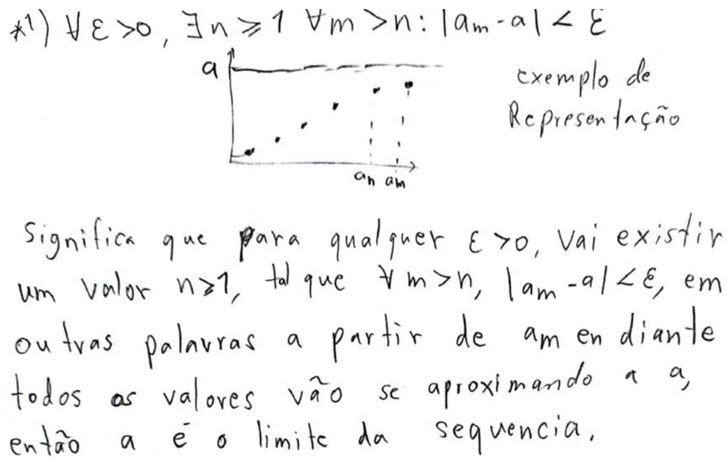
In this activity, we propose the discussion of two definitions for the limit of a sequence in the context of being a problem “created” by a fictitious student. It involves what we presented in the delimitation of the conceptual field about the fact that definition (A) implies definition (B), but that the opposite is not true, i.e. (B) does not imply (A). It's a situation that allows us to discuss problems involving understanding the definition of limit and, at the same time, highlight important aspects of the conceptual field involved in constructing this concept.

For this activity, the participants were given one hour to take notes on the sheets given to them with all the details requested. They were told that they could take pictures or copy the resolutions into their notebooks, as the sheets would be collected later. At the end, they discussed each group's resolutions, trying to highlight the choices, whether the definition was (*1) or (*2), and what arguments were used to justify this choice. Below are the discussions and analyses of the data generated.

## Description and analysis of the situation

After solving the activity, the participants were invited to present their thoughts on the activity, and together, in the chart, they argued about their choices.

When we look at Pedro's writing, we see that he tried to present the two definitions first with a geometric representation followed by a description, in natural language as requested in the activity, of what the definition (*1) and the definition (*2) meant. As we can see in Figure 1 and Figure 2.

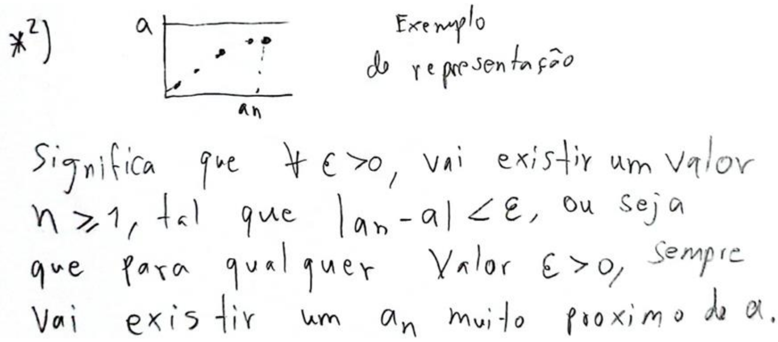

*1) $\forall \varepsilon > 0, \exists n \geq 1\ \forall m > n: |a_m - a| < \varepsilon$



Significa que para qualquer $\varepsilon > 0$, Vai existir um valor $n \geq 1$, tal que $\forall m > n$, $|a_m - a| < \varepsilon$, em outras palavras a partir de $a_m$ en diante todos os valores vão se aproximando a a, então a é o limite da sequencia.

Figure 1.

*First part of Pedro's resolution. Research data.*


*2)



Significa que $\forall \varepsilon > 0$, vai existir um valor $n \geq 1$, tal que $|a_n - a| < \varepsilon$, ou seja que para qualquer Valor $\varepsilon > 0$, sempre Vai existir um $a_n$ muito proximo de a.

Figure 2.

*Second part of Pedro's resolution. Research data.*

Pedro tries to detail the two definitions, and we can see that in definition (*1) he considers the fact that for all $\varepsilon > 0$ there is $n \geq 1$, such that for all $m > n$ we will have $|a_m - a| < \varepsilon$, explaining that from a $a_m$, which in this case should be *m,* all the values will become close to $a$. In the case of definition (*2), he explained the difference between the two definitions in these descriptions, even confusing $a_m$ with *m* and $a_n$ with *n*

When asked if he agreed with the student's argument proposed in the activity, Pedro wrote:

> I think the two expressions can represent the idea of limit. I think the first is more formal, but the second is more practical. I think the student is clear about the idea of approximation. The second would be a simpler option.

Pedro is concerned with understanding the idea of approximating the limit of a sequence, considering the fact that there is a term $n \geq 1$, such that the values of the sequence will approach $a$. As we saw in the discussion about the conceptual field, in the discussion about definitions (A) and (B), this is not enough to guarantee that the limit exists and that it will be $a$. This is a common mistake in which students mobilize theorems in action that have been effective in dealing with other situations, but which end up causing problems when they are mobilized for other situations, as they involve other aspects of the concept (Burigato, 2019).

In this sense, Vergnaud (1990) argues that the presentation of a definition is insufficient for students to understand all the aspects involved in the construction of a concept. It is through situations, in which the concept is the object of study, that it will be possible for the subject to understand the nuances involved. In the activity in question, we see the possibility of an explicit theorem in action mobilized by Pedro in which "$\forall \varepsilon > 0$, if there is $n \geq 1$, such that the values of the sequence will approach $a$, the limit of the sequence exists and will be $a$.". This is a false theorem in action, as it does not apply to the study of the limit of any sequence. However, he mobilizes it as being correct, probably because of other situations in which this theorem has been mobilized and there has been no problem, as is the case with most of the activities experienced by the students throughout their teaching.

We return to Job and Schneider's (2014) argument about the dichotomy between working with the formal and intuitive aspects of limits, in which most of the activities experienced by students involve intuitive aspects that can contribute to a lack of understanding of the deductive aspects involved in the definition. This favors the construction of knowledge in the wrong way, as in the case of the false theorem in action mobilized by Pedro.

After the participants had written down their solution, they were invited to discuss with everyone else, in the blackboard, how they thought about the situation. As we had participants with various experiences, we had a diversity of speeches and arguments. Let's look at some examples, which we'll identify as participants 1 and 2:

> Participant 1: I can only read the expressions, I know the mathematical symbology, but I can't assign meaning to them.

Participant 2: I would choose the first definition, which is more complete because for me it includes the second, which, I believe, is missing something.

These discussions ended up making some participants feel that the definition was not applicable. One of them argued that he hadn't used the definition for anything during his course and that, as a result, it only seemed to be useful for showing properties. Given these observations, we asked the participants a question: When we find the limit, how can we tell that it is unique? Everyone answered that it is the definition of limit that guarantees that if the limit exists, it is unique. With that, one of the participants, Pedro (fictitious name) asked to present what he had thought about the definitions presented.

Pedro: For me the second definition is enough because from a certain *n* all the images will approach the limit *a*, for every epsilon greater than zero.

He argues as he writes the graphical representation in the blackboard, and we can see his writing in Figure 3.

Figure 3.

*Pedro's production of the definition (*1).*

Again, we see a confusion with the elements of the domain of the sequence, in which he writes as: $a_1, a_2, a_m, a_n, \ldots$. Next, Pedro says that this will also happen with the definition (*2), "*That's my point of view*", and says that he will explain why. He starts to make another representation and goes on to explain:

Peter: We have the same idea, *a* few points, and the *a* which is our limit. Here it says (he refers to the definition) that for every epsilon, there will be $a_n$. The epsilon is the proximity to the limit (Figure 4). The idea is that this epsilon is very small, so there will always be one $a_n$ that leads to values very close to the value of *a*, the limit, and if I choose a smaller epsilon, there will also be one $a_n$ that will have values close to *a*. If the epsilon is bigger, it's further to the right (in this case, it's more to the left, Figure 4) and if the epsilon is very small, the values are further to the right (more to the right and closer to the value of *a*, which is the limit), and as it is for any epsilon, there will always be one $a_n$. So, for me, this definition is also valid.

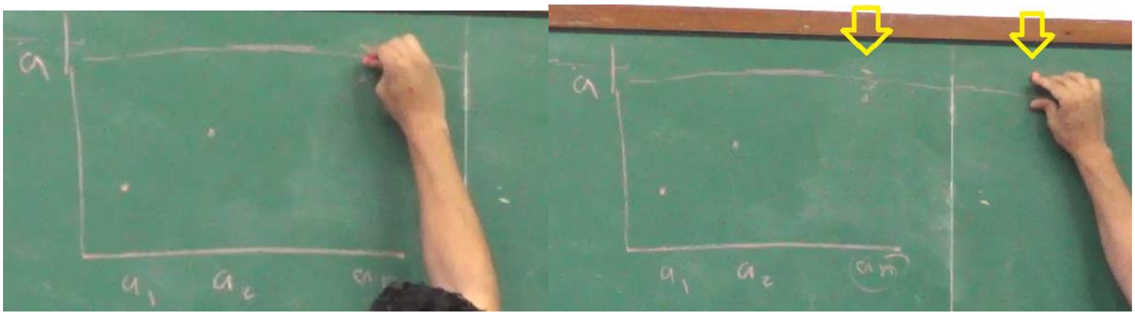


Figure 4.

*Pedro's production of the definition (*2).*

In Figure 4 we can see the representations he made while explaining why he believed that the second definition was sufficient to define the limit. There were further discussions between the participants, as one of them thought that what Pedro did was for a very specific case, in this case the representation he used to explain his point of view. In addition, they corrected the values that Peter had inserted in the representation, such as: $a_1, a_2, a_m, a_n, \ldots$ for the correct values *1, 2, 3, n, ...*

Figure 5.

*Pedro's production of definitions (*1) and (*2).*

Pedro was asked why he chose to put the *m*, in the first representation, in that place, why he didn't put it a little before or after. We asked him this question in an attempt to recall the elements of the conceptual field involved in the definition of the limit of a sequence, as presented earlier in (A).

He argues that according to definition (*1) *m* is greater than *n*, so what matters is that *m is* to the right of *n*, is greater because it all depends on the epsilon, and for each epsilon chosen, there will be an *n* that satisfies both definitions. Let's look at the reproduction of the dialog:

> Researcher: So, which of the two definitions do you choose?
> Pedro: Both because everything depends on the epsilon.
> Researcher: So what is different about the *m* in the first definition?
> Pedro: The difference here is that if I choose epsilon, there will be this *n* such that for all the *m*, after *n*, the values of the sequence will be close to *a* - Figure 6 - that's the idea, from now on, all the values are close to *a*.

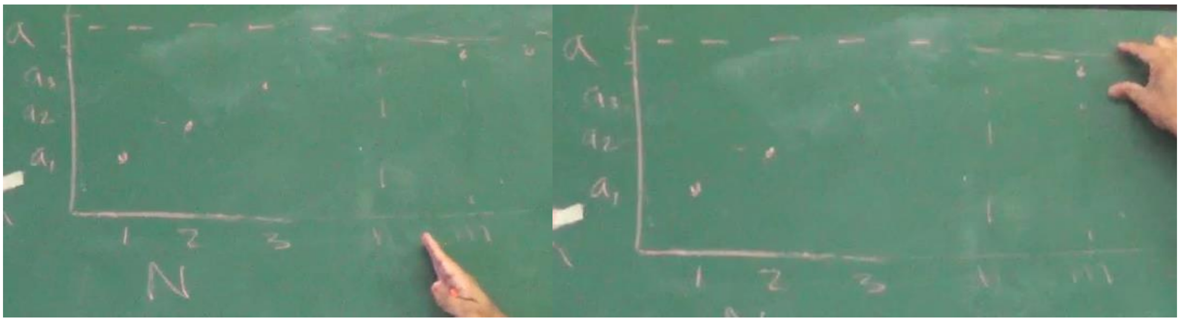


Figure 6.

*Pedro's production of definitions (*1) and (*2).*

Pedro: And here (*2) says that there will always be an *n*, where all the values are very close to $a$. Unlike (*1) which talks about *n* further on.

In these arguments, we see that Pedro is once again using the theorem in action that he had used in the activity sheet he had initially handed in. He justifies his actions by using graphs and finger gestures to explain that the limit exists because we can choose a minimal epsilon and that there will be an *n* that satisfies it. The formal aspects that we mentioned in the delimitation of the conceptual field of definition (A), do not seem to be mobilized in their actions, but rather geometric representations, which, as previously mentioned, are not interesting for the study of sequences. The professor needs to be aware of his didactic choices regarding the use of certain representations:

> [...] more generally, while ostension can be effective in certain cases, it is often based on a misunderstanding, as students do not "see" the same thing as the professor. Moreover, it is not always suitable for defining a mathematical object in a functional way (Schneider, Job, 2016, p.96).

The discussion with the group continued with the proposal that the participants try to agree on the definition.

Researcher: Very interesting, but we need to decide, guys, because the definition has to be the same for everyone, how do we do it? Participant 2 chose (*1) and Pedro chose both. What criteria do we use to choose? For example, (*2) is simpler and can deal with many situations, so why don't we choose it?

Participant 2: But this depends a lot on the analysis because I don't know if a particular example of a sequence will be covered simply by the second one. So the first one is safe, and I can always use it in any case.

As there was an impasse, they were asked to analyze the case of the sequence $a_n = (-1)^n + \frac{1}{n}$, in which we have the possibility of having even and odd numbers, Figure 7.

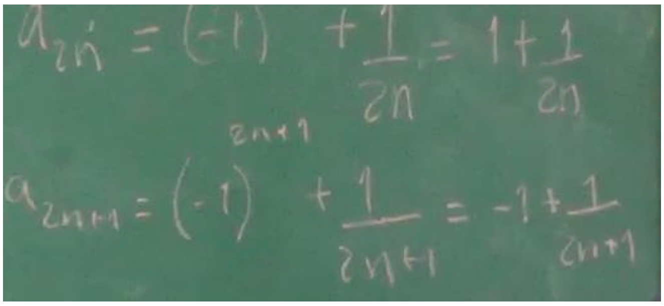


Figure 7.

*Pedro's representation of the example of a particular sequence.*

Pedro writes the two cases, Figure 7, for even numbers and odd numbers in the chart and argues that this sequence has no limit. He was then asked about the definition (*2) he had chosen. What happens if we choose *a=1*, for example? Would it be possible, given an epsilon, to find an *n* that would satisfy it? Pedro looks at the situation in the chart and says yes, that although the sequence has no limit, using the definition (*2) we could conclude that the limit of the sequence would be one, which is a mistake. He argues what we brought up in the previous discussion about the elements of the conceptual field, that definition (*1) is the definition for the limit of a sequence, and definition (*2) works for particular cases.

In Pedro's explanation, followed by Figure 6, we see how he seeks to justify his choice, in this case the use of the false theorem in action that he had mobilized. The graphical representations end up being a means for him to justify his choice to defend the use of the two definitions. It is only after being confronted with the example of the non-convergent sequence that he better analyzes the elements involved in the definition (*1). In fact, Job and Schneider argue about this when they talk about the students' work:

They are stuck with definitions that are "descriptions" of what they see in the behavior being studied. They cannot imagine their definitions as something to be chosen to allow evidence, despite the many contradictions pointed out by the professor. Students see a definition as a description of some mental concept that they believe each of them shares (Job, Schneider, 2014, p.640).

This fact is aggravated by the lack of activities in which students can confront the elements involved in the definition, in this case situations that work with the definition of limit, analyzing not only examples that work, but also counterexamples.

These elements, modeled in theorems in action, allow us to understand the knowledge that can be mobilized by the student when dealing with about introducing the definition of the limit of a sequence. And with this, we can study how the concept of limit lives on, in the ecological sense, in this institution and thus deepen our studies in the MER (Chevallard, 1999) on limits, which are still in the process of being constructed.

Next, we present our final considerations for the study carried out.

## Final considerations

In this article, we study a situation experienced by a subject about the definition of the limit of a sequence. To do this, we mobilized the theory of conceptual fields to analyze its construction, seeing how situations, operational invariants, and the representations used influence the way in which students understand a concept (Vergnaud, 1990).

The study of the situation shows how the construction of the concept requires a long time and reinvestment. In the case presented, the subject had already studied the limit of a sequence intuitively and by formal definition, using quantifiers, but the operative invariants of the schemes mobilized for the given situation were related to particular examples and, to a large extent, to graphical representations of sequences that had limits. The interesting thing is that, even in these cases, graphical representation is not a relevant resource, after all, these functions are defined in the set of naturals, which, in a way, does not favor the use of such representations. In other words, sometimes students need to break with aspects that conflict with important points in the construction of a concept to expand their schemes.

It is important to remember that in mathematics, it is enough to present a counterexample to prove that a result is false. However, presenting several examples in which the result is true is not an argument that makes it valid. In the case of sequences, calculating their limit, if it exists, requires more general arguments based on real analysis.

This study, with its definition, will be significant for the continuity of our project. In this sense, it has been shown that working through the definition of limits by sequence can be a pertinent proposal to help develop the concept of limits, which is currently widely presented through numerical tables, graphical representations, and algebraic functions. It is interesting because, in addition to presenting elements of mathematical intuition with the convergence of sequences, it enables the development of deductive work and the construction of the concept of limit with mathematical demonstration, which corroborates the idea of developing a MER that includes elements of intuition and deduction.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