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Prolegomena for building a reference epistemological model for teaching calculus: What are models? What is calculus?

Prolegómenos para la construcción de un modelo epistemológico de referencia para la enseñanza del cálculo: ¿Qué son los modelos? ¿Qué es el cálculo?

Prolégomènes pour la construction d'un modèle épistémologique de référence pour l'enseignement du calcul : Que sont les modèles ? Qu'est-ce que le calcul ?

Prolegômenos para construção de um modelo epistemológico de referência para o ensino de cálculo: O que são modelos? O que é cálculo?

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Abstract

This article proposes a theoretical reflection on models and Calculus, pointing out preliminary notions and basic principles for building a reference epistemological model to teach this subject. To this end, we examine some models present in our daily lives to suggest a generalization of the term. Next, we present the representative model proposed by the anthropological theory of the didactic to subsequently define the dominant and the reference epistemological models as well as the didactic model of reference. Having overcome these definitions, we develop a brief history of the term to clarify what we understand today by Calculus and how it has been used over time. The results of the theoretical-bibliographical research will be valuable for didactic researchers in constructing a reference epistemological model for teaching Calculus.

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Keywords: Models, Reference epistemological model, Calculus.

Resumen

Este artículo propone una reflexión teórica sobre qué son los modelos y qué es el Cálculo, señalando nociones preliminares y principios básicos para construir un modelo epistemológico de referencia para la enseñanza de esta asignatura. Para ello, examinamos algunos modelos presentes en nuestra vida cotidiana, en un intento de sugerir una generalización del término. A continuación, presentamos el modelo representativo que propone la teoría antropológica de lo didáctico para, posteriormente, definir el modelo epistemológico dominante, el modelo epistemológico de referencia y el modelo didáctico de referencia. Superadas estas definiciones, desarrollamos una breve historia del término para aclarar qué entendemos hoy por Cálculo y cómo ha sido utilizado a lo largo del tiempo. Los resultados de la investigación teórico-bibliográfica serán de utilidad para que los investigadores didácticos construyan un modelo epistemológico de referencia para la enseñanza del Cálculo.

Palabras clave: Modelos, Modelo epistemológico de referencia, Cálculo.

Résumé

Cet article, soumis à *Educação Matemática Pesquisa* magazine, propose une réflexion théorique sur ce que sont les modèles et ce qu'est le calcul, en soulignant les notions préliminaires et les principes de base afin qu'un modèle épistémologique de référence pour l'enseignement du calcul puisse être construit. Pour atteindre cet objectif, nous avons examiné quelques modèles présents dans notre vie quotidienne, pour tenter de suggérer une généralisation du terme. Ensuite, nous présentons le modèle représentatif proposé par la Théorie Anthropologique de la Didactique, pour définir ultérieurement le modèle épistémologique dominant, le modèle épistémologique de référence et le modèle didactique de référence. Après avoir surmonté ces définitions, nous avons développé un bref historique du calcul pour clarifier ce que nous entendons par calcul aujourd'hui et comment ce terme a été utilisé au fil du temps. Les résultats de la recherche théorique et bibliographique seront utiles pour que les chercheurs en didactique puissent construire un modèle épistémologique de référence pour l'enseignement du calcul.

Mots-clés : Modèles, Modèle épistémologique de référence, Calcul.

Resumo

Este artigo propõe uma reflexão teórica sobre o que são modelos e o que é Cálculo, apontando noções preliminares e princípios básicos para que se possa construir um modelo epistemológico de referência para o ensino desta disciplina. Para atingir esse objetivo, examinamos alguns modelos presentes no nosso cotidiano, numa tentativa de sugerir uma generalização do termo. A seguir, apresentamos o modelo representativo proposto pela teoria antropológica do didático, para, posteriormente, definir modelo epistemológico dominante, modelo epistemológico de referência e modelo didático de referência. Ultrapassadas essas definições, desenvolvemos uma breve história do termo para esclarecer o que entendemos hoje por Cálculo e como foi utilizado ao longo do tempo. Os resultados da pesquisa teórico-bibliográfica serão úteis para que os pesquisadores em didática possam construir um modelo epistemológico de referência para o ensino do Cálculo.

Palavras-chave: Modelos, Modelo epistemológico de referência, Cálculo.

All matter in the universe is made up of atoms. The word atom comes from Greek and means "indivisible" or "that which cannot be broken." This microscopic particle is the fundamental unit of all substances, a unit that, for a long time, was thought to be indivisible into smaller units, a belief that no longer holds. Currently, we know that the atom is composed of two distinct regions, the nucleus and electrosphere, where we find, for example, neutrons, protons, and electrons.

Over time, several atomic theories were formulated, each presenting a different atomic model. The first idea of an atom was based on philosophical deductions, and the Greeks imagined that matter could be divided into smaller units until a unity that could no longer support further divisions was reached.

In the early 19th century, John Dalton believed the atom was a solid, indivisible sphere. At the end of that century, Thomson presented the atom as a uniform sphere with a positive charge, impregnated with electrons (the negative electrical charges compensated for the positive ones, and the whole remained neutral). Thomson's model became known as "plum pudding." Around 1914, Rutherford conceived the atom as a dense, positively charged nucleus with negatively charged electrons rotating around it, in a region called the electrosphere (the planetary model). This model was improved by Bohr, who divided the electrosphere into seven layers, called valence layers.

In short, atomic models are attempts to represent the main components of the atom and its structure. They are not reality but a representation of it.

In 1953, James Watson and Francis Crick presented another important scientific model: the double helix structure of DNA. In Watson and Crick's model, each DNA molecule comprises two strands of nucleotides linked by nitrogenous bases through hydrogen bonds.

We could mention many other types of scientific models, and it would be appropriate to evaluate what characterizes them as such. Why are they called "models"?

In 1996, Gilles Willet, a research professor at the Department of Information and Communication at Laval University (Quebec), who focuses on communication within organizations in the phenomenon of telecommunications and its technologies, theories of communication, and the right to communicate, published the article *Paradigme, théorie, modèle, schéma: qu'est-ce donc?*, [Paradigm, theory, model, and schema: what is it?] to clarify these terms by emphasizing their differences. Julien Cartier, in turn, in *Qu'est-ce qu'un modèle?* [What is a model?] (2019) explains very simply how we can find a definition for this term.

Our initial focus in this study is characterizing the models. We have already mentioned two critical scientific models: the atomic model, its evolution, and the model of the DNA molecule, none of which we will not study here. We are interested in three crucial models of science teaching: the dominant epistemological model (DEM), the reference epistemological model (REM), and the didactic model of reference (DMoR), which are often used at random, without a criterium. Like Willet's, our claim is modest: we just intend to clarify the terms and bring elements that allow us to understand why they are called models.

Next, we will concentrate on defining Calculus, providing a brief historical evolution of the concept. To encourage our discussion, let us see what the *Online Portuguese Dictionary* (https://www.dicio.com.br/) presents:

> Meaning of Calculus
> masculine noun
> Solving problems involving numbers.
> Operation performed to find the result of combining several numbers; computation.
> [Arithmetic] Art of solving elementary arithmetic problems. [...]
> [...]
> Mental calculation. Arithmetic calculation performed in your head.
> Algebraic calculus. Calculation that is done with algebraic expressions.
> Arithmetic calculation. Performing arithmetic operations.
> Diferential calculus. Calculus related to derivatives and differentials.
> Integral calculus. Calculation related to integrals.
> Etymology (origin of the word calculus). The word calculus derives from the Latin "calculus, i", meaning small stone. [...]

Therefore, we understand that this reflection is essential for theorizing the elaboration of the REM and answering questions such as: What tools should be used to develop reference epistemological models? What methods should be adopted to build those models? What criteria should be used to test and evaluate the proposed REM?

We have structured our writings as follows: after clarifying what a model is, we will move on to the analysis of three important models of science didactics: the dominant epistemological model (DEM), the reference epistemological model (REM) and the didactic model of reference (DMoR). Having overcome these definitions, we develop a brief history of calculus to explain what we understand by calculus today and how this term has been used over time. The results of the theoretical-bibliographical research will be helpful for researchers in didactics to build a reference epistemological model for teaching calculus.

Model: an attempt at generalization

A vast body of literature is dedicated to defining the term "model." However, experts have not agreed on a definition, and this article does not aim to achieve the synthesis they have been seeking for decades.

Cartier (2019, section *Comme disait Saint-Augustin*) highlights that many people know what a model is, but when one must define it, a problem arises. The author points out that the difficulty of this undertaking is due primarily to the polysemy of the word itself, both in the scientific field and everyday language, a polysemy that is perfectly illustrated by the diversity of objects that we find labeled with the term "model."

Nonetheless, according to Cartier (2019, section *Comme disait Saint-Augustin*), it is safe to say that a consensual definition should be broad enough to encompass as many cases as possible. Hence, it states that a model is a representation, meaning it can represent almost anything. Most importantly, the model represents –completely or partially– what scientists call "theory."

Bacharach (1989, p. 496) says a theory is an "expression of a relationship between units observed, directly or by approximation, of the empirical world." Science is concerned with so abundant and disjointed facts that it becomes necessary to articulate them. This articulation is the work of a theory. "A thought that integrates facts," according to Cartier (2019, p.1). The primary purpose of a theory is to answer questions of *how*, *when*, and *why*. Cartier (2019) still argues that facts are outside of us, while theory –as an object of thought– resides within us.

But what about the model? According to the author, the model is at the interface between theory and facts. It constitutes a means of materializing the theory, of making it comprehensible, communicable, and explainable. It is a means that allows us to reflect on, to question our thinking. Thus, far from being a pure abstraction or an artifice disconnected from reality, the model is constructed from facts and is enriched by constant comings and goings between thought and reality (Cartier, 2019).

Following Cartier's (2019) thinking, the main advantage of this model definition is that it allows several objects to be classified as models (from a geographical map, through mathematical formulas, to the student who is the 'model' student and represents the expected behavior).

Modeling means creating, making, or designing a model. Just like "forming according to a model," as, for example, in the sense expressed in "the painter molded the features of the muse." We draw attention to the fact that, in the Portuguese language, the terms *modelar* [modeling] and *modalizar* [modelize] are not confused (https://www.dicio.com.br/):

> Meaning of *Modelar* [Modeling]
> [...]
> direct transitive verb
> To make a model or mold of a piece; to forge: to shape clay.
> [Figurative] To form according to a model: the painter shaped the muse's features.
>
> Meaning of *Modalizar* [Modalize]
> direct transitive verb
> Impose modalities to: Modalize teaching. To vary, to give another aspect to: The pastor modalized the liturgy.

Modalize, therefore, means imposing modalities. It is related to mode, modal, a proper way of doing something. We did not find the term "*modelizar*" [modelize] in Portuguese. The word "*modelagem*" [modeling] also derives from "model": it is nothing more than "the action or effect of modeling" (https://www.dicio.com.br/). Mathematical modeling, for example, putting it simply, boils down to creating a mathematical model capable of explaining a specific phenomenon.

As stated earlier, this representation may be partial. Cartier assures that in the representation of the DNA molecule (model mentioned at the beginning of this text), much information is omitted (length of the sequence, nature of the molecules, etc.), and it is not necessary to represent thousands of nucleotides in order to understand the DNA model.

Therefore, the author says that we must accept that "the relative poverty of a model is consubstantial with its operationality" (Cartier, 2019).

We will now proceed concisely and schematically to present some considerations regarding the definition of the word model brought by Gilles Willet (1996), similar to Cartier's (2019).

Table 1.

What is a model? (Willet, 1996, pp. 10-13)

What is a model?
A model is a projection of a theory. However, as it refers to a more limited range than theory, the model is less applicable.
A model represents only some of the characteristics of the object or phenomenon studied, which are expressed as a set of systematic propositions concerning observations and measurements made on specific aspects of an object or phenomenon.
A model is not as elaborate as the object, phenomenon, or process it is supposed to represent and explain.
Models facilitate the explanation and popularization of a theory by providing knowledge that would otherwise remain complicated or ambiguous in a simple way.
No model is sacred. A model is just a simplified, relative, incomplete, and temporary representation of a part of reality or a phenomenon. It is never either the real or the phenomenon studied.
No single model can be applied to all levels of analysis and all research objectives. Models must always be checked and compared with the circumstances, situations, and cases to which they apply and must be transformed into results.
A model should be specific enough to correctly represent certain aspects of its purpose. However, it should not be too detailed, as it should be generalizable to more than one observed situation.
The art of modeling requires skills such as abstraction, deduction, evaluation, and a deep knowledge of already known models.
To develop a model, it is necessary to know how to abstract the most important aspects of the studied reality, that is, to form them intellectually.
One must be able to deduce or infer consequences and predictions from the proposed model. The inferences that can be drawn from a model depend on the general context of the situation analyzed and represented and on the theoretical statement that underpins it. The richness of the meaning of a model will depend on the relevance of the deductions, assumptions, and predictions to which it has given rise.
Models are not neutral. They often form the basis of our perception of the world and condition our ways of acting and behaving.

Thus, if we wish to build a model for teaching Calculus, we must have, in short, something that represents particular characteristics of the object or phenomenon (Calculus), that is capable of facilitating the explanation and popularization of something related to the subject, that is simplified, relative, incomplete representation and can be compared to others, and that is specific enough to correctly represent specific aspects of its purpose (teaching Calculus). Having the above established, we can delve deeper into the didactical models.

The representative model of the anthropological theory of the didactic

As its name suggests, the anthropological theory of the didactic (ATD) is a theory "of didactics." Yves Chevallard (mathematician to whom the ATD is attributed) argues that didactics should be defined as the science of disseminating knowledge. Therefore, doing didactics would mean doing research, producing pieces of knowledge, and organizing these pieces in terms of "bodies of knowledge." The idea behind this science is that someone makes an effort so that someone else can learn something. According to Chevallard (2013), it studies works that, socially, are didactic gestures that are made –or can be made– about them, as well as the effects of the study and learning encounter associated with these gestures. Therefore, "didactic" refers to "the set of gestures in a given society" (Chevallard, 2013, p.1), which is the object of the study of didactics.

Knowledge, in turn, spreads among people and institutions. However, knowledge and its practices (know-how) are equally important in this process. Human activities involve both knowledge and the practical part of this knowledge, which Chevallard (2013) managed to represent through one word: praxeology. Praxeology –praxis (practice) + logos (knowledge)– is the keyword of the anthropological theory and all human behavior can be represented by this term.

Praxeology comprises four notions: task (T), technique (t), technology (θ), and theory (Θ), modeled by the quartet $[T, t, \theta, \Theta]$. The praxeological model is capable of decomposing all human behavior. We will analyze each variable separately.

Any human activity can be broken down into a succession of tasks. A task is almost always identifiable in a given language by an action verb, such as walk, sing, shout, cry, calculate, draw, etc. This type of task can be of a particular type when associated with an object (for example, "draw an equilateral triangle").

Continuously, completing tasks requires implementing a technique (an art, a know-how), that is, putting into practice a specific "way of doing." Every technique must be constructed. Task and technique form the so-called know-how block (praxis block, practical).

Pure and simple practice does not persist long without being grounded in something. It is based on a so-called "technological discourse," a technology (discourse –logy on technique –techno). Technology, in turn, requires that certain aspects that are not justifiable but contain meaning be explained, which is why it needs to be based on a theory. The second block, the knowledge block (logos), is then formed by combining technology and theory.

Praxeology [T, t, θ, Θ]

[task, technique, technology, theory]

Let us then try to describe, in praxeological terms, any human conduct: making a cake, solving an equation, or planting a tree. The conduct described in "making a cake" encompasses a task represented by the action verb "to make". For this task to be carried out, we must, in turn, implement a technique. There is a technique for placing the ingredients (placing them in a particular order, at a specific temperature and quantity). Furthermore, one technology requires, for example, that the flour is not beaten too vigorously and that the baking powder is added last, after all the ingredients. All of this can be justified by a theory (of chemistry, physics, or even mathematics).

Research that adopts the anthropological theory of the didactic as a theoretical framework cannot distance itself from these postulates, as teaching mathematics is a human activity and, therefore, can always be described in praxeological terms.

For scholars of the anthropological theory of the didactic, defining the four elements of praxeology is already commonplace, and it is difficult to find a theoretical framework for ATD that does not refer to the praxeological model. However, since the objective of this article is to discuss the models, we could not fail to mention it.

It is also common for research in didactics to cite three other models, which we will discuss below.

Dominant epistemological model (DEM), reference epistemological model (REM) and the didactic model of reference (DMoR)

We will now adopt as the core of our study the article written by professors Berta Barquero, Marianna Bosch, and Josep Gascón, *Las tres dimensiones del problema didáctico de la modelización matemática* [The three dimensions of the didactic problem of mathematical modeling] (2013), where they address two important models for research in didactics: the dominant epistemological model and the reference epistemological model.

As the title suggests, the authors analyze the fundamental dimensions of the didactical problem of mathematical modeling (MM). However, what the teachers propose in the text can be used as an analytical strategy for any research in didactics. Let us explain.

When we aim to investigate something (or formulate a problem to be investigated didactically), we must analyze the dominant ideas in a given culture (in school culture, for example). Thus, the authors suggest that

> [...] to transform the problem (of mathematical modeling) to begin to formulate it as a research problem in didactics within the scope of the ATD, we must question the ways of interpreting [...] MM, that is, the epistemological model of the dominant MM, not only in school institutions but also in the noosphere. (Barquero, Bosch, & Gascón, 2013, p.3)

They explain that official programs, textbooks, teachers' recommendations, and teaching materials are productions of the noosphere. Let us also add, within the Brazilian context, the norms, laws, the Curriculum Parameters, the Common Curriculum Base, etc. Everything that is "dominant" in a given culture.

And they continue:

> We refer to epistemological models or ways of interpreting and describing Euclidean geometry, school algebra, MM, proportionality, or statistics that are predominant in school institutions but also in the noosphere and in institutions that produce mathematical knowledge. (Ibid., p.4)

They are epistemological models or *ways of interpreting and describing any mathematical knowledge in play*. This "dominant" way of interpreting is generally assumed uncritically ("Teaching problems are formulated, usually, assuming –and without questioning– the notions and the dominant ideas in the school culture mentioned.") (Ibid., p.3).

The dominant epistemology (or the dominant epistemological model) in a given culture cannot go unnoticed in a didactic investigation, such is its influence on the teaching-learning process.

> Epistemology represents the combination of the terms *episteme* (science) and *logos* (study, discourse), meaning discourse or study about science. Epistemology is defined by Runes (1998) as a branch of philosophy that investigates knowledge-related aspects: origin, structure, methods, and validity. For Lalande (1999, p.313), epistemology is a "critical study of the principles, hypotheses, and results of various sciences, aimed at determining their logical origin, value and objective importance. (OLIVEIRA, Ivanilde Apoluceno de. Epistemologia e Educação: bases conceituais e racionalidades científicas e históricas [Epistemology and Education: conceptual bases and scientific and historical rationalities]. Petrópolis, RJ: Vozes, 2016, p.17).

These are representative attitudes of this dominant model (called "applicationism" by Berta Barquero, Marianna Bosch, and Josep Gascón): assuming that mathematical models preexist and apply to all scientific systems and assuming that neither models nor systems evolve. Therefore, we notice a dominant model of analysis and interpretation of mathematical knowledge.

However, this model of description and interpretation needs explaining. According to the professors, it is necessary to deconstruct and reconstruct the praxeologies that are intended to be analyzed. In other words, it is necessary to question this dominant way of describing and interpreting. At this point, the reference model is presented:

> It is called a reference epistemological model (REM) and always has a provisional character. This is the instrument with which researchers can deconstruct and reconstruct the practices whose intra-institutional and inter-institutional diffusion they intend to analyze. Thus, the REM becomes an instrument of emancipation of teaching and didactic science that allows us to question how institutions involved in the didactic problem interpret mathematical knowledge. (Barquero, Bosch, & Gascón, 2013, p.5)

The reference epistemological model, the professors argue, decisively conditions the breadth of the research field, the didactic phenomena that will be "visible" to the researcher, the types of research problems that can be posed, and the provisional explanations that can be proposed, that is, the type of solutions that will be considered "admissible" (Ibid., p.5).

Epistemological dominant or reference models are adopted as hypotheses; they are not definitive, and they can and should be modified.

Mariana Bosh and Josep Gascón (2010, p.55) also state that the need for mathematics didactics to develop its epistemological models of mathematical knowings is a fundamental contribution of the theory of didactic situations (absorbed by the ATD). As for the anthropological theory, according to the authors, praxeology serves as a model for both mathematical knowledge and teaching activities for disseminating and studying this knowledge.

Next, the teachers bring a crucial reflection on the models in didactics by stating that "every organization or didactic praxeology that exists in a given institution is supported and strongly conditioned by the epistemological model of mathematics that is dominant in said institution" (Bosh & Gascón, 2010, p.60). As stated above, teachers receive this "dominant" form uncritically (which characterizes the teacher's "spontaneous" epistemology, according to Brousseau. Spontaneous because it occurs naturally, as a spontaneous reproduction of the dominant epistemology). Therefore, teachers must urgently develop epistemological models that serve as a reference for the development of new praxeologies:

> In the works cited, we have explained reference epistemological models (REMs) specific to each of the mathematical areas considered: elementary algebra, functional limits, mathematical modeling, numbering systems, and magnitude measurements. These models, elaborated by math didactics for analysis and didactic design, must be

> considered relative and provisional reference systems for the researcher. We have used them, in each case, as instruments of analysis of the epistemological model of mathematics dominant in the school institution and as auxiliaries to characterize the spontaneous teaching models supported by the epistemological model mentioned above. This result is essential for designing, managing, and evaluating proposals for new didactic organizations. (Ibid., p.60)

We know that anthropological theory describes human activities in terms of praxeologies or praxeological organizations, which leads to the possibility of adopting the expression dominant *praxeological* model (DPM) rather than the dominant epistemological model (DEM) and reference *praxeological* model (RPM) instead of reference epistemological model (REM), to analyze the praxeologies that surround that *episteme*, that knowledge. However, we understand that DEM and REM appear more frequently in searches.

Professors Bosh and Gascón continue to recognize that both the "naive" epistemological models (as Brousseau referred to them) and the REM used in the investigations are not purely epistemological in the classical sense of the term. We should consider them "epistemological-didactic."

> Indeed, the first works with an anthropological focus (Chevallard, 1991, 1992) already presented the need to substantially expand the epistemology to integrate it into its object of study, together with the genesis and development of knowing, its teaching, utilization, and institutional transposition. This expanded, in parallel, the same notion of the didactic phenomenon and, consequently, the object of didactic study [...] (Bosch & Gascón,2010, p.61).

Thus, the didactic of the sciences must also develop, according to the authors, its didactic models of reference (DMoR), which can be considered an "expansion of reference epistemological models" (Ibid., p.61).

> One of the essential functions of the use of these models is to constitute, for the researcher in didactics and for his/her subject, an instrument of emancipation regarding the different institutions that form part of his/her object of study: the mathematical institution, the class, the school institution and society (Chevallard, 2007; Bosch & Gascón, 2007). In particular, it should serve to question, analyze, and evaluate (instead of uncritically accepting) the types of dominant models in these institutions](Ibid., p.61).

Praxeology, DEM, REM, and DMoR are therefore characterized, leaving us with a brief discussion on what we understand by Calculus. And this is what we will do next.

Calculus, a brief history

> Calculus is a way of calculating, so mathematicians sometimes talk about the "calculus of logic," "calculus of probability," and so on. But we all agree that there is actually only one pure and simple Calculus, and it is written with a capital C (Crilly, 2017, p.78).

In this section, we will use as the main reference the book *Histoire du Calcul,* by René Taton, published in 1946 by Presses Universitaires de France, in Paris. This work covers the history of what we know as Calculus, starting from arithmetic and going through numerical calculation, algebraic, trigonometric, and probabilistic calculus.

Interestingly, Calculus books (with a capital C) usually begin with a chapter dedicated to numbering. Let us see, for example:

> Piskounov, N. (1990) *Cálculo Diferencial e Integral* [Differential and Integral Calculus]. Edições Lopes da Silva.
>
> CHAPTER I - NUMBERS, VARIABLES, FUNCTIONS.
> § 1. Real numbers. Representation of real numbers by points on the numerical axis.
> § 2. The absolute value of a real number.
> § 3. Variable quantities and constant quantities.
> § 4. Domain of definition of a variable.
> § 5. Ordered variables. Increasing variables and decreasing variables. Limited variables.
> § 6. Functions.
> § 7. Different ways of expressing functions.
> § 8. Main elementary functions. Elementary functions.
> § 9. Algebraic functions.
> § 10. Polar coordinate system.
>
> Leithold, L. (1994) O *Cálculo com Geometria Analítica* [Calculus with Analytical Geometry]. Editora HARBRA.
>
> CHAPTER 1 – REAL NUMBERS, FUNCTIONS, AND GRAPHS.
> 1.1 Real Numbers and Inequalities.
> 1.2 Lines and Coordinates.
> 1.3 Circumpherences and Graphs of Equations.
> 1.4 Functions.
> 1.5 Function Graphs.
> 1.6 Trigonometric Functions.

The following question seems pertinent: Do we consider teaching real numbers, for example, to be teaching Calculus?

René Taton (1946) explains that modern beings live surrounded by many numbers, from those used in commerce, salaries, and taxes to temperatures and lengths, and wherever they go, there is an infinity of operations and calculations to be carried out. However, even though it is about mathematics, the calculus mentioned here does not refer to elementary operations to find a result. On the contrary, it has other objectives and much more specific purposes.

However, the Calculus we know today was limited to arithmetic and algebra for a long time, gaining an extension in the 17th century with the creation of differential and integral calculus (Taton, 1946). Leithold (1994) says that:

> Some ideas of Calculus can be found in the works of ancient Greek mathematicians from Archimedes's times (287-212 BC) and in early 17th-century works by René Descartes (1596-1650), Pierre de Fermat (1601-1665), John Wallis (1616-1703), and Isaac Barrow (1630-1677). However, the invention of Calculus is often attributed to Sir Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) as they began to effect the generalization and unification of the subject. Other 17th and 18th-century mathematicians also contributed to the development of Calculus, such as Jakob Bernoulli (1654-1705), Johann Bernoulli (1667-1748), Leonhard Euler (1707-1783), and Joseph L. Lagrange (1736-1813). However, it was not until the 19th century that the processes of Calculus received a solid foundation from mathematicians such as Bernhard Bolzano (1781-1848), Augustin L. Cauchy (1789-1857), Karl Weierstrass (1815-1897), and Richard Dedekind (1831-1916).

Indeed, differentiation and integration are essential and are, according to Crilly (2017, p.78), "the twin peaks of Calculus as established by Newton and Leibniz. The words are derived from Leibniz's *differentialis* (taking the differences or "parting") and *integralis* (the sum of the parts or "putting together")". Differentiation and integration are, therefore, two sides of the same coin.

But the history behind differential and integral calculus is long and impossible to exhaust here. This topic just intends to provoke reflection on something that seems commonplace to us but which has historical justification and a reason for being; after all, "it wasn't always like this." Everything can be explained and justified:

> The word "calculate" is a diminutive of "*calx*", which, in Latin, means "stone." In the past, it meant "to sum using pebbles." Mathematicians' contributions to the birth of Calculus are countless. MOAR (2003) ensures that many, such as Cavalieri, Barrow, Fermat, and Kepler, used Calculus concepts to solve various problems. However, at that time, there was no logically structured construction, that is, each author had their proposal for how the content was structured, making it difficult to interrelate the content. The development and improvement of techniques associated with Calculus occurred with Newton and Leibniz, who gave rise to the most important foundations for teaching Calculus, such as the formalization of derivatives and integrals. According to MOAR (2003), Calculus can be divided into two parts: one related to derivatives or differential and integral calculus, and another related to integrals or simply integral calculus. (Torres, & Giraffa, 2009, p.1-2)

Professor Gabriel Loureiro de Lima (2008) explains that the subject Differential and Integral Calculus was introduced in the Brazilian curriculum in 1810 in the mathematics course at the Royal Military Academy of Rio de Janeiro and was based on the book *Traité Traité Élémentaire de Calcul Différentiel et du Calcul Intégral* by the French Sylvestre François Lacroix (1765-1843), who considered that the concept of function was the starting point for the development of Calculus (Lima, 2008, p.3).

Thus, as previously stated, we do not intend to exhaust the history of the subject Differential and Integral Calculus but to provoke reflection on the need for justification, the search for the reason for being, with which the teacher should be concerned when starting a class on any mathematical object. Developing a reference epistemological model for teaching Calculus requires the teacher to have a critical attitude to escape from "naive epistemological models," as Brousseau called it, as cited by Bosh and Gascón (2010).

Knowing Calculus requires the student to know some essential mathematical concepts, such as algebra and geometry. Students should be able to easily perform operations (which is why the books start with whole numbers). Function is another key concept that should be explored in depth, especially continuous functions, trigonometric functions, and their graphs. After that, the concept of limit could be introduced. Limit, integration, and derivation are the three pillars of today's Calculus.

A model for teaching Calculus

The theoretical reflection proposed in the previous pages contributes to the primary objective of this thematic publication, which is the construction of a reference epistemological model for Calculus teaching (here understood as the range of concepts encompassing the notion of limit, derivation, and integration). To exemplify the models mentioned, we will present a task from the Differential Calculus component involving calculating limits by definition, which has been one of the most complex tasks for undergraduate students.

Before presenting the task, we introduce the definition of limit, so we can later infer the epistemological model that supports it.

II. Definição de limite

24. Seja I um intervalo aberto ao qual pertence o número real a. Seja f uma função definida para $x \in I - \{a\}$. Dizemos que o limite de $f(x)$, quando x tende a a, é L e escrevemos $\lim\limits_{x \to 1} f(x) = L$, se para todo $\varepsilon > 0$, existir $\delta > 0$ tal que se $0 < |x - a| < \delta$ então $|f(x) - L| < \varepsilon$.

Figure 1.

Definition of limit from a Calculus textbook (Iezzi, Murakami, & Machado (2013, p.23)

As we see, the definition of the limit of a function is a product of the arithmetization of analysis that characterizes the dominant epistemological model of differential calculus. According to Cornu (2002, p. 153), this notion is central and permeates all mathematical analysis.

As an example, here is a task that evokes the definition above:

Task 1: Demonstrate, using the definition of limit, that $\lim\limits_{x \to 1}(3x + 2) = 5$.

The cognitive process indicated by the verb expressing the task (demonstrate) greatly reflects the previously mentioned epistemological model, given that it is based on a logical-formal structure that underpins mathematical analysis. More forcefully, we can state that it is a structure based on Logicism[4] (Amaral, 2020), a pulsating movement in the 19th century.

The praxeology initiated by the task above is complemented by the technique presented in Figure 2 below:

[4] "The term Logicism refers to a tendency, program, or doctrine that reduces mathematics to logic. It is commonplace to find in the literature that Frege and Russell were the first proponents of such a view.17 Endorsing this, we must remember Carnap's words, who, in a well-known passage, defined this program in the following terms: Logicism is the thesis that states that mathematics can be reduced to logic, and is therefore part of it. Frege was the first to disclose such a view. In the majestic book, *Principia Mathematica*, the English mathematicians A. N. Whitehead and B. Russell produced a systematization of the logic from which they built mathematics." (AMARAL, L. A. D. . *A filosofia da matemática de Kant no (novo) tribunal da razão: alguns aspectos do anti-intuicionismo no século dezenove e uma variante neokantiana* [Kant's Philosophy of Mathematics in the (New) Court of Reason: Some Aspects of Nineteenth-Century Anti-Intuitionism and a Neo-Kantian Variant]. SYNESIS (ON LINE) , v. 12, p. 67-87, 2020.)

19. Usando a definição, demonstre que $\lim_{x \to 1} (3x + 2) = 5$.

Solução

Devemos mostrar que, para qualquer $\varepsilon > 0$, existe $\delta > 0$ tal que:

$0 < |x - 1| < \delta \Rightarrow |(3x + 2) - 5| < \varepsilon$

Notemos que:

$|(3x + 2) - 5| < \varepsilon \Leftrightarrow |3x - 3| < \varepsilon \Leftrightarrow 3|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \frac{\varepsilon}{3}$

Assim, se escolhermos $\delta = \frac{\varepsilon}{3}$, teremos:

$\forall \varepsilon > 0, \exists\, \delta = \frac{\varepsilon}{3} > 0 \mid 0 < |x - 1| < \delta \Rightarrow |(3x + 2) - 5| < \varepsilon$

De fato, se

$0 < |x - 1| < \delta = \frac{\varepsilon}{3} \Rightarrow |x - 1| < \frac{\varepsilon}{3} \Rightarrow 3|x - 1| < \varepsilon \Rightarrow$

$\Rightarrow |3x - 3| < \varepsilon \Rightarrow |(3x + 2) - 5| < \varepsilon$

Figure 2.

Technique for solving Task 1 (Iezzi, Murakami, & Machado (2013, p.27)

The technological-theoretical discourse presents elements that will point to a direct application of the definition. In this way, the formal definition of limit itself is the technological discourse for calculating the limit through definition. However, this does not seem so natural since this exact definition is also the technology for calculating limits in other contexts, such as problem situations that can be solved using the notion of the limit of a function.

From the praxeological analysis presented, we can point out the characteristics of the dominant epistemological model and the form presented in the textbook. From this analysis, we observe that the definition provided does not uphold a technological-theoretical discourse for diverse tasks that do not explicitly reference the calculation of limits through the definition. This was the concern of Chevallard and many collaborators regarding the inseparability of the technical-practical and technological-theoretical blocks, an inseparability not achieved in countless praxeologies presented in differential calculus textbooks.

The dominant epistemological model mentioned shapes the praxeologies in the initial part of Calculus courses. Therefore, it defines a dominant praxeological model that sometimes seems abandoned in the other courses but is strongly required to formalize the notion of the limit of a function.

Cornu (2002) warns us that we must think about a didactic model of reference that supports how such knowledge will be disseminated in a given institution. First, this researcher highlights that there is a difference between definition and concept and that such a distinction is didactically important. This is because, according to him, remembering the definition of limit is different from understanding its fundamental conception. The notion of approximation, usually presented before the definition and related to a dynamic notion of limit, expresses how this idea is implemented to solve real problems, not depending directly on the definition. However, a teaching model based exclusively on this approximation principle may lead students to believe that they have understood the definition without having acquired the implications of the formal concept.

Provisional considerations

We bring our understanding of the didactic model closer to that considered in the field of research into teaching natural sciences, in which “the didactic model is a mediating scheme between reality and the teacher’s thinking, a structure in which knowledge is organized” (Chrobak, 2006 apud Júnior & Marcondes, 2010 p. 101 -116). The nature of this model is provisional and changeable, and it approximates a reality (in which teachers and students are inserted), being a resource to develop and support teaching practice.

Thus, this brief theoretical reflection brought preliminary notions and basic principles (prolegomena) to assist teachers who teach differential and integral calculus. We examined the models and characterized the dominant epistemological model, the reference epistemological model, and the didactic model of reference. Regardless of how and when they appear in research and investigations, they will always be what they actually mean. They are models, in the correct sense of the term, one dominant (the prevailing, preponderant, influential, predominant) and another used as a reference (alluded to, mentioned, or used as an example).

That said, we hope these writings can help teachers and researchers construct reference epistemological models for teaching Calculus. Mathematics education requires such a change in teaching this subject.

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