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Double integral, quadric surfaces and the works of Antoní Gaudi: Possibility of elaboration of an epistemological model of reference

Doble integral, superficies cuádricas y las obras de Antoní Gaudí: posibilidad de elaboración de un modelo epistemológico de referencia

Intégrale Double, surfaces quadratiques et œuvres d'Antoní Gaudi : possibilité de développer un modèle de référence épistémologique

Integral dupla, superfícies quádricas e as obras de Antoní Gaudi: possibilidade de elaboração de um modelo epistemológico de referência

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Abstract

In order to teach the mathematical object double integral, we developed a reference epistemological model (REM) to construct a teaching device called study and research path (SRP), which was applied in a course with thirteen engineering and mathematics undergraduate students from two public institutions Bahia countryside. Theoretically, we supported our research in the anthropological theory of didactics, and the methodological processes were based on the structures that govern the SRP. In the analysis, we highlighted that there were differences between the REM constructed before and after the execution of the SRP; the students appropriately selected the mathematical objects that helped them answer the guiding question of the SRP and articulated other areas of knowledge to ensure a "good answer" to the proposed question.

Keywords: Double integral, Epistemological reference model, Study and research path.

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Resumen

Para la enseñanza del objeto matemático integral doble, desarrollamos un modelo epistemológico de referencia (MER) para construir un dispositivo de enseñanza denominado recorrido de estudio e investigación (REI), que fue aplicado en un curso con trece estudiantes de grado de ingeniería y matemáticas de dos instituciones públicas del interior de Bahía. Teóricamente, sustentamos nuestra investigación en la teoría antropológica de lo didáctico, y los procesos metodológicos se basaron en las estructuras que rigen el REI. En el análisis, destacamos que hubo diferencias entre el MER construido antes y después de la ejecución del REI; los estudiantes seleccionaron adecuadamente los objetos matemáticos que los ayudaron a responder la pregunta orientadora del REI y articularon otras áreas de conocimiento para garantizar una "buena respuesta" a la pregunta propuesta.

Palabras clave: Integral doble, Modelo epistemológico de referencia, Recorrido de estudio e investigación.

Résumé

Afin d'enseigner l'objet mathématique intégrale double, nous avons développé un modèle épistémologique de référence (MER) pour la construction d'un dispositif d'enseignement appelé parcours d'étude et de recherche (PER), qui a été appliqué dans un cours avec treize étudiants de licence en ingénierie et en mathématiques de deux institutions publiques de l'intérieur de Bahia. Sur le plan théorique, nous avons fondé notre recherche sur la théorie anthropologique du didactique et les processus méthodologiques ont été basés sur les structures qui régissent le PER. En analysant les résultats, nous soulignons qu'il existe des différences entre le MER construit avant et après la mise en œuvre du PER ; les étudiants ont sélectionné de manière appropriée les objets mathématiques qui les ont aidés à répondre à la question qui a généré le PER et ont articulé d'autres domaines de connaissances pour garantir une « bonne réponse » à la question proposée.

Mots-clés : Intégrale double**,** Modèle épistémologique de référence, Parcours d'étude et de recherche.

Resumo

Com o intuito de ensinar o objeto matemático integral dupla, elaboramos um modelo epistemológico de referência (MER)para a construção de um dispositivo de ensino denominado percurso de estudo e pesquisa (PEP), que foi aplicado em um curso com treze estudantes das engenharias e da licenciatura em matemática de duas instituições públicas do interior da Bahia.

Teoricamente apoiamos nossa pesquisa na teoria antropológica do didático e os processos metodológicos foram pautados nas estruturas que regem o PEP. Em análise aos resultados, destacamos que houve diferenças entre o MER construído antes e depois da execução do PEP; os estudantes selecionaram de forma apropriada os objetos matemáticos que os ajudaram a responder à questão geratriz do PEP e articularam outras áreas de conhecimento para assegurar uma “boa resposta” para o questionamento proposto.

Palavras-chave: Integral dupla, Modelo epistemológico de referência, Percurso de estudo e pesquisa.

Double integrals, quadric surfaces, and Antoní Gaudi' works: A possibility of developing a reference epistemological mode

Teaching practice requires teachers' prior planning, with the study of the teaching object and how it will be taught to students. Chevallard (1999) called this process mathematical organization and didactic organization, respectively.

Our emphasis will be on the study of the epistemological dimension of double integrals and on how to structure a mathematical organization for the development of a study and research path (SRP). In this sense, the object is an entity to be perceived, which can be in the form of definitions, propositions, theorems, symbols, and rules that may or may not be visualized and materialized. Visible entities are called ostensive objects and are generally represented through graphs, words, symbols, maps, and others. Non-ostensive objects are those formed by someone's mental schemes—not physical or material, of which an idea, a definition, and a thought are examples.

However, there is an intertwining between the two objects because, for a non-ostensive object to be present, communicated, or represented, it must use ostensive objects. In other words, the ostensive object discloses the non-ostensive object. Therefore, when studying an object, we access the idea and its representation simultaneously and establish associations with other objects to weave a neural network of thoughts that are perceived by representations.

In the scientific field, an object has its reason for existing. It responds to a request, a demand, and emerges based on a new observed problem. However, when validated, it moves from the category of common sense to the scientific category, and an entire structure (origin, limit, logical value, principles, field of action, and rules, among others) is outlined to prove its existence. By critically analyzing structures and proving them using scientific methods, it acquires the status of an epistemological object.

For example, a person who studies the mathematical object "double integrals" is unraveling all or part of the structures validated in the scientific canon. The double integral is an epistemological object belonging to the science and mathematics field, used to calculate the measurement of an area. However, the double integral is a simplified and idealized mental image that allows itself to be represented –with greater or lesser precision– to describe the behavior of a system of ideas, representations, and symbols connected with other mathematical and non-mathematical epistemological objects.

These objects are broadly interconnected and enable new objects to originate, adding relevant information to a subject when associated. Metaphorically, an object can be linked to many others, like the branches of a tree. We call a model this set of epistemological objects that preserve common characteristics and are part of the same system. Weyne (2009, p.12) defines a model as “a simplified representation or interpretation of reality, or an interpretation of a fragment of a system, according to a structure of mental or experimental concepts.”

The author also argues that the construction of a mathematical model, specifically, will be “the approximate and selective representation of a given situation that can be expressed in mathematical terms” (ibid, 2009, p.12). When selecting a given circumstance, there are always choices to be made to suppress some mathematical objects and add others, whether belonging to the mathematical universe or not.

When we refer to an epistemological model, we highlight a scientific model mediated by theoretical and empirical evidence of theoretical objects accepted by peers and subject to correction of hypotheses whenever necessary.

A problem is often solved through hypothesis-raising, which consists of abductive reasoning, free from judgments or initial proofs. The hypothesis is articulated with arguments that ensure it is coherent and can be confirmed as valid (even if momentarily).

A hypothesis can be attested as scientific and, over time, be revealed as false or added with new information, which are natural processes of science. In this sense, an epistemological model can dominate a scientific community until something or someone modifies that system.

When we isolate an excerpt from the current system to respond to a given problem, we often create an epistemological model that will serve as a reference for that context, called the reference epistemological model (REM).

In this text, we describe a REM developed for the application of a teaching device called a “study and research path” (SRP) to teach the mathematical object “double integrals.”

The avert of the REM

Chevallard (2005), upon realizing that educational institutions kept a current epistemological model[4] for teaching mathematical objects based on a “monumentalist” teaching –one in which mathematical works are visited as statues, stagnant– proposed a break from this paradigm to present the perspective of “questioning the world” (Chevallard, 2015).

However, this paradigm shift required changes in how mathematical objects were taught, placing them as non-static knowledge, in movement with the world, and, for this reason, questionable. Realizing that mathematics and its teaching –as well as science– need to seek answers to emerging questions, Chevallard (2009a) developed a teaching/research device called

[4] Gascón (2021, p. 14) states that “in previous works, we have called *dominant epistemological model* (DEM) which we call here *current epistemological model* (CEM) in an educational institution.”

the study and research path (SRP), in which he describes how a group of students looks for answers to a question proposed a priori, called the generating question.

To answer the guiding question, students must conduct investigations using different means and media, producing questions and intermediate answers until they can reach an answer proven by solid arguments. One of the aspects considered relevant in this process is that students come into contact with the need to propose answers, which inserts them into procedures for validating the scientific method.

However, the SRP demands a new structure for working with mathematical objects, as the current epistemological model may not correspond to the responses and questions that emerge from students and, mainly, the linear form in which the mathematical and school curriculum is arranged. For this reason, we emphasize that the current paradigm is broken and that there is a need to evaluate and overcome the possible restrictions when we propose to take action around questions.

In this sense, the construction of a REM requires the analysis of new phenomena that emerge from the need to break with the dominant model in institutions in order to incorporate different tasks, technical supports, and materials that can contribute to the "emancipation of the didactics of mathematics" (Gascón, 2014, p.106). However, it also demands that the teacher studies the epistemological object and its relations with other knowings to create a hypothesis for a mathematical epistemological model that will guide him in conducting the SRP. This model serves as a reference for teachers regarding the possible paths students can take when seeking answers.

As a hypothesis, it can be modified during the course, sometimes expanded, sometimes simplified, as the REM guarantees the existence of various mathematical and non-mathematical objects that students can mobilize. Broad knowledge of the study of the mathematical object enables the teacher to know its reason for existing, its relationships with other fields of knowledge and with the internal field of mathematics, the history that permeates the emergence of the object, and its relationship with the social and cultural context of the time in which it was constructed.

The REM is an alternative model to oppose the current epistemological model, and intrinsically, it is an action that gives voice and autonomy to the teacher because when creating their generating question and selecting the mathematical objects that will "supply" the SRP, they become co-creators of the curriculum. In this process, knowings not proposed in official documents or for the age group of a specific student may emerge, requiring the teacher to perceive the world in a more critical and questioning way and make quick decisions, which may lead the current school culture to change. For Gascón (2014, p.100), “the teacher must analyze the epistemological models of mathematics that dominate the institutions involved critically and, thus, free him/herself from the uncritical assumption of these models.”

One of the restrictions for the implementation of the SRP methodology begins with the disruption from the school culture/beliefs that “a good teacher goes to the board and explains the content, while the students copy it in their notebooks,” and, when they do not do so, they are “wasting time” or promoting small talk in class. The subjectivity of this type of interpretation is veiled because it is an institutionalized, deep-rooted collective thought that the teacher who escapes from that model is deluded and optimistic. Morin (1989, p. 189) emphasizes that “educators must begin reforming their thinking, despite institutions trying to block their initiatives, because one day, their ideas will prevail.”

Another relevant aspect of preparing a REM is that it identifies the teacher’s personal characteristics and his/her relationship with the epistemological object and the institution. An autonomous and creative teacher often does not adapt to a technicist institution or one that prefers to uphold the current teaching model. In this sense, Chevallard (1996) clarifies that the set of relationships established by a person (X) in an institution (I), represented by R (X, I), determines the existence of the object. These relationships change over time due to the influence of the various institutions the teacher will be part of during his/her professional life.

The REM provides knowledge of the mathematical object and its association network with other knowledge to highlight the relevant aspects for formulating didactic problems that will contemplate the epistemological, economic, ecological, and language dimensions (Brandão, 2021). The epistemological dimension circumscribes the didactic problem in the study of knowledge of the epistemological object, the economic dimension delimits a minimum effective and economical unit to teach this object, the ecological dimension provides the conditions and restrictions for the study of that object in institutions, and the language dimension characterizes how this object is communicated, visualized, signified, and interpreted in different institutions.

Thus, the construction of the REM and the SRP contributes to didactic emancipation, as it promotes chaos in the educational environment –a movement of resistance against the current model that demands self-regulation– to make new possibilities of education, communication, and questioning of the world emerge.

When we structure the study and research path for teaching double integrals, we must study that epistemological object and its relationships with other knowledge. This idea led us to construct a reference epistemological model that we present in the following section.

The REM developed for teaching double integrals

We searched books on differential and integral calculus for the dominant (or current) epistemological model (DEM) on the double integral used in Brazilian higher education institutions and found that authors construct definitions without questioning processes or applications around the topic and follow with demonstrations of properties and exercises solved through algorithmic technique procedures. Figure 1 registers how a differential and integral calculus book (Leithold, 1996, p.1024) presents the DEM.

Definition: Let f be a function defined on a closed rectangular region R. The number L will be the limit of sums of the form $\sum_{i=1}^{n} f(\xi_i, \gamma_i)\Delta_i A$ if L satisfies the property that for all $\varepsilon > 0, there\ is\ a\ \delta > 0$ such that for every partition Δ, for which $|\Delta| < \delta$ and for all possible selections of the point (ξ_i, γ_i) in the i-th rectangle i = 1, 2, ..., n,

$$|\sum_{i=1}^{n} f(\xi_i, \gamma_i)\Delta_i A - \mathrm{L}| < \varepsilon.$$

If such a number exists, we write:

$$\lim_{|\Delta| \to 0} \sum_{i=1}^{n} f(\xi_i, \gamma_i)\Delta_i A = L$$

If a number L satisfies Definition 18.1.1, we can show it is unique. The demonstration is similar to the proof of Theorem 2.1.2 concerning the uniqueness of the limit of a function.
Definition: A function *f* of two variables will be said to be integrable in a closed rectangular region *R* if *f* is set in *R* and the number *L* from Definition 18.1.1 exists. This number L will be called the double integral of f in R, and we can write:

$$\lim_{|\Delta| \to 0} \sum_{i=1}^{n} f(\xi_i, \gamma_i)\Delta_i A = \iint_R f(x, y)\,dA$$

Figure 1

Definition of double integral (Leithold, 1996, p.1024)

For implementing the SRP device based on the paradigm of questioning the world, the DEM exposed would not allow us to associate the definitions of the mathematical object and its pragmatic applications since the language used is restricted to the internal field of mathematics.

In search of other references for what we had as our intention, we checked the publications addressing double integrals. We found that regarding the mathematical objects “double integrals” and “quadric surfaces,” there is a shortage of publications where they are applied in situations that are adverse to mathematics and the proposed theme. Research such as that by Silva and Morretti (2018a; 2018b) on quadric surfaces and by Henriques (2006) on multiple integrals bring valuable contributions to objects individually, focused on the internal field of mathematics and the use of technological artifacts, but do not reflect their applications in undergraduate courses for non-mathematicians. Regarding the applications of these mathematical objects, Romo-Vázquez (2010a) and Romo-Vázquez and Chavez (2017) discuss activities involving the Laplace transform and differential equations for future engineers but do not reflect in the same direction we propose.

As the DEM and research on the subject did not encompass the aspirations we had regarding the teaching of double integrals, we organized a course entitled "(Se) Integre Duplamente à Superfície Quádrica" [Doubly Integrate the Quadric Surface] in which 13 students, six mathematics teaching degree students and seven in engineering (civil, environmental, and electrical), participated in the five face-to-face meetings.

For the SRP generating question, we formulated question Q_o, "How have Gaudí's works withstood weather conditions?" based on Antoni Gaudí's works. This Catalan architect used quadric surfaces and elements of physics to build them.

For the development of the SRP, we built a mathematical and a didactic organization; however, for this study, we emphasize the mathematical organization for the constitution of the REM.

In the first moment of studying the epistemological object –double integrals– the teacher/researcher prioritized investigating all possible relationships that could emerge when students research and establish a hypothesis, an inference from a reference epistemological model developed before the SRP.

Guided by this hypothesis, we needed to know the mathematical ecosystem of which the double integral is part, that is, the network of objects associated with it. Based on this bias, we created the following teaching problem: How to organize immediate[5]/ostensive and dynamic/non-ostensive objects that characterize double integrals in calculating the measurement of quadric surfaces to be used in a mathematical model that allows the production of meanings, abductive reasoning, and differentiated praxeologies that meet the interests of engineering and mathematics undergraduates?

For this path, we sought to build the mathematical ecosystem that revolves around the double integrals, composed of two trophic levels[6]: at the first level, it weaves a web of relationships with mathematical objects –numbers, functions, spatial geometry, quadric surfaces, matrices, coordinates– that feeds the constitution of the object under study. At the second level, it plays a secondary role, as it starts to feed other branches of mathematics (such as topology), other areas of knowledge (engineering, architecture, physics, among others), and

[5] Peirce (2005, p.79-80) distinguishes objects into immediate and dynamic. Immediate objects can be visualized and are ostensive, represented by symbols; dynamic objects are those that are not visible, are in the interpreter's mind, and are non-ostensive.

[6] We build an analogy with ecology, in which trophic levels are those that produce their own food and those that do not. In this sense, all objects that help define the double integral, such as functions of several variables and partial derivatives, among others, feed the double integral, are the producers. While objects that use the double integral as a tool, such as the center of mass and Green's theorem, belong to the second trophic level, are the consumers.

other mathematical objects, such as triple integrals, surface integrals, differential equations, complex variables, series.

It is possible to identify the existence of biotic and abiotic factors[7] that make up each trophic level and are conditioned by immediate/ostensive and dynamic/non-ostensive objects. At the first trophic level, we presented the mathematical objects that "feed" the double integrals: numbers, functions, geometry, surfaces, matrices, coordinates, and Lebesgue integrals. Each one has its biotic function: for numbers, the notions of measurement, quantity, and order; for functions of one and two variables, limits, derivatives, and integrals; in geometry, the plane, spatial, and analytical; in surfaces, conics and quadrics; for coordinates, cylindrical, spherical, and Cartesian; for matrices, vectors, determinants, and linear systems; for the Lebesgue integral, integrals, vector space, and subspace.

At the second level, double integrals feed branches in mathematics and other areas. For branches of mathematics, topology, vector fields, complex variables, differential equations, and triple integrals, the biotic factors are thus organized: in topology, functions, and derivatives; in vector fields, vectors, line integrals, surface integrals; in complex variables, functions, derivatives, and integrals; in differential equations, equations, derivatives, integrals, and series, and for the triple integral, integrals. As abiotic factors: in topology, they are topological space, the Gauss-Bessot theorem, Morse theory, the Hopf index theorem, and the knot theory; for vector fields, Gauss's theorem, Green's theorem, path independence, Stokes's theorem; for complex variables, hyperbolic functions, Cauchy's theorem, Euler's equation, Laplace's equation, Laurent series; for differential equations, equations, derivatives, theorem of homogeneous equations, method of the coefficient to be determined; for triple integrals, spherical and cylindrical coordinates, and the Jacobian transformation.

For other areas, the double integral supports physics, chemistry, biology, engineering, and economics. The integral is the biotic function of all, and they differ in terms of the abiotic function: in physics, they are scalar measure, vector measure, mass, and moment of inertia; in chemistry, gas pressure, volume, area, chemical reactions; in biology, cardiac capacity, contrast dilution, growth rate; in engineering, flow of vibratory energy in structures, volume, moment of inertia, center of mass, electromagnetism, thermology; in economics, future value, continuous income flow, quality control.

[7] We consider biotic factors as mathematical objects that feed the double integral and interact with the first trophic level; they are the producers. Abiotic factors are external factors that influence the existence or use of the double integral; they are the properties and theorems that belong to calculus or other areas of knowledge.

Figure 2 presents the mathematical object, the trophic levels, and the factors that help construct the double integral and summarizes the relationships between mathematical objects, properties, definitions, and theorems that contributed to constructing the REM.

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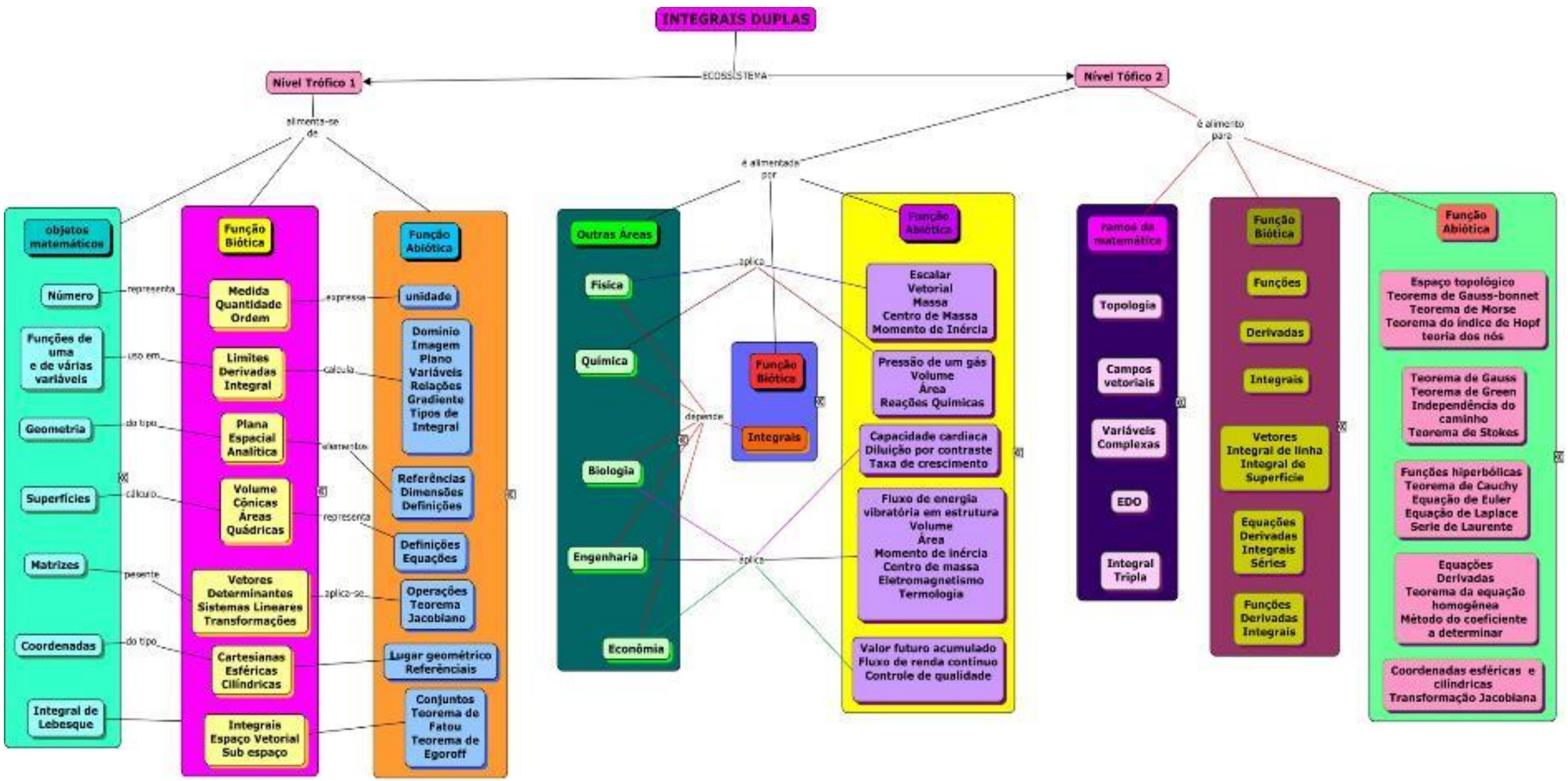

Figure 2.

Double integral ecosystem (Brandão, 2021, p.133)

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We still needed to look for elements to solve the didactic problem. For this, the study of the ostensive and non-ostensive objects around double integrals emerged as essential entities in constituting the REM. Figure 3 depicts the first designed REM that the professor/researcher created before developing the SRP.

Figure 3 begins with “area,” which associates numbers as a measure of magnitude related to scalar and vector physics to calculate mass and underpin vector geometry. The area also expresses the reason for the existence of integrals, which makes it possible to calculate the measurement of volume and relates functions with limits, derivatives, and analytical geometry, which, in turn, contribute to matrices, determinants, and linear systems and are associated with spatial geometry and plane geometry. Together, they help to form functions of several variables that use quadric surfaces that partially originate from conics. The study of mass, functions of several variables, and the expansion of integrals give rise to double integrals, which feed into surface integrals, other areas of knowledge, and the calculation of volume measurements in three-dimensional spaces. In physics, double integrals are used to calculate the mass center and the inertia moment.

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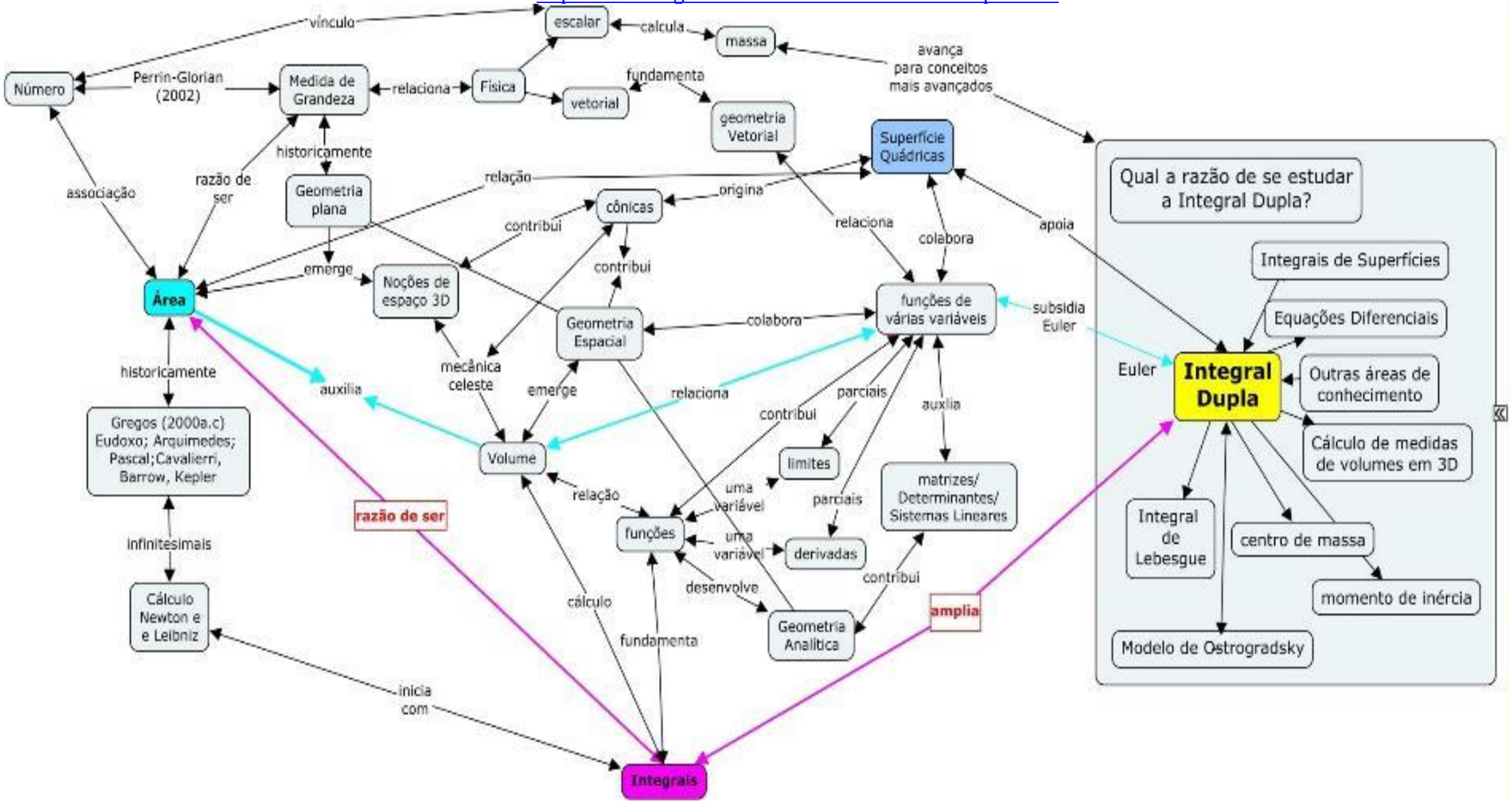

Figure 3.

Ostensive and non-ostensive objects associated with double integrals (Brandão, 2021, p.138)

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This initial hypothesis for the REM is quite broad and presents rich elements to feed the SRP with new questions encompassing mathematical and non-mathematical epistemological objects. For the teacher/researcher, the REM opens up a range of paths students can bring when developing the SRP, minimizing the uncertainties and vulnerabilities they are subjected when proposing a question beyond the mathematical field.

Since an inference can be partially or fully verified/falsified, it gives us a way back to adjust what did not produce solid argumentation or did not actually apply in a given circumstance. The REM proposed in Figure 2 sought to contemplate the most mathematical epistemological objects directly or indirectly associated with double integrals.

However, didactic predictability is not a characteristic of classroom spaces; not everything that is planned occurs in its entirety, and it is during the teaching process that some elements change. Vernet (1975) and Chevallard (1982) called this process “internal didactic transposition.”[8]

The development of the SRP mobilized knowledge is presented in Figure 3, but not completely, as we observed that some mathematical objects have been suppressed during the course. Thus, the REM was modified in double integrals teaching based on students’ questions and the search for answers.

The REM from students’ research

The paths students take when conducting research are countless and mark the unpredictability of the knowledge they acquire and relate. As teachers, we are often asked unusual questions to which we do not always know the answer. This vulnerability can occur when we structure mathematical and didactic organizations, and at this point, the teacher must take a quick attitude and immediate action to handle the unexpected.

In the previously prepared REM, articulations of ostensive and non-ostensive objects were made from the perspective of the teacher/researcher, who sought to create a network of probable connections that the students on the course could mobilize. However, students were more succinct and achieved some of this REM.

In this article, we will not describe the development of the SRP[9]. However, we will register some specific speeches/writings that prove students’ articulations of the mathematical epistemological objects that constituted the later REM.

In the first meeting, the conversation with the students focused on their impressions of mathematics, the place it occupies in the world, whether it can be considered creative, and whether it is present in nature and manufactured constructions. In this context, the teacher asked

[8] From the original: “La transposition didactique désigne donc le passage du savoir savant au savoir enseigné” (CHEVALLARD, 1982, p. 6). Our translation: “The didactic transposition designates the transition from mathematical knowledge to taught knowledge.”

[9] For a more in-depth description of the SRP, see Brandão (2021), in: https://repositorio.pucsp.br/jspui/handle/handle/24405

P_1: *What is the relationship between the sunflower, the daisy, and the peacock?* Student E_{10} answered: *There are elements of Geometry in its physical structure, I think* ***plane geometry****,* ***spatial geometry****, and fractals.* Then, E_2 replied: *All of them have an* ***area****!* The teacher returned a question to class P_2: *How would you calculate the area of a peacock, which is very irregular?* Student E_3 said: *In this case, I would separate each little part to try to get as* ***close as possible****. As for* ***integrals,*** *you make the rectangles and calculate each little piece, making an estimate.* E_7 added: *But even so, it would be an approximation, so it uses a* ***limit****.*

The first non-ostensive epistemological objects emerge from the students' speech. This sequence continues in the second meeting when they bring the results related to the first question, Q_o: How would Gaudí's works withstand the weather conditions?

We clarify that, at the end of the first meeting, the students formulated three intermediate questions: Q_1: Who was Antoní Gaudí? Q_2: What were his works? Q_3: What relationship is there between these works and mathematics? This research resulted in some associations with other mathematical objects portrayed in the speech fragments below.

> *E_5: At the beginning of the research I did I saw a* ***prism****, after that* ***ellipsoids****,* ***paraboloids****,* ***hyperboles****,* ***hyperboloids****, and* ***hyperbolic paraboloids****. The latter I saw at the key to the doors of some houses he built, and on the balcony of the Sagrada Família.*
> *E_8: He talked about the hyperbolic paraboloid on the balcony, but we saw it* ***in the structural calculations****. We found an article that said that he used* ***geometry*** *for the structural conception of his works.*

In this part, the students mobilized the mathematical objects "conics" and "quadric surfaces," but the relationship with other epistemological objects that belong to engineering courses appeared –structural calculation. Following the discussions, the teacher asked P_4: *What is the difference between area and surface?* We recorded some of the answers.

> *E_9: When I talk about surface, I think about area. The area has width and length, and we can also see the surface.*
> *E_1: The area is more about length and width. When you talk about surface, it involves* ***three or more dimensions****.*

In this dialogue, the students tried to distinguish an area from a surface, and in this search, they came across the space made up of n variables, which is associated with the epistemological object functions of several variables.

We would like to highlight that at the beginning of this second meeting, there was a round table with professionals from several areas of knowledge: an electrical engineer, an environmental engineer, a civil engineer, a physics teacher, a mathematics teacher, and an architect. The electrical engineer discussed the importance of the wiring between the poles for conducting electricity and stated that the curve the wires form is a catenary. When the students met with the teacher to discuss the answers to Q_1, Q_2, and Q_3, student E_8 mentioned: *The electrical engineer spoke of* ***catenaries****, and I saw that Gaudí used catenary arches a lot in his works.*

After the teacher explained catenaries briefly, she asked if they had researched anything on this topic. Student E_7 answered: *I remember seeing the **equation** of the catenary, which is a **hyperbolic cosine**. There is something with Euler raised to x power minus Euler's number raised to -x power, and everything divided by two.*

In the third meeting, the students had two moments: in the first, they attended a lecture with Professor Afonso Henriques on 3D printers to study geometric sieves. The students were then asked to use acrylic paste and barbecue sticks to reproduce some of the solids found in Gaudí's works. During the process, some elements, such as balancing force, emerged from conversations between students, who were distributed into groups. One of the registers was from student E_{11}, who said: *We need to make a base with the sticks.* E_4 mentioned: *The sticks won't hold, they'll fall!* Student E_3 solved the situation by suggesting: *Let's make a square base, then it supports the solid. The square base will give the **balance** to the solid.*

Indirectly, students began to construct elements (the notion of equilibrium, partition of a solid into smaller regions, the square base) to answer the generating question Q_o, and the first signs of epistemological objects, mass center, inertia moment –which will be fundamental for the answer– appear.

When students presented their solids, the teacher asked them how they would calculate the volume of those works. After a brief silence, student E8 replied: *As I did to calculate the square integral, only now that there is height I would make prisms, parallelepipeds. Is that right?* After which student E_5 said: *Ah! We obtained the **double integral**. Hence the name of the course! (doubly integrate the quadric surfaces).*

In the fourth and fifth full-time meetings, students completed some activities and answered questions Q_4: What is the relationship between double integrals and quadric surfaces? and Q_5: What is the relationship between the double integral and the resistance of Gaudí's works? In the end, they answered Q_o.

For the first activity –which required calculating the measurement of a strip of vegetation suppressed from a forest that had the shape of a hyperbolic paraboloid– some epistemological objects such as the partial derivative, polar coordinates, and volume appeared in the discussions of the exercises. We recorded the statements of some course participants:

*E_6: If the vegetation is on the hyperbolic paraboloid, we will calculate the **volume**.*
E_{11}: Yes, for this, we use the double integral with dx and dy.
E_{13}: Did you establish the domain of the original integral?
E_6: I used polar coordinates.

In the solution of the second activity –the calculation of the mass center where the power transmission tower should be installed– it was possible to detect the use of moments of inertia in relation to the abscissa axis and the ordinate axis in E_8's statement: *Remember that when you use **Mx** in the formula, y is in the equation, and when you use **My** in the formula, it is x.* In this statement, the student refers to the algebraic representations for the moment of inertia equations registered by $M_x = \iint yf(x,y)dA$ and $M_y = \iint xf(x,y)dA$.

With these statements, we can see that some mathematical and non-mathematical epistemological objects emerged in each path taken in the development of the SRP. Finally, the REM constituted was synthesized in the map registered in Figure 4.

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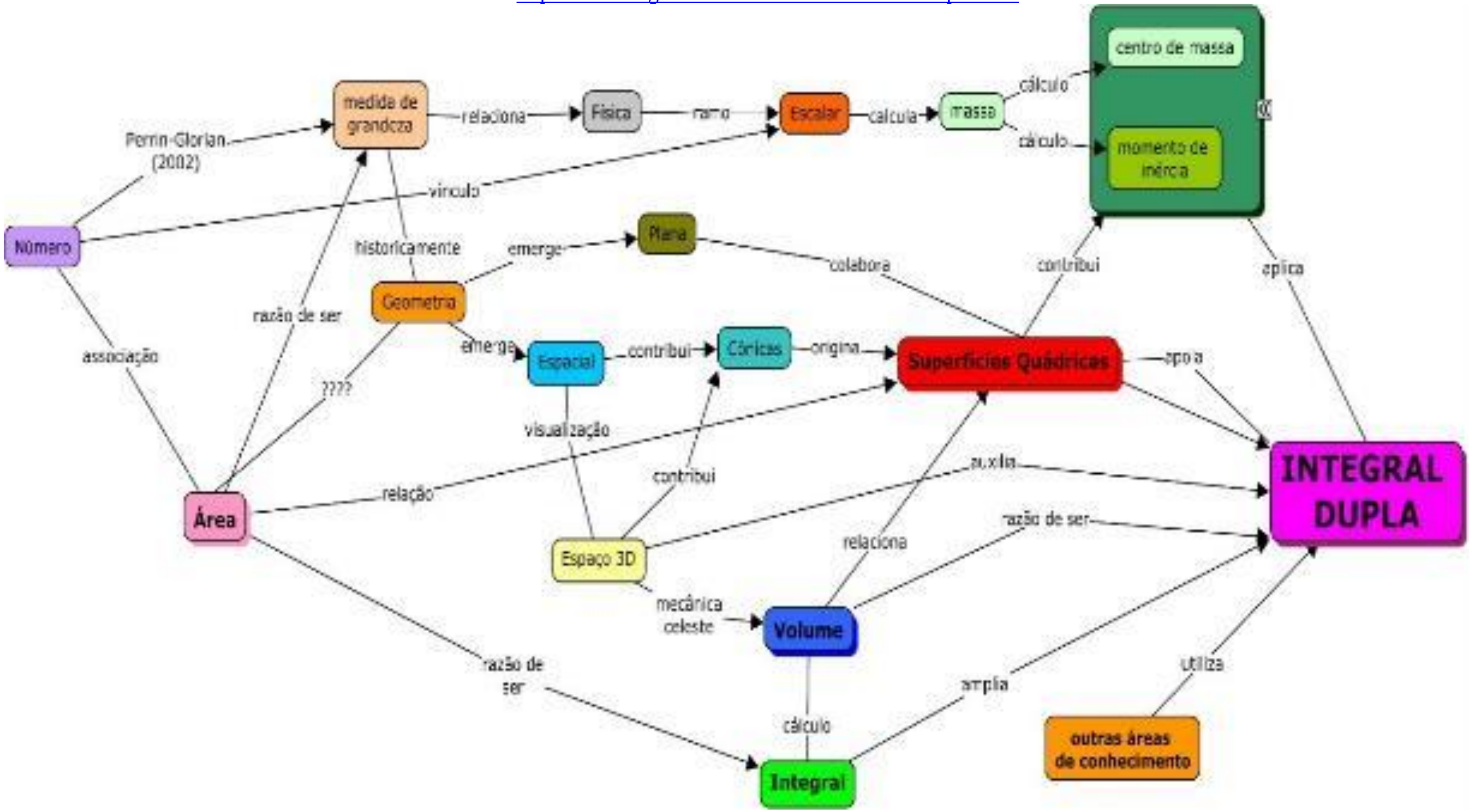

Figure 4.

The REM that guided the SRP (Brandão, 2021, p.139)

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This conceptual map allowed us to compare the dominant models, the previous hypothetical one and the one recorded in Figure 4. As a result, we noticed that the ideas arising from the formation of the mathematical definition recorded in the DEM appear in students' speeches. Without being expressed in the same way as in the differential and integral calculus book, some mathematical objects were suppressed from the previous REM (differential equations, matrices, linear systems, determinants, Lebesque integral, surface integral) due to the connections students made. We verified that the course participants (physics, engineering, mechanics) registered correlations with other areas of knowledge (structural calculation, structural balance, the catenary, and the wires between the power poles).

Regarding students' response to Q_o: How have Gaudí's works withstood the weather conditions? we selected two answers that meet the REM created after the SRP by the objects students researched and related:

> *E_{10}: This morning, I talked about Gaudí's Basilica, which also has this shape (referring to quadric surfaces). I stood there thinking and then I started to understand the relationship between the center of mass, the relationship between beam constructions in the discipline of resistance of materials that uses moments of inertia, and the mass center; and then you see architects concerned about this, not only in their projects but in those of other architects.*
> *E_8: I find it interesting how he joined several solids together, and it should be seen as an element of resistance because it is not a union of solids, as we say: "made in any way." It was made with resistance and balance in mind.*

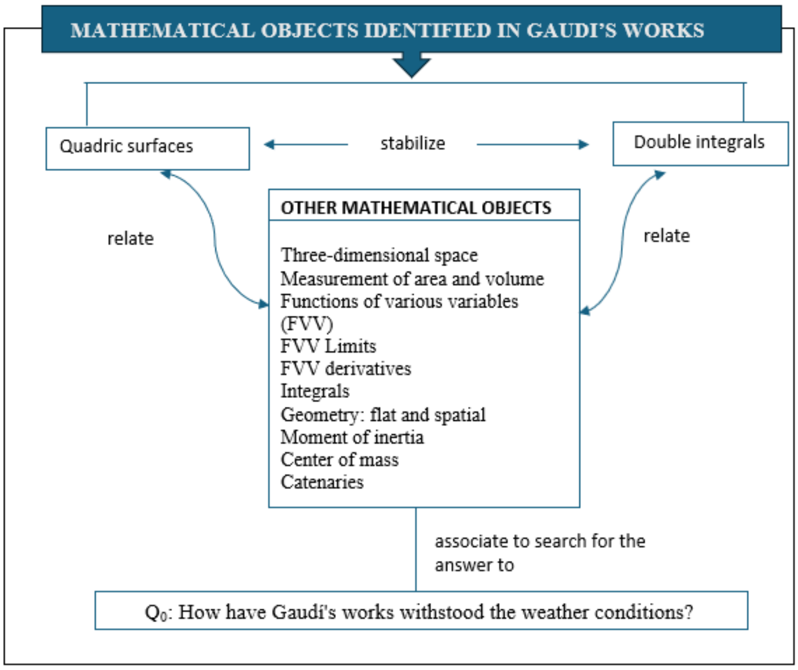

Figure 5

Construction of the alternative reference epistemological model (by the authors)

Based on the didactic problem: *How to organize immediate/ostensive and dynamic/non-ostensive objects that characterize double integrals in calculating the measurement of quadric surfaces to be used in a mathematical model that allows the production of meanings, abductive reasoning and differentiated praxeologies that meet the interests of engineering and mathematics undergraduates?* We infer that every student's answer is relevant for manifesting the learning processes. One should not underestimate what a group of students can provide as an answer. At the end of the SRP, we saw that they could make connections, synthesize answers, and associate mathematical and non-mathematical objects appropriately and coherently.

After the expositions, considerations, and analyses made on the epistemological models, we dared to construct an alternative reference epistemological model (AREM) for teaching double integrals based on the developed SRP, presented in Figure 5.

In the AREM developed, the mathematical objects prominently identified in Gaudí's works were concentrated in three-dimensional surfaces, specifically, in quadric surfaces; however, all the structural calculations to keep them stable and balanced were related to the concepts of center of mass and moments of inertia that are applications of double integrals.

However, these two objects are related to other mathematical objects: three-dimensional spaces, areas, and volume measurements, functions of several variables, limits, derivatives, integrals, plane and spatial geometry, the moment of inertia, the mass center, and catenaries, which, associated, allow us to seek an answer to the initial question Q_0: How have Gaudí's works withstood the weather conditions?

Some considerations and perspectives

In this text, we addressed how epistemological objects, more specifically mathematical ones, can be classified as ostensive and non-ostensive, how we interpret the definition of the mathematical epistemological model, and how it can be constructed linked to a didactic problem and the development of a study and research path.

Regarding the epistemological model, we clarify how we understand the dominant (or current) epistemological model and the reference epistemological model to elaborate a mathematical organization in which we select the mathematical objects that preserve relations with the double integral and the possible areas of knowledge that use it as a tool. However, this model served only as a compass for the development of the SRP, as the students, when following their paths in search of the answer to the generating question Q_0, chose the mathematical objects that they thought were most appropriate to achieve their goal.

Therefore, we can highlight those students reduced the previous REM by mobilizing a smaller amount of mathematical objects to constitute the REM after the SRP, and raised questions related to the course in which they were enrolled to promote pragmatic meanings (structural calculation, wires between posts forming the catenary), that is, those meanings arising from practice. This situation shows that by providing a teaching methodology in which students engage in the creation of scientific arguments to prove their hypotheses, teachers and professors can feel independent to build an alternative epistemological model.

For future research, we suggest that the SRP be developed in the classroom to verify whether, with a larger group of students and a more restricted chronology, other objects will be requested for the composition of the REM.

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