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Teaching knowledge mobilized and re-elaborated by mathematics teachers from the perspective of lesson study and semiotic representations

Conocimientos didácticos movilizados y reelaborados por profesores de matemáticas desde la perspectiva del estudio de la lección y las representaciones semióticas

Savoirs pédagogiques mobilisés et réélaborés par les professeurs de mathématiques sous l'angle de l'étude des leçons et des représentations sémiotiques

Conhecimentos docentes mobilizados e reelaborados por professores de matemática na perspectiva do *lesson study* e das representações semióticas

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Abstract

This study aims to analyze the re-elaboration of mathematical knowledge for teaching, based on the experience of a training process based on Lesson Study, with the contribution of the Theory of Semiotic Representation Registers. Qualitative in nature and with action research characteristics, the Critical Events data analysis approach was used. The LS experience demonstrated to the teachers the different phases of organizing and implementing a class. He also highlighted the importance of collaboration between teachers in building effective pedagogical practices. Regarding the use of the theory of semiotic representations, teachers who started the training process far from its foundations came to realize the need to work with the different representations of the concept of related function, and the challenges in carrying out the necessary conversion processes to effective conceptual mastery in mathematics. In this way, it is considered that the articulation between both theories contributed to the re-elaboration of mathematical knowledge for teaching function.

Keywords: Mathematical knowledge, Class studies, Teaching, Function.

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Resumen

Este estudio tiene como objetivo analizar la reelaboración del conocimiento matemático para la enseñanza, a partir de la experiencia de un proceso de formación basado en el Estudio de Lecciones, con el aporte de la Teoría de los Registros de Representación Semiótica. De carácter cualitativo y con características de investigación-acción, se utilizó el enfoque de análisis de datos de Eventos Críticos. La experiencia de LS demostró a los profesores las diferentes fases de organización e implementación de una clase. También destacó la importancia de la colaboración entre docentes en la construcción de prácticas pedagógicas efectivas. Respecto al uso de la teoría de las representaciones semióticas, los docentes que iniciaron el proceso de formación lejos de sus fundamentos se dieron cuenta de la necesidad de trabajar con las diferentes representaciones del concepto de función relacionada, y de los desafíos para llevar a cabo los procesos necesarios de conversión a efectivos. Dominio conceptual en matemáticas. De esta manera, se considera que la articulación entre ambas teorías contribuyó a la reelaboración del conocimiento matemático para la función docente.

Palabras clave: Conocimiento matemático, Estudios de clase, Enseñando, Función.

Résumé

Cette étude vise à analyser la réélaboration des connaissances mathématiques pour l'enseignement, à partir de l'expérience d'un processus de formation basé sur l'étude des leçons, avec l'apport de la Théorie des Registres de Représentation Sémiotique. De nature qualitative et présentant des caractéristiques de recherche-action, l’approche d’analyse des données sur les événements critiques a été utilisée. L'expérience LS a montré aux enseignants les différentes phases d'organisation et de mise en œuvre d'une classe. Il a également souligné l'importance de la collaboration entre les enseignants dans la construction de pratiques pédagogiques efficaces. En ce qui concerne l'utilisation de la théorie des représentations sémiotiques, les enseignants qui ont commencé le processus de formation loin de ses fondements ont pris conscience de la nécessité de travailler avec les différentes représentations du concept de fonction associée, et des défis liés à la réalisation des processus de conversion nécessaires vers des représentations sémiotiques efficaces nécessaires à la maîtrise conceptuelle en mathématiques. De cette manière, on considère que l’articulation entre les deux théories a contribué à la réélaboration des connaissances mathématiques à des fins pédagogiques.

Mots-clés : Connaissances mathématiques, Études en classe, Enseignement, Fonction.

Resumo

Este estudo tem como objetivo analisar a reelaboração dos conhecimentos matemáticos para o ensino, a partir da vivência de processo formativo baseado no *Lesson Study(LS),* com a contribuição da Teoria dos Registros de Representação Semiótica. De natureza qualitativa e com características de investigação-ação, utilizou-se a abordagem de análise de dados de Eventos Críticos. A vivência do LS evidenciou para os professores as distintas fases de organização e efetivação de uma aula. Ressaltou ainda a importância da colaboração entre professores na construção de práticas pedagógicas eficazes. Com relação ao uso da teoria das representações semióticas, os professores que iniciaram o processo formativo distante de seus fundamentos, chegaram a perceber a necessidade de trabalhar com as distintas representações do conceito de função afim, e os desafios para a realização dos processos de conversão necessários ao efetivo domínio conceitual em matemática. Dessa forma, considera-se que a articulação entre ambas as teorias colaborou para a reelaboração dos conhecimentos matemáticos para o ensino de função.

Palavras-chave: Conhecimento matemático, Estudos de aula, Ensino, Função.

Teaching knowledge mobilized and re-elaborated by mathematics teachers from the perspective of lesson study and semiotic representations

Our understanding of the complexity of the knowledge necessary for teaching practice has advanced since the 1980s. Since then, discussions have been deepened, and several authors have started to consider "that the teaching profession is too complex to be fully and definitively constituted in initial training" (Maldaner, 2000, p.5, our translation). Gatti *et. al*. (2019, p.37, our translation) add that teacher training should move beyond "a cognitive perspective related to mastering the content and start to integrate pedagogical, methodological, historical-cultural and psychosocial training". Regarding Mathematics, studies have been conducted focusing on the specificity of the field.

In this sense, this article, which stems from ongoing research for a doctoral thesis, is relevant because it encompasses contributions from theories of the didactics of Mathematics for the development of a range of knowledge necessary for training mathematics teachers. The various training processes in the field are expected to provide the foundations for a type of teaching practice that provides students with conditions to construct knowledge and mathematical skills, as instructed by Base Nacional Comum Curricular (BNCC, 2018), the national curriculum guidelines in effect in Brazil.

Based on mathematics teachers' formative needs, the objective of this study has been defined: to analyze the re-elaboration of mathematical knowledge for teaching arising from a training experience based on Lesson Study, with contributions from the Theory of Registers of Semiotic Representation (TRSR).

The integration of these theories in mathematics teacher training also emphasizes the relevance of this study, since the literature survey (Rodrigues; Teixeira, 2020; Cardoso; Fialho; Barreto, 2023; Cardoso; Barreto; Pinheiro, 2024) revealed a lack of studies related to the integration of Lesson Study (LS) and mathematical knowledge for teaching, especially when it comes to the concept of functions. As for the intersection with the principles of TRSR, we could not identify any research study on the theme. It is also important to highlight that we decided to explore LS because the literature considers that formative processes based on this theory favor the development and mobilization of mathematical knowledge (Cardoso; Barreto; Pinheiro, 2024).

Mathematical knowledge for teaching was analyzed according to the categories proposed by Ball, Thames and Phelps (2008), who deepened the knowledge proposed by Shulman (2014), specifically focusing on mathematics teachers. The TRSR was used because since Mathematics is an abstract science (Duval, 2009), it requires the exploration of different representations for the construction of a mathematical object. According to the author (Duval, 2011a, 2011b), there is no conceptual construction without perceiving the difference between the concept and its corresponding representations, which makes them indispensable for teaching work.

The theoretical and methodological foundations of the formative process

Since analyzing mathematics teacher training is still a challenging theme within the Brazilian context, this article uses three different theories as references, whose foundations are discussed here.

Structuring of the formative process

In this study, Lesson Study played a role in structuring the formative process, according to proposals by Lewis, Perry and Hurd (2009) as these authors focused their studies specifically on the continuing training of Mathematics teachers, which is close to the objective of this research.

For training based on LS, the authors proposed the development of cycles. Experiencing each cycle requires performing four stages with specific goals and actions to be performed by those involved in it: i) investigation; ii) lesson planning; iii) execution of the planned lesson; and iv) reflection. The structure of the cycle should be adapted to the conditions presented by the institutional reality and the group of teachers experiencing the formative process, including the specification of the length of each stage according to the interests and needs presented by each context. The cycle can be repeated during the formative process depending on the group's goals and conditions.

In the investigation stage, the teachers should study guidelines, official curricula and lessons related to the foundations of the mathematical concepts being studied. From the same perspective, they should solve problems, share and discuss their solutions while taking students' thinking into consideration. In this stage, there is the construction of the foundations for structuring the lesson related to the LS cycle.

The second stage – lesson planning – requires teachers to craft a lesson plan with goals, methodology and procedures, highlight the skills students are expected to develop, choose the tasks to be performed and predict the solutions students will likely come up with in order to anticipate the difficulties they might encounter. This planned lesson should include the participation of all members of the group experiencing the formative process. From among them, they will choose the teacher who will conduct the lesson and the group of students to whom it will be taught.

The third stage – the execution of the planned lesson – involves teaching the lesson itself. The selected teacher teaches the lesson and coordinates the work; the others observe it and collect data about the class while avoiding intervening in their peer's work. The observed data should be recorded to be used in a collective reflection on the experience.

In the final stage of the cycle – reflection –, the group shares and discusses the collected data, including aspects related to the teacher's actions, students' reactions and the learning of the selected concept. In this occasion, they synthesize what was learned from their experience with the cycle, which allows them the possibility to redesign a new cycle.

Lewis, Perry and Hurd (2009) believe that performing the four stages may generate three types of intervening changes: changes in teachers' knowledge and beliefs, changes in teachers' professional community, and changes in teaching and learning resources. The authors acknowledge that "instructional improvements" only occur gradually. Therefore, in this article, we discuss the re-elaboration of teaching knowledge, which was possible based on experiencing one LS cycle.

Mathematical Knowledge for Teaching

In order to deepen the types of knowledge, we used contributions by Ball, Thames and Phelps (2008). These researchers established the scope of the knowledge necessary for working specifically with Mathematics, harnessing two of the knowledge types suggested by Shulman (2014) as necessary for teaching. Thus, they defined: i) content knowledge and ii) pedagogical content knowledge.

According to Ball, Thames and Phelps (2008), content knowledge is composed of: common content knowledge; specialized content knowledge; horizon content knowledge. Common content knowledge is revealed through mastering the concepts of Mathematics, without an intrinsic link to teaching. It should be mastered by teachers and also by other professionals within different scopes. Specialized content knowledge, however, is exclusive to teaching. It is the knowledge that enables teachers to perceive the different mathematical ideas that are present in a concept and integrate them in teaching strategies. Ball; Thames; Phelps (2008) named it "skills related to explaining and justifying". Horizon content knowledge allows teachers to understand how mathematical contents are distributed over school grades, the relationship that should be established among them in each grade and with other curriculum subjects.

The second cluster, pedagogical content knowledge, focuses more on the pedagogical aspects of the Mathematics discipline and is formed by: knowledge of content and teaching; knowledge of content and students; knowledge of content and curriculum. Knowledge of content and teaching requires "interaction between specific mathematical comprehension and comprehending the pedagogical aspects that affect students' learning" (Ball; Thames; Phelps, 2008, p. 401, our translation). Therefore, teachers should consider the content, the skills they expect students to develop, the procedures that they will adopt and make decisions related to planning. Knowledge of content and students promotes the perception of the articulation between mathematical knowledge and what students are capable of developing. Thus, the teacher can anticipate the solutions that might be presented by the students and perceive their constructions and mistakes to generate appropriate teaching strategies. Finally, knowledge of content and curriculum refers to mastering the set of guidelines and instructions contained in the official documents that form the guiding framework in educational materials, curricula and textbooks, which should be the foundational guidelines for their practice.

Theory of registers of semiotic representation and teaching knowledge

As Mathematics is considered an abstract science, it requires the use of signs (Duval, 2009), which, when appropriately organized, form the semiotic representations of each mathematical concept. The author states that working with semiotic representations of mathematical objects is essential for the development of the concept itself. Based on such relevance, discussing the foundations of the Theory of Registers of Semiotic Representation, as proposed by Raymond Duval (2009, 2011a, 2011b), became necessary knowledge, as a part of Mathematics teachers' formative process.

Duval (2009) advocates that teaching work should take into consideration all types of semiotic representations capable of expressing the concept that is being approached – in the case of this study, the concept of functions. To do so, the author considers that it is necessary for the knowing subject to perform three cognitive activities: formation, treatment and conversion.

Formation consists of developing a representation of the concept, that is, "the way objects are presented to us (…), the issue surrounding 'how we can access them by ourselves'" (Duval, 2011a, p. 15, our translation). Representations can be formed in different registers: numerical, algebraic, graphical, tabular, drawings, etc.

Treatment is the cognitive activity that should be developed in order to search for a solution to a mathematical problem. The knowing subject starts with the representation, which has already been formed, and searches for the solution through transformations performed within the register in which the representation is formed. It is called starting register (Duval, 2011a).

Conversions are also transformations between representations of the same mathematical object, but here they occur in different registers – the starting register and the target register (Duval, 2009). Therefore, converting means identifying the signifying elements in the starting representation register and reorganizing them into another register, the target register. This activity both requires and fosters the perception of the difference between the signifier and the signified. Unlike other cognitive activities, this is not governed by rules and relies on identifying and differentiating the representations of the same mathematical object.

The aspects promoted by the TRSR are relevant knowledge for teaching practice. Therefore, in this text, we aimed to highlight which types of knowledge emphasized by Ball, Thames and Phelps (2008) this theory interconnects with and to whose realization it contributes. We reinforce that applying the TRSR to the classroom and to teacher training enables us to harness several types of knowledge necessary for teaching practice in the field of Mathematics. Moreover, it is important to stress the types of knowledge – in their epistemological and didactical conception – among those pointed out by Ball, Thames and Phelps (2008) which display higher or lower potential for being developed based on the use of the aforementioned theory.

Common content knowledge is the most popular type of knowledge in Mathematics teaching degree programs. Therefore, it is common for high school teachers to master it. In this research study, it was not possible to detect any moments when this knowledge was developed by the teachers, so it will not be analyzed here.

On the other hand, the articulation of specialized content knowledge has been intensely observed, since this type of knowledge refers to a deep comprehension of a specific concept, which needs to go beyond what is taught to students. By highlighting the need for perceiving different representations, the interconnections among them and the cognitive activities that should be developed with students, the TRSR provides teachers with elements from epistemological, methodological and cognitive points of view, which favors the development of an analytical view of one's own knowledge and of the obstacles that students may face, that is, difficulties faced by teaching practice.

According to Duval (2011a, p. 15, our translation), in addition to the importance of comprehending the abstract nature of mathematical objects, it is necessary to understand "how objects are presented to us". Thus, the author highlights that it is important for Mathematics teachers to be aware of the epistemological and cognitive role that semiotic representations play in their own mathematical knowledge.

It is necessary for Mathematics teachers to understand not only the concept of functions, but also the types of registers of semiotic representations through which a function can be represented, such as graphical, algebraic, tabular, figural and in their native language. Furthermore, it is important to emphasize the relevance for teachers to comprehend the functioning of each register of representation and how it is related with other registers, that is, the interconnections among different registers of semiotic representation are a core foundation for mathematics learning.

From the same perspective, the interconnection between specialized content knowledge and knowledge of content and teaching highlights the importance of the TRSR for mathematics teaching and learning processes. Knowledge of content and teaching requires teachers to combine specific Mathematics knowledge with knowledge of pedagogical and didactical aspects that could influence student learning.

Duval (2011a, p.9) argues that "[…] specific comprehension problems that students face in Mathematics learning" originate not only in the issues related to the pedagogical organization of tasks, but also "in the particular epistemological situation of mathematical knowledge". These ideas reinforce the influence of the TRSR on teachers' comprehension of the difficulties encountered by students in the Mathematics discipline, which reveals its intense relationship with knowledge of content and students.

By understanding the cognitive functioning of students' thinking in Mathematics learning, Mathematics teachers will be able to identify the difficulties their students face in problem-solving and the potential gaps in their cognitive activity, that is, the formation, treatment and conversion between registers of semiotic representation. This identification is important because it can serve as a diagnostic tool for developing pedagogical practices aligned with students' specific needs.

Through the categorization of different forms of representation of mathematical objects, such as graphical, algebraic, tabular, figural and other types of register, and the interconnection among these registers, Duval (2009) proposed theoretical aspects that influence mathematics teaching and learning processes. This influence can be observed in official curriculum documents, which interconnects the TRSR with knowledge of content and curriculum.

The importance of representations is observed in BNCC (2018), the national curriculum guidelines in effect in Brazil, across all stages of Basic Education. In high school, this idea is more clearly evident in the document, which states that students should "develop more and more sophisticated representations and procedures" (BNCC, 2018, p. 529, our translation). Conceptual terms developed by the TRSR are observed in the document:

> The skills that are directly linked to representations assume the development of *registers to evoke a mathematical object. […] in this field, it is possible to verify unequivocally the importance of representations for the comprehension of facts, ideas and concepts, since access to mathematical objects occurs through them***.** In this sense, in Mathematics, using *registers of representation* and different languages is necessary for understanding, solving and communicating the results of an activity (BNCC, 2018, p. 529, our emphasis and translation).

Thus, the TRSR provides solid theoretical foundations for analyzing content and, consequently, for curricular guidance, because the BNCC (2018) is the document that offers guidelines on teaching and learning processes in Brazil. This allows teachers and curricula to consider various ways through which knowledge can be represented and understood by students. The essence of this idea is that one should not consider only "what" to teach, but also "how to teach" it.

The relationship between the TRSR and knowledge of content and curriculum enables interconnecting the theory with horizon content knowledge as the following aspects are an important part of teaching work: understanding how contents are distributed across each school year, how they interconnect with one another, and what prior and subsequent knowledge is essential to comprehend a specific content. Each of these elements makes use of symbols and visual representations to communicate mathematical concepts. They are essential to communicate and understand mathematical ideas, and their correct interpretation is central to learning and to the application of mathematical concepts.

It is important to highlight that these forms of knowledge are not linear. They are part of a set of ideas with interconnected knowledge, enabling a wide comprehension of the educational process. Another relevant aspect is that the TRSR is integrated into knowledge, revealing that these forms of knowledge are larger categories and that they have other characteristics that define them. Furthermore, it aims to promote a dynamic and integrated approach to Mathematics teacher training. Thus, it is advocated that the TRSR is important teaching knowledge within the context of mathematics teaching and learning, and it provides essential elements to the curriculum, while being in line with the principles stating that the theory offers theoretical and methodological constructs for the comprehension and development of Mathematics teaching, thus highlighting the importance of various representations and promoting a specialized understanding of mathematical knowledge.

Methodological Approach

This article stems from intervention research with a focus on action-intervention, following the guidance proposed by Ponte (1994, p. 6, our translation). The author stated that

"[…] the discussions and decisions related to the development of an investigation are deeply shared by the investigator and other participants".

The intervention consisted of a continuing education formative process for working with affine functions. It lasted 60 hours, which were divided into 17 sessions. The process constituted one LS cycle and focused on working with the concept of affine functions, while guided by the principles of the three aforementioned theories.

The participants of the process were three Mathematics teachers who taught Mathematics at a vocational high school in the state of Ceará, Brazil. The process was carried out between May 2022 and January 2023 across 17 sessions, which were distributed as follows: introductions and negotiations with the group (3 sessions); investigation (8 sessions); planning (3 sessions); execution of the lesson (1 session) and post-lesson reflection (2 sessions). To implement the class, we chose a first-year high school group, because this is the year when the concept in question is allocated the most curriculum time.

All sessions were recorded in audio, and a field journal was written. Oral data transcription was performed through *Transkriptor* software, and the reports were reviewed by the researchers. The participants also transcribed the Critical Events, that is, those that provide data that "[…] may generate insights into participants' explicit and implicit meanings within an educational context" (Powell; Francisco; Maher, 2004, p. 26, our translation).

The research was approved by the Ethics Committee of the State University of Ceará, Brazil, with a substantiated opinion[3], and based on the theoretical and/or methodological principles proposed by Lewis, Perry and Hurd (2009), Ball, Thames and Phelps (2008) and Duval (2009, 2011a, 2011b).

Data analysis and discussion

During our initial contact and negotiations with the group, we discussed elements of the formative process proposal. There was a proposal for exploring functions because it is an integrative concept and a constituent element of a conceptual field (Merli, 2022). The teachers considered working with exponential, quadratic and affine functions. They chose affine functions because, in their own words:

> **Teacher C**: [...] I find it interesting because for external assessments, affine functions are more palpable.
> **Teacher B**: The students asked for a review of this content...
> **Teacher B**: [...] for those who studied it long ago...
> **Teacher C**: Or for those who didn't study it deeply (Formative process, 2022).

[3] 63721822.2.0000.5534.

It is possible to understand that the teachers made a choice based on the importance of affine functions for external assessments, and because the students had difficulty with this content. The decision also took into consideration that by exploring the concept of affine functions, the students could keep exploring and improving their knowledge of other types of functions.

During the investigation sessions, it was possible to notice the development and re-elaboration of teaching knowledge, as well as the mobilization of the TRSR as mathematical knowledge for teaching. It is fundamental to highlight that analyzing the mobilization of mathematical knowledge for teaching permeates the comprehension that different forms of teaching knowledge are harnessed in an intertwined way, and they are interconnected and linked. It occurs because these different forms of knowledge are inseparable, and none of the elements is totally isolated, that is, it is not possible to conceive the idea that the mobilization of teaching knowledge can be linear; actually, it is a complex web of articulated and related knowledge.

From the perspective of teaching, semiotic representations and their role in teaching and learning processes were gradually understood across the formative process as essential elements for teacher planning. Although the teachers considered the texts "an easy read", Teacher C said: "[…] I had difficulty understanding some technical terms specifically used by the author […]". This opinion was shared by the other students, who admitted to not having studied the TRSR deeply in initial or continuing teacher training.

In light of this, the researchers asked the teachers what their initial understanding of the notion of semiotic representations was. They answered as follows:

> **Teacher C**: [...] the very etymology of the word suggests that semiotics has to do with what you see […] with how I can relate an image to a concept, to an object […] So, it is the way that will represent what the object is. It's one more way of representing visually, and it has a certain organization, too. It's not something done like a mere sketch […] So, there's a whole set of patterns in semiotics […]
> **Teacher B:** [...] I understood it when the professor exemplified it with writing a text: you start with words, then build sentences, then build paragraphs, then you have the full text. I think that now it's clearer than what I was thinking before.
> **Teacher A**: Now we can understand it [semiotic representation] better, mainly because within the semiotic representation itself, we see that there are three functions, and each function has, as its name suggests, its own particularity (Investigation Stage, 2022).

It was possible to notice that the teachers still had an incipient understanding of semiotic representations, even though their perceptions varied in terms of details and levels of depth. Their answers showed the development of ideas, such as the distinction between representation and object, and the perception of the different functions of representations, moving closer to what Duval (2009) proposed. Furthermore, we especially highlight that the variety of understandings and diversity of perspectives presented by the teachers pointed out the multiple facets of the TRSR concepts that could be explored within educational contexts.

In addition, we emphasize that the teachers still did not appear to understand the importance of exploring different representations simultaneously, since they did not focus on the existence of different registers. These gaps are expected because they did not have the necessary contact with the theory during their training. Their lack of mastery of representations indicates weaknesses in specialized content knowledge and knowledge of content and teaching.

As the discussions progressed, the teachers gradually appropriated what they called "technical terms", as shown by their perceptions of registers of semiotic representations:

> **Teacher A:** About registers [...] to some extent, it's like vocabulary. The bigger a person's vocabulary in Mathematics, the more they can develop and understand things better.
> **Teacher B**: I like to mention the example Teacher C gave [...] the more they [the students] can see that the same number can be represented in fractional form, as a decimal, and even graphically, the deeper their understanding of that subject. And it says here that the more representations they produce, the more apprehension they have, and I agree with that. If they can represent the same thing in various ways, it means they understand what they're doing.
> **Teacher C**: [...] that is, they will increase their mastery of that object. They will not be limited regarding their ability to understand or visualize the object or the situation that is being explored (Investigation Stage, 2022).

The teachers' comments show their comprehension of the difference between the concept and their signifiers. In this sense, the three teachers shared the view that using various registers and representations is central to Mathematics teaching and learning. They agree that using different forms of representation helps students to effectively develop the concept in question. These ideas are in line with what Duval (2009, 11a) proposed when he stated that an approach based on multiple representations is an important pedagogical strategy for learning and for the development of mathematical skills in students.

Considering the teachers' answers, it is possible to perceive the mobilization of knowledge of content and teaching and knowledge of content and students because they emphasize a pedagogical approach aligned with the importance of semiotic representations and with the context which students are part of, especially in Teacher B's and C's discourses. Although the participants referred to different representations, none of them mentioned any cognitive activity to be developed by the learning subject up to that point. The teachers only mentioned it when analyzing a problem situation, which will be discussed later.

Moreover, the discussions also highlighted the importance of students to understand the meanings of the symbols omitted in mathematical studies. They are usually omitted when it is necessary to perform algebraic operations. The symbols omitted in Mathematics are not visually apparent to students, making it necessary to use pedagogical practices that bring to light the symbols and numbers omitted in the representation to facilitate students' comprehension. Teacher A explained that when it is a positive case, it is easier for students to visualize it. However, when it comes to, for example, the symbols and numbers omitted from a second-degree polynomial function, students' difficulties increase considerably.

Teacher B added that "they also get really confused when you add x and x, or when you write x times x. They can't understand it [...]". Teacher B also said that there are several procedures in Mathematics, and that it confuses students. Teacher A pointed out that in the case of affine functions, there are at least two symbols that may confuse students: coefficients *a* and *b*. Teacher C stated that if students are really confused and do not understand well these symbols, they will have real difficulty in successfully understanding the concept of affine functions.

These comments allow us to perceive the mobilization of knowledge of content and students. The teachers explain their students' main difficulties and the confusion that arises when they have to deal with symbols, which are sometimes omitted in mathematical treatments. When teachers comprehend the importance of understanding students' difficulties and how they can be related to the multiplicity of mathematical procedures involved in a problem-solving process, especially when it comes to underlying symbols and operations, it is possible for them to develop pedagogical strategies to overcome their students' inability to comprehend the semiotic representations that are not so evident.

In light of the evidence presented, it is possible to infer that teachers' ideas and perceptions regarding the TRSR progressed. They said that the theoretical elements proposed by the theory had already been present in their pedagogical practice, even if unconsciously. Their statements confirmed that the TRSR is important knowledge for teaching that can improve teacher training, highlighting its characteristic of encompassing multiple forms of knowledge. By gradually becoming aware of the signifier and the signified, teachers can develop specific pedagogical practices so as to foster the interconnection of different types of semiotic representations, thus re-elaborating knowledge of content and teaching, specialized content knowledge and knowledge of content and students.

During the discussion of a problem situation, the group of teachers perceived students' possible difficulties with affine functions and detailed the pedagogical strategies to address them. Observing symbols and representations and how they are interconnected in the problem and developing adaptations and modifications to the problem situation were considered important tasks in the formative process. The problem the teachers chose for the discussion can be seen below.

> **5.** (UFG-GO) A baker makes 300 loaves of bread an hour. Based on this data, write:
> **a)** the rule representing the number of loaves made (*p*) as a function of time (*t*); $p = 300t$
> **b)** how many loaves are made in 3 hours and 30 minutes?
> 1,050 loaves

Figure 1
Problem situation chosen by the teachers for the discussion (Prisma FDT textbook, 2020, p. 69).

Teacher A argued that their choice was based on the fact that it showed a common situation for students and that it could attract their interest and motivate them. Teacher B agreed with A's idea and added that "[…] it's more aligned with students' lives".

The teachers emphatically stated that one of the reasons for students' difficulties was their focus on the notion of variables **x** and **y** to the detriment of other letters that can represent the same idea: **t** and **p,** in the case of the problem. "For them, everything is **x** and **y**. We must make this differentiation. Actually, in the **x** and **y** relationship, (they must) understand what means to be **x** and what means to be **y**" (Teacher A). In this sense, the teachers concluded that students' initial difficulties stemmed from the perception and distinction of the variables involved in the process. The teacher's comments, especially Teacher A's, showed progress in terms of specialized content knowledge and knowledge of content and students, both because he is the head teacher of the first-year classes and due to his active participation in the discussions.

Another aspect that was emphasized was the term "rule", which appeared in the problem statement and was highlighted as a likely cause of confusion among students. The statement does not express "formation rule" clearly. The teachers commented on it:

> **Teacher A**: [..] I think they need to know what the question is asking them, what the rule of a function is. It's a formation rule.
> **Teacher B:** Not for us, who have studied it, but for a student who's starting to read it, it might be hard.
> **Teacher A:** Yes. For example, I'm exploring tests with open-ended questions. I asked "what's the function that represents this or that situation?" […] (Investigation Stage, 2022).

Their comments also emphasized the importance of adapting teaching to students' comprehension level and reinforced the relevance of conceptual understanding over the mere memorization of formulas. The teachers showed that they understand their students' difficulties, revealing knowledge of content and students linked to teaching affine functions.

The researchers asked about the cognitive activities students may develop during the problem-solving stage. Teacher A offered this answer about item *a*: "It's treatment, isn't it? [pause] No, it's conversion. Conversion […] from native language to algebraic representation". In this case, Teacher A reflects on the cognitive activities that could be developed by the students and concludes correctly by saying that conversion is the activity that may be performed, showing signs of mastery of the specialized content knowledge of the TRSR.

Item *b* was considered a likely focus of student difficulty because it involved time with two units of measurement (3 hours and 30 minutes). "In this case, [the student] may notice that thirty minutes are half an hour" (Teacher A). Another strategy discussed was the possibility of using the calculation of proportion, indicating that 1 hour is to 60 minutes, just as (*t)* hours is to 30 minutes. "Proportion could be confusing for them. And after they do it, they need to add it to 3" (Teacher A). Teacher B agreed by stating that, in fact, students might consider that the number obtained from calculating the proportion was (*t*) itself and replace it in the formation rule without adding the 3 hours already mentioned in the problem statement. The group considered this pedagogical strategy a challenge, which reaffirms that teachers mastered knowledge of content and students.

Teacher B highlighted another possible pedagogical strategy: students could craft a table in which hours were fractioned in halves by using mathematical logical thinking. "Guys, if I know the baker makes 300 loaves of bread in an hour, how many will he produce in half an hour? They should understand that it will be half of that number." From the teacher's perspective, students would be able to visualize it in "another way". With this in mind, it is important to emphasize that knowledge of content and students and knowledge of content and teaching are interwoven, considering that the teacher aims to combine the pedagogical strategy with a supporting semiotic representation to overcome the difficulties that students might face.

The group agreed that adding the students' construction of a table evidences the mobilization of one more register of semiotic representation. Furthermore, they considered that, in the numerical register, the sum of time (*t*) and 0.5 (corresponding to 30 minutes) will not be difficult for students.

Nevertheless, Teacher B argued that some students might have difficulty in performing the treatment of item *b*, which involves the multiplication $p = 300, 3.5 = 1050$ loaves of bread, especially because it contains a decimal number. For this reason, the teachers agreed that maybe it would be necessary to propose a discussion for this type of multiplication, confirming that the teachers mastered knowledge of content and students.

Another adaptation suggested by the teachers was to add a representation in a graphical register. The justification for this suggestion was that the problem will provide greater articulation among the multiple semiotic representations of the affine functions concept, as shown in the Table 1.

Table 1

Reformulated problem situation (Research collection, 2022).

Problem situation (UFG-GO-adapted) A baker makes 300 loaves of bread per hour. Based on this data, do the following: a) Build a table, considering the following times: 1 hour, 2 hours, 3 hours, 4 hours and 5 hours, with the corresponding number of loaves of bread. b) The formation rule that represents the number of loaves of bread made (*p*) as a function of time (*t*); c) Using the formation rule, how many loaves are made in 6 hours? d) Considering the table from item *a*, identify 4 ordered pairs. e) Represent graphically the relationship between loaves of bread and time.

The teachers' mobilization of knowledge – especially of specialized content knowledge, knowledge of content and teaching and knowledge of content and students – in the investigation stage was observed while they discussed the theme as a group. We highlight the importance for teachers to conduct discussions, critical analysis and reflections, which progressed gradually throughout the formative sessions with the group, as they engaged and exchanged ideas when working to overcome the challenges that might arise in the classroom.

It is important to highlight that the development, re-elaboration and mobilization of knowledge were achieved by each teacher in a different way. The word *re-elaborate* is used here because the teachers themselves stated that they already used some of theoretical elements proposed by Duval (2009), but they did not have deep knowledge of the theme. By understanding the importance of semiotic representations for mathematical activity, the teachers expanded their perspective regarding what had already been developed by them in their daily practice.

Lesson planning stage

In the lesson planning stage, we highlight the collective work and the collaboration among the three teachers to develop the plan. Since the problem had to be modified, as previously discussed, Teacher A proposed that the lesson should follow the methodology described below, and the others agreed with it. They decided to select six problems from the first-year textbook. The researchers asked what methodology they would use to explore problems in the classroom. This is the conversation that follows:

> **Teacher C**: [...] in the beginning, we need to explain the lesson proposal based on a problem and divide the students into groups. Each group will have a problem to solve, and, at the end of the class, they will present what they were able to develop.
> **Researcher**: But will each group have a different problem?
> **Teacher C**: Yes.
> **Teacher A**: I have an idea. First, let's solve a problem with the whole class […] then, we give one problem to each group.
> **Teacher C**: All right.
> **Teacher A**: We will conduct the whole development and, as they answer, we will write it on the board.
> **Researcher**: In this case, will you explore one or two problems in a group?
> **Teacher A:** One problem. Only one problem [...].
> **Teacher C**: That's it. Six or seven groups with six students in each.
> **Researcher:** [...] How would you choose the problems?
> **Teacher C**: From the textbook.
> **Teacher B**: We should change the variables.
> **Teacher C**: [...] it's good because they'll see functions being applied, not just the same old thing again.
> **Teacher A**: And we could see some topics in the textbook. The topics that will be approached.
> **Researcher:** And what will the first problem be? This one here? (the one with the loaves of bread)
> **Teacher C**: It could be that one. The starting point and the reformulated version (Planning Stage, 2022).

The teachers adapted the problems according to the five items from Table 1. They aimed to enhance them with variations in the use of semiotic representations. They also prioritized the adaptation and modification of the variables involved in the problems, while always focusing on the difficulties the students might face when trying to solve them, especially how values could impact the graphical construction. This adaptation took two formative sessions. For the aforementioned aspects, the teachers harnessed knowledge of content and teaching, knowledge of content and students and specialized content knowledge. These forms of knowledge were perceived as the teachers adopted a reflective approach to the teaching of affine functions, while observing the challenges that would be faced by students both in terms of conceptual aspects and mathematical procedures, and in the semiotic representations and cognitive activities related to each problem. We highlight that Teacher C was chosen to teach the lesson because he was the most experienced one in the group. Although he was not the head teacher of the selected class, he was considered "a good teacher" and accepted by "most students".

The conversation transcribed above showed signs of a methodological approach in teachers' discourse, especially Teacher A's and C's, when they suggested teamwork, problem-solving and discussing an initial problem with the whole class, which confirmed their mastery of knowledge of content and teaching. The teachers delved deeper into aspects of the methodological approach by highlighting didactical resources and the time allocated for students to do the tasks.

> **Teacher A**: [...] How long will it take the groups to solve it?
> **Teacher C**: I think thirty minutes, at most.
> **Teacher A**: […] The first two groups to achieve the solution usually take longer, then the other teams understand it. But we can't leave their solutions on the board. What should we do?
> **Teacher C**: Posters?
> **Teacher A**: Yes! We can even compare the graphs.
> **Researcher**: What would they [the students] write in the posters?
> **Teacher A**: The formation rule and the graph.
> **Teacher C**: That's it. We could ask them to discuss the other items and the poster they made.
> **Teacher A**: [...] They explain the coefficients and the graph and make a presentation [...] but it would be good if they had the other problems, so they could understand what each team is presenting.
> **Teacher C**: So, we will have to print sheets with the questions for each group. [...] leave some space for them to solve it.
> **Teacher A**: I think this would be very interesting. Very good (Planning Stage, 2022).

The teachers' discourses emphasize the mobilization of knowledge of content and teaching as they discussed the possibilities for teaching affine functions. The teachers' reflections show their engagement in developing strategies to mediate students' comprehension. This continuous reflection on how to teach affine functions reveals not only their mastery of the content, but also their ability to adapt and contextualize teaching in order to meet students' needs and skills.

Moreover, the group dedicated to discussing the skills and abilities proposed by the BNCC that could be developed in the planned lesson.

> **Researcher**: I want you to think for a while. Do you think we'll be able to develop all of them [the abilities]?
> **Teacher C**: We'll have to select them. [...] (Planning Stage, 2022).

In the following discussions, according to Teacher C, it was necessary to verify the specific abilities, too. The teachers understood that planning a lesson did not mean that it would contemplate all skills, and that several classes, with different goals, may be necessary in order for a skill to be achieved (Table 2). Besides, we highlight that the teachers mentioned having received continuing training on the BNCC through training programs offered by SEDUC-CE (Ceará State Department of Education, in free translation) and by the school management, so they were familiar with the curriculum and current official documents.

Table 2

Skills selected by the teachers (BNCC, 2018, our translation).

(EM13MAT101) Interpreting economic, social and natural science-related situations involving the variation of two quantities through analyzing graphs of functions and rates of change while being supported or not by digital technologies. (EM13MAT302) Solving and proposing problems whose models are first and second-degree polynomial functions, in different contexts, involving or not digital technologies. (EM13MAT401) Converting algebraic representations of first-degree polynomial functions into representations on the Cartesian plane, identifying cases in which behavior is proportional, with or without using algebra and dynamic geometry software/applications. (EM13MAT501) Investigating relationships among numbers in tables to represent them on the Cartesian plane, identifying patterns and formulating conjectures to generalize and express this generalization algebraically, while recognizing cases when the representation corresponds to a first-degree polynomial function.

The skills selected by the teachers, as seen in Table 2, showed signs of mastery of knowledge of content and curriculum, especially by teacher C. Even though Teachers A and B participated actively in the discussion, Teacher C was the most proactive participant in the conversation. The teachers discussed the lesson goals:

> **Teacher A:** I think the first thing is to identify a first-degree function […]. Identify a first-degree function.
> **Teacher C**: A polynomial function (Planning Stage, 2022).

The group agreed that it would be interesting to set a goal related to the comprehension of representations and problem-solving, as detailed below.

Table 3

Objectives proposed by the teachers (Research collection, 2022)

- Comprehending/identifying a first-degree polynomial function. - Comprehending the various semiotic representations of first-degree polynomial functions and representing them (Native language, Algebraic, Tabular and Graphical representations). - Interpreting problem-situations in different contexts.

The clarity the teachers showed by presenting a well-defined methodological approach while highlighting the skills to be developed in the classrooms and the goals that were established confirmed their mastery of knowledge of content and teaching. By communicating the pedagogical strategies in a precise and organized way, the teachers showed not only their understanding of mathematical concepts, but also their ability to transpose this knowledge into teaching practice. These ideas can also be observed in their choice of didactical resources, when teacher C pointed out the need to print the problems:

> **Teacher A**: In this case, all groups will receive all problems and a poster board.
> **Teacher C**: That would be good.
> **Teacher B**: They will produce the poster.

Researcher: You should also hand out paper so that they can solve the problems. [...]
Teacher C: We need around 10 poster boards. Rulers, too (Planning Stage, 2022).

Besides the resources mentioned above, such as poster boards, rulers and paper, Teacher C emphasized the use of textbooks: "we used the textbook for planning, so it should be in the references". The teacher defined that he would primarily use the whiteboard because he considered it an interesting methodological strategy to engage students' participation.

The planning stage was important not only for promoting the mobilization and re-elaboration of teaching knowledge, especially knowledge of content and teaching, knowledge of content and students, specialized content knowledge and knowledge of content and curriculum, but also for bringing the teachers closer to one another and to the researchers. The group strengthened its relationships and drew closer to what Lewis, Perry and Hurd (2009) highlighted about the collaborative work fostered by LS.

Lesson execution

This stage consisted of the execution of the lesson according to the plan developed by the teachers. It was a very important moment in the formative process, in which the teachers performed the pedagogical practice that had been discussed as a group.

In the beginning of the class – a discussion of the concept of Functions and Affine Functions –, Teacher C approached the theme (Affine Functions/first-degree polynomial functions) and explained the three goals proposed for the lesson (Table 3), discussed representations of the concept of functions and explained various examples on the white board. In this stage, the students participated actively in the development of the class and demonstrated that they understood the representations used.

The teacher asked the students "what an affine function is" and the specific types of function. In addition, he asked for examples of coefficients to write the formation rule of affine functions. However, he was careful enough to give different names to the ordinate, as can be seen in the examples constructed by the teacher and the students together:

E.g.1: f(x) = -2x +9 (coefficients: a = -2; b = 9);

E.g.2: g(x) = 3x (coefficients: a=3; b=0);

E.g.3: h(x) = 5 (coefficients: a = 0; b=5).

The teacher emphasized that the symbols f(x), g(x) and h(x) represented the same idea, that is, they represented the *y* in the function. According to Duval, it is possible to say they are representations of the same mathematical object. Moreover, the teacher highlighted that the function h(x) was a specific type, a constant function. He approached aspects of the first-degree polynomial function, reinforcing its relevance for other fields of knowledge, even after high school. In addition, he discussed with the students the representation in the graphical register. As the examples were given, the students followed the solution by answering actively.

As the teachers had anticipated in the lesson plan, the students showed understanding of the formation rule and treatment of affine functions in the algebraic register. They demonstrated their perception of the concept of ordered pairs, in which order matters (x,y). It is important to emphasize that, until that moment, the students did not seem to mind being observed by the group of teachers. Besides, they were participating actively, as the teachers had anticipated.

During the development of the representation in the graphical register, the teacher commented the concepts of positive and negative coordinate axes, scales and the origin of the axes. Other concepts discussed by the teacher were "abscissae" and "ordinates", which the students found strange. The teacher used the graphical representation to explain said concepts by saying that one of the axes represents the ordinate and the other, the abscissa. Thus, he asked: "how do you create the graph?". Two students answered: "by connecting the points"; "by drawing a straight line".

The teacher said that the straight line is formed by infinite points (ordered pairs) and explained the concept of increasing and decreasing behaviors in the graph of the function. We highlight that the teacher displayed mobilization and mastery of specialized content knowledge, knowledge of content and teaching and knowledge of content and students. He was clear when communicating the concept of affine functions and the semiotic representations involved. Moreover, he encouraged students' participation by asking them questions during the development of the class and demonstrated sensitivity towards students' needs. This approach guided the discussion in an interactive way by involving the students with questions and stimulating their active participation, as had been anticipated by the teachers.

Besides, it is relevant to emphasize the condition of complexity, interweaving and interconnectivity of knowledge, which were observed in the teacher's pedagogical practice through a simultaneous and inseparable mobilization. The teacher expressed his mastery of specialized content knowledge and, at the same time, adapted his discourse to students' needs, thus stimulating their participation and answering students' questions. In addition, he used an accessible methodological approach aiming to promote students' conceptual understanding of affine functions.

The class progressed towards its third stage, solving problem situations as a group. The students had previously been divided into groups by the head teacher, Teacher A, who said the criterion adopted for the division was to promote heterogeneity within groups, so that there were students from different levels of knowledge and understanding in each group. Six groups were created. It was necessary to perform the activity in another environment, with more space, so that students could discuss and make the posters, which had been foreseen in the lesson plan. The size of the classroom did not accommodate group work for a 40-student class, which means a lack of opportunities for students to exchange knowledge.

In that stage of the lesson, it was hard for Teachers A and B to remain in their observation roles, so they went beyond what the LS approach prescribes for the members of the group. As the discussions within student groups generated high demand, they supported Teacher C, who was conducting the lesson, in the pedagogical practice.

It is relevant to highlight the teachers' mastery and mobilization of knowledge of content and teaching. Their pedagogical practice, focused on solving problem situations in groups, enabled them to observe the mobilization of knowledge by organizing the student groups according to specific criteria (heterogeneity), defining tasks to be performed (solving problem situations and making posters), choosing the appropriate environment to conduct the activity (a larger space to enable conversation and development) and adapting the plan during the execution of the lesson (teachers participated actively due to high demand from students).

Other important aspects were teachers' decision-making, interacting with students during the activity, overcoming challenges (such as the time allocated for the activity and the difficulty presented by two groups in making their posters) and adapting the planned strategies in face of the circumstances, which confirmed their mastery of knowledge of content and teaching.

In the fourth stage of the lesson – a collective discussion about graphical and algebraic representations of the problems –, four teams managed to conclude the solution and finish the posters within the allocated time, while two groups had difficulty. The students' representation in the graphical register showed their learning and understanding of some of the elements that constitute this representation, but it is also possible to observe the complexity of the interconnection between the algebraic and graphical registers of affine functions, as stated by Duval (2011a, 2011b). Therefore, it is necessary to explore semiotic representations more deeply so that students understand the explicit and implicit interconnections involved in the problem situation. Furthermore, the role of the teacher in the process should be highlighted as central to student learning.

In the assessment stage of the class, the students recognized it was "a very good" class and emphasized the representations they studied. They also pointed out that the class had different characteristics from those usually observed. In face of the modifications that were made for the execution of the lesson, including moving students to another environment, this comment leads us to infer that the students referred to the possibility for discussing in groups and presenting each team's production to exchange the knowledge they had generated. This shows the importance of what LS proposes in terms of planning and teachers' collective participation for students to develop promoted mathematical concepts.

Teacher C's mastery of knowledge of content and teaching was confirmed. He was always engaged and motivating during the execution of the lesson. Besides, he followed the collectively developed plan, adapting it as necessary. This reflects that a lesson plan should not be immutable. It is changeable and depends not only on the teacher, but also on students.

Another point that needs to be highlighted is the teacher's mastery of specialized content knowledge and knowledge of content and students. The teacher was consistently attentive to students' needs, adapting the plan and answering questions as needed. The interconnection of knowledge during pedagogical practice became more and more evident, that is, the mobilization of teaching usually happens simultaneously and in an interconnected way.

Post-lesson reflection stage

The post-lesson reflection stage aimed to promote a discussion about the executed lesson from different perspectives: those of the lead teacher, the peer teachers and the researchers. About the execution of the lesson plan, Teacher C said: "[…] I'll start talking about the time. […] In a general way, everything we planned was executed, but time didn't allow us to do everything completely". The problem with time occurred mainly because of the groups who received problems related to decreasing functions. When teachers were planning, they did not realize that this concept could present more difficulties, which reveals their shortcomings in both knowledge of content and teaching and knowledge of content and students.

The researchers asked the teachers what aspects had presented more difficulties to the students, and Teacher A answered: "Actually, it was a nice surprise. I think they didn't have much difficulty. What made it harder about the graph was that it involved decimal numbers, decimal points, […] it made the construction of the graph more difficult […]". This comment shows that although they took an active role at some moments of the class, the teacher was still able to observe the main difficulties faced by the students.

As for suggestions for changes to the class, Teacher A stated that: "[we could] add one more question, one that doesn't involve the rule of three. We could review questions with decimal points [decimal numbers]". The teacher referred to the need for a question that allowed students to understand that affine functions were not always linked to the rule of three, which occurred in the question presented in Table 1.

Teacher C also considered it necessary to add "another moment to present some type of diagnosis. Although we have one part in the group and the part in which they express their ideas, there should be some feedback at the end. A test, for example, so we'll be able to see if we reached everyone […] if they achieved a certain level of comprehension". This proposal generated the following discussion:

> **Teacher B:** I agree with the modification Teacher C proposed. An individual test for each one.
> **Teacher C**: Yes, an individual test so we can see what each one achieved. Because we saw what the team achieved, but not each student individually.

> **Teacher A**: [...] I have a suggestion. For example, if we explored two problems, we could firstly present a problem for everyone. There would be a general problem and an individual problem. We could show them the problem and tell them to solve it. We could review the subject, then give them some time to solve it individually […] without starting anything about the subject, then we could follow the plan Teacher C already followed. It made a difference. But we could start with a question for students to try to solve based on their previous knowledge.
> **Teacher C**: [...] there is a point I'd like to draw attention to: the class is really large, so it would be hard for us to have a more individual interaction and get feedback from students who have difficulty. I didn't know if they were actually managing to do it. That's it, because the class was large […] (Reflection Stage, 2022).

The post-lesson reflection moment allowed the teachers to reflect on various aspects, including the execution of the plan, students' difficulties and possible modifications for future classes. Teacher C highlighted time as an obstacle for completing the planned activities, while Teacher A observed that, although students had some difficulty in constructing graphs with decimal numbers, they had a positive result globally.

Suggestions for changes included more practical examples and performing individual evaluations to better understand each student's progress, especially because it was a large group of students, which hindered individual interactions. Their reflections emphasized the importance of adapting teaching to better meet students' needs and skills and confirmed the mobilization of specialized content knowledge, knowledge of content and teaching and knowledge of content and students.

Furthermore, the teachers produced a written assessment of the main contributions the formative process made to teacher training, as can be seen in the following Table.

Table 4

Teachers' individual assessments of the formative process (Research collection, 2023)

Teacher A Mainly in the observation of small processes, it becomes more evident that difficulty in understanding a topic is linked to difficulty in communication, whether due to small gaps or to the type of mathematical language used.
Teacher B In this formative process, I learned many things, and I can highlight researching and searching for new methodologies to support Mathematics teaching. Perceiving that a teacher should be a researcher and be open to new methodologies, and that these new methodologies can be effective in teaching were great learning points. As for my reflections on teaching, they are in line with the contributions I received, with the idea of always being a researcher-teacher because it helps a lot in my teaching practice. Education needs research, new methodologies.

Teacher C
First, I'd like to thank you for the opportunity to participate in the study and, through it, learn more about semiotic representations and the Lesson Study methodology, which I didn't know anything about. Therefore, the best for me was learning about these methodologies and studies that can improve our teaching practice more and more. As for semiotic representations, all the work related to generating various representations and conversions among different forms of register made me reflect more on how I can organize my classes better. And it also led me to wonder if I'm really learning correctly all the necessary steps to help my students learn. Based on these reflections, I could bring the student's view into my practice, so I can raise questions to make my students think about what they're studying and if they understand it clearly, so they can manifest their mastery of the uses and representations of the concepts presented, resulting in satisfactory learning. About the Lesson Study methodology, it's important to emphasize that the teacher should always be open to try new things, that research is a central part of lesson planning, because knowledge changes, and we need to follow these changes. Therefore, it is indispensable for teachers to plan their classes in an interconnected way, with the stages and each goal you aim to achieve. This was something I was used to, but due to what is required from a teacher, we can't always plan classes the way we would like to. Finally, I highlight the professional improvement that the whole experience promoted, because the interconnection among us teachers was really interesting, not only for our study sessions, but especially when we planned the lesson, took care of every little detail and tried to anticipate every possible situation, until the time to teach the class came. It all occurred satisfactorily and everything we did contributed to the success of the process.

According to the teachers' feedback, the contributions from the formative process positively impacted each teacher individually and as a group. Their comments revealed that the process generated significant understanding, highlighting the relevance of effective communication, constant research, methodological innovation and cooperative work in order to improve teaching practice. Their positive feedback emphasizes the efficiency of the formative process in strengthening both teachers' individual skills and collaboration within the group, as can be observed in the Table 5.

Table 5

A summary of contributions from the formative process to teacher training (Produced by the authors, 2024)

Aspects	**Teacher A**	**Teacher B**	**Teacher C**	**Whole group**
Important aspects in training	- Changes to mathematical representations. - A detailed observation of students' answers. - Recognizing the complexity of previous stages in the learning process.	- Importance of proper planning. - Appreciation of post-class feedback. - Discovery of semiotic representations.	- Studying the suggested topics. - Knowledge of forms of register of semiotic representations and practicing the Lesson Study methodology.	- Emphasis on semiotic representations. - Recognizing the importance of lesson planning and execution, highlighting Lesson Study and the flexibility for adaptations.

Contributions to teaching	- Comprehension of semiotic representations and their impact on student learning. - Sensitivity to perceive students' difficulties and taking a proactive approach to addressing them.	- Integration of acquired knowledge into pedagogical practice. - Appreciation of planning rather than only of lesson execution.	- Collective engagement and flexibility to conduct studies, have discussions and plan activities.	- Appreciation of planning, lesson execution and post-lesson feedback. - Importance of teamwork and teachers' engagement.

Finally, the teachers' answers show a reflexive comprehension of the benefits provided by the formative process, ranging from improvement in communication to a comprehensive understanding of lesson planning and execution, and greater sensitivity to students' perceptions and difficulties during the teaching and learning process, confirming the mobilization of mathematical knowledge for teaching, with contributions from the TRSR in the formative process based on Lesson Study.

Conclusion

This paper aimed to analyze the re-elaboration of mathematical knowledge for teaching within a formative process based on Lesson Study, with contributions from the TRSR. It was possible to perceive that the formative process did not reinforce only content knowledge. It also improved pedagogical skills, thus promoting continuing teacher training and the mobilization, development and re-elaboration of mathematical knowledge for teaching with contributions from LS and the TRSR.

The investigation meetings allowed the teachers to immerse themselves in the Theory of Registers of Semiotic Representations, affine functions and analysis of pedagogical issues, especially harnessing specialized content knowledge, knowledge of content and teaching and knowledge of content and students. The group discussion on a problem situation highlighted the integration between theory and practice and emphasized the importance of mastering the content, understanding students and adapting teaching methods. The collective engagement and recognizing the importance of semiotic representations indicated progress in teacher education and in Mathematics teaching practices.

During planning and discussions involving the teachers, we highlight advances in the development and re-elaboration of specialized content knowledge, knowledge of content and teaching, knowledge of content and students and knowledge of content and curriculum. It is also relevant to mention collective work, a contextualized adaptation of problems and reflections on the methodology and didactical resources. The teachers showed commitment to teamwork by adapting the problems to semiotic representations of affine functions and searching for strategies to facilitate students' development of concepts. It was evident that they were in line with the BNCC, familiar with educational goals and willing to meet curriculum guidelines.

Based on the analysis of the lesson and teachers' knowledge, we highlight the interconnection of specialized content knowledge, knowledge of content and teaching and knowledge of content and students during their pedagogical practice. Teacher C showed his mastery of affine functions and had the ability to use teaching strategies that were appropriate for students' characteristics, promoting engagement and adapting his approach according to the group's needs. He displayed accessible communication skills and stimulated students' participation, which contributed to the mathematics teaching and learning processes.

The teachers' post-lesson reflections emphasized the need for adapting the plan, identifying students' difficulties, individual assessment, emphasis on communication and pedagogical research, as well as the benefits brought by collaboration and professional growth, which revealed the mobilization of specialized content knowledge, knowledge of content and teaching and knowledge of content and students. These reflections are important because they generate discussions aimed at improving pedagogical practice, especially highlighting the impact of collaborative work to generate significant Mathematics learning.

The study reported in this paper revealed significant advances in the development, re-elaboration and mobilization of mathematical knowledge for teaching through the integration of LS and the TRSR, which contributed to the continuous education of Mathematics teachers. There was progress in understanding mathematical concepts and interconnecting theoretical knowledge and pedagogical practices for teaching affine functions. As a suggestion for future research, we propose establishing collaborative groups that incorporate Lesson Study stages and integrative cycles supported by semiotic representations.

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